GEORGIA TECH SCHOOL OF MATHEMATICS

MATH 1502

CALCULUS II
Final Exam
December 16th 2005

First Name : _________________________ **Last Name :** ___________________________

WARNING :
1. Check that you have 18 pages
2. Put your name NOW ! on each of the 18 pages
3. read carefully, read the comments in italic, take your time, do not panic and double check what you write. Take the time to write in plain English the theorems and results you are using to justify your answer.

Analysis (24 pts)	
Linear Algebra (30 pts)	

Analysis

1. Give the Taylor series about $x = 0$, of

$$\frac{1}{1-x^2} =$$

2. Give the Taylor series about $x = 0$, of (*Hint : the previous formula can be used*)

$$\ln\left(\frac{1+x}{1-x}\right) =$$

3. Compute the limit of

$$\lim_{n\to\infty} \frac{n+\sqrt{n^2+\sqrt{3}}}{n+21} =$$

4. Compute
$$\lim_{n\to\infty} \frac{n}{(n^2+1)} + \frac{n}{(n^2+2^2)} + \cdots \frac{n}{(n^2+n^2)} \ =$$

5. Are the following series convergent and why ?

(a)

$$\sum_{n=1}^{\infty} \frac{n^{167}}{n!}$$

(b)

$$\sum_{n=1}^{\infty} \frac{1}{n^{.999}}$$

(c)

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1.001}}$$

6. Is the following series convergent and why ?

$$1 - \frac{1}{\ln 2} + \frac{1}{\ln 3} + \cdots + (-)^{n-1}\frac{1}{\ln n} + \cdots$$

7. What is the radius of convergence of the series
(*What test are you using ?*)

$$\sum_{n=1}^{\infty} \frac{(3)n!}{(n!)^3}\, x^{3n}$$

Radius $=$

Test $=$

8. Give a power series representation for the proper integral

$$\int_0^x \frac{\ln(1+t)}{t}\,dt \;=$$

9. Give the following limit : what is the rule you are using ?

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

Rule :

Linear Algebra

1. For which values of the parameters $a, b \in \mathbb{R}$ has the following system of linear equations a unique solution, no solutions or an infinite number ?

$$\begin{aligned} x - y + z - w &= -1 \\ 2z - w &= 0 \\ x - y + z &= -2 \\ ax + y - z - 2w &= b \end{aligned} \tag{1}$$

Unique solution a : b :

No solution a : b :

An infinite number a : b :

(*Use this page for row-reduction of the system (1)*)

2. Let M be an 5×7 matrix. What is the sum ?

$$\mathrm{Rank}(M) + \dim \mathrm{Ker}(M) \ =$$

3. Let W be a subspace of $\mathbb{R}^{13}$. Then compute the dimension of $W^\perp$ in terms of the dimensions of W

$$\dim(W^\perp) \ =$$

4. Let $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 3 & -3 \\ 0 & 1 & -2 & 2 \end{bmatrix}$ (*Use back pages to row reduce* A)

(a) What are the rank of A and the dimension of its kernel ?

$$\mathrm{Rank}(A) \ =$$

$$\dim \mathrm{Ker}(A) \ =$$

(b) Give a basis of $\mathrm{Im}(A)$

(c) Give a basis of $\mathrm{Ker}(A)$

(*Use this page for row-reduction of* A)

5. Compute the inverse of $B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$. (*Use back page to row reduce* B)

$$B^{-1} =$$

Check

$$B \cdot B^{-1} =$$

(*Use this page for row-reduction of* B)

6. Let the subspace $V \subset \mathbb{R}^4$ be spanned by

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \text{ and } \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Compute an orthonormal basis of V
(*Hint : use the Gram-Schmidt procedure*)

Basis : $\underline{u}_1 =$ $\qquad\qquad$ $\underline{u}_2 =$

(b) Compute the matrix of the projection onto V
($Hint$: $P = \underline{u}_1 \cdot \underline{u}_1^t + \underline{u}_2 \cdot \underline{u}_2^t$)

$P =$

(c) If $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, compute the distance from b to V.

7. Let $D = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

   (a) Compute the eigenvalues of D (*Hint : compute the characteristic polynomial and find its roots. The roots are obvious integers.*)

(b) Compute the corresponding eigenvectors.