GEORGIA TECH SCHOOL OF MATHEMATICS

MATH 1502

CALCULUS II

Final Exam : 2 hours & 50 minutes

December 12th 2008, Section K

First Name : ____

Last Name : ____

Students : please do not write anything in the box below!!

I.1 2 pts		II.1 2 pts		III.1 2 pts	
I.2 2 pts		II.2 2 pts		III.2 2 pts	
I.3 2 pts		II.3 2 pts		III.3 6 pts	
I.4 3 pts		II.4 4 pts		III.4 6 pts	
I.5 3 pts		II.5 8 pts			
I.6 8 pts		II.6 6 pts			

WARNING :

Put your name on the top of each page.

Read carefully, read the comments in italic, take your time, do not panic and double check what you write.

Write the result cleanly and use the blank pages for your calculations.

Take the time to write in plain English the arguments used to get or justify the answer.

I- Analysis

1. Expand $f(x) = (1 + x)^{1/2}$ as a power series in x.

2. Compute the following limit *(Indicate the rule used to do so)*

$$\lim_{x \to 0} \frac{x^2}{1 - \cos x}$$

Ruled used :

3. For which values of α is the following integral convergent or divergent ?

$$\int_0^\infty t^{\alpha-1}\, e^{-t}\, dt$$

Convergent for $\alpha \in$

Divergent for $\alpha \in$

4. Is the following series converging ? *(Give the test used to conclude)*

$$\sum \frac{1}{\sqrt{2k(k+1)}}$$

Converges ☐ *Diverges* ☐

Test used :

5. Is the following series converging *(Indicate the test used to conclude)*

$$\sum \frac{(k!)^2}{(2k)!}$$

Converges ☐ *Diverges* ☐

Test used :

6. Give the interval of convergence of
(Hint : analyze the convergence inside and on the boundary of the interval)

$$\sum_{k=0}^{\infty} \frac{x^k}{(2k+1)^{1/3}}$$

Interval of convergence =

(Use this page for your calculation)

II- Linear Algebra

1. Indicate whether the following function is linear or not and why

$$g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y + x \\ 0 \\ x - y \end{bmatrix}$$

Linear ? YES ☐ NO ☐

Why ?

2. Find all solutions (a, b, c) of the equation $A^2 = B$ whenever

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

3. Let $\mathbf{x}$, $\mathbf{y}$ be two vectors in $\mathbb{R}^n$ such that $|\mathbf{x}|^2+|\mathbf{y}|^2+|\mathbf{x}||\mathbf{y}| = |\mathbf{x}+\mathbf{y}|^2$. What is the angle between them ?

Angle $=$

4. Draw the image the unit square by the matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and compute its area.

Area $=$

5. Consider the system of linear equations

$$\begin{aligned} ax - y &= b \\ -x + ay - z &= 0 \\ -y + az &= 0 \end{aligned}$$

(a) Give the set of values of a, b for which this system have a unique solution.

(b) Then give this solution for any such values of a and b

(c) Give the set of values of a, b for which this system have no solution ?

(d) Give the set of values of a, b for which this system have an infinite number of solutions ?

Unique solution $a\,,\; b\; =$

Solution $=$

No solution $a\,,\; b\; =$

$\infty\#$ solutions $a\,,\; b\; =$

6. Compute the inverse and the determinant of B below
   (Hint : use the same row reduction method for both the inverse and the determinant)

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\det B \ =$

$B^{-1} \ =$

Check : $B \cdot B^{-1} \ =$

(Use this page for your calculation)

III- Problem

In this problem A denotes the matrix below and it will be asked to compute its double QR-factorization. It is advised to use row reduction on the question 1 and to check carefully the result in order to use it in the next questions.

$$A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ -1 & 0 & 0 & 1 \\ 1 & 2 & -2 & 3 \end{bmatrix}$$

1. Give a basis of $\mathrm{Im}(A)$. *(Hint : row reduce A and conclude)*
   (Use back page for your calculation)

   Basis of $\mathrm{Im}(A)$

2. Give a basis of $\mathrm{Ker}(A)$. *(Use back page for your calculation)*

   Basis of $\mathrm{Ker}(A)$

3. Write A as $Q_c R$ (QR-factorization) where Q_c is an isometry
(Hint : use 1 and a Gram-Schmidt orthonormalization procedure to get Q_c ; keep track of the size of both matrices Q_c and R)

$$Q_c =$$

$$R =$$

(Use this page for your calculation)

4. Give the QR-factorization of R^t (see 3) in the form $R = TQ_r^t$. Check that T is invertible

   (Hint : keep track of the size of both matrices Q_r and T)

$$Q_r =$$

$$T =$$

(Use this page for your calculation)