Georgia Tech — School of Mathematics

Math 1502D

Calculus II

Final Exam : 2 hours & 50 minutes

December 9th 2009, 1502D

First Name : ____

Last Name : ____

Students : please do not write anything in the box below!!

I.1 2 pts		II.1 3 pts		III.1 2 pts	
I.2 3 pts		II.2 6 pts		III.2 4 pts	
I.3 3 pts		II.3 2 pts		III.3 3 pts	
I.4 4 pts		II.4 4 pts		III.4 6 pts	
I.5 3 pts		II.5 5 pts		III.5 5 pts	
I.6 5 pts					

WARNING :

Put your name on the top of each page.

Read carefully, read the comments in italic, take your time, do not panic and double check what you write.

Write the result cleanly and use the blank pages for your calculations.

Take the time to write in plain English the arguments used to get or justify the answer.

I- Analysis

(Estimated time 53 minutes)

1. Compute the following limit *(Indicate the rule used to do so)*

   (Give the result here and make your computation on next page)

$$\lim_{x\to\pi/2} \frac{1+\cos 2x}{1-\sin x} =$$

Ruled used :

2. For which values of α is the following integral convergent or divergent ?

   (Give the result here and make your computation on next page)

$$\int_0^\infty \frac{x^{\alpha-1}dx}{1+x}$$

Convergent for $\alpha \in$

Divergent for $\alpha \in$

(Use this page for your calculation)

3. Give the Taylor series of the following function near $x = 0$

   (Hint : use the Taylor series for $\cos$ *and* $\sin$*)*

$$x \sin x - 2(\cos x - 1) =$$

4. (a) Is the following series converging? *(Give the test used to conclude)*
(Give the result here and make your computation on next page)

$$\sum_{k=2}^{\infty}(-1)^k \, \frac{k^2+3k+1}{(k-1)^2}$$

Converges ☐ *Diverges* ☐

Test used :

(b) Is the following series converging *(Give the test used to conclude)*
(Give the result here and make your computation on next page)

$$\sum_{k=2}^{\infty} \frac{k^2+3k+1}{(k-1)^{7/2}}$$

Converges ☐ *Diverges* ☐

Test used :

(Use this page for your calculation)

5. Give the interval of convergence of

   (Hint : analyze the convergence inside and on the boundary of the interval)

$$\sum_{k=0}^{\infty} \frac{(x-2)^k}{\sqrt{k^2+3k+1}}$$

Interval of convergence $=$

Convergence on the left

YES ☐ NO ☐

Convergence on the right

YES ☐ NO ☐

(Use this page for your calculation)

6. (a) Compute analytically the integral

$$I = \int_1^3 \frac{dx}{x}$$

.

(b) Use the trapezoidal rule to compute $\ln 3$ numerically with $n = 4$ interval.

(c) Compare with $\ln 3 = 1.098$ by giving the error.

(Give the result here and make your computation on next page)

Integral $I =$

Trapezoidal value for $\ln 3 \; =$

Error $=$

(Use this page for your calculation)

II- Linear Algebra

(Estimated time 56 minutes)

1. Let $\mathbf{u} = \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$, $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \left(\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\right)$ be three vectors in $\mathbb{R}^3$.

   (a) Compute the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (b) Compute the area of the triangle formed by the origin and the extremities of $\mathbf{u}$ and $\mathbf{v}$.

   (Hint : compute their cross product, be aware that only the triangle is concerned!)

   (c) Compute the volume of the parallelepiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

(Give the result here and make your computation on next page)

Angle $(\mathbf{u}, \mathbf{v}) =$

Area of triangle $(\mathbf{u}, \mathbf{v}) =$

Volume $(\mathbf{u}, \mathbf{v}, \mathbf{w}) =$

(Use this page for your calculation)

2. Consider the system of linear equations

$$\begin{aligned} y + 2z &= 0 \\ x + 2y + z &= 1 \\ ax + y &= b \end{aligned}$$

(a) Give the set of ALL values of a, b for which this system have a unique solution.

(b) Then give this solution for any such values of a and b

(c) Give the set of ALL values of a, b for which this system have no solution ?

(d) Give the set of ALL values of a, b for which this system have an infinite number of solutions ?

(Give the result here and make your computation on the back pagex)

Unique solution $a\,,\ b\ \in$

Solution $=$

No solution $a\,,\ b\ \in$

$\infty\#$ solutions $a\,,\ b\ \in$

(Use this page for your calculation)

3. Let A be an $m \times n$ matrix with rank r.

   (a) What are the dimension of its kernel and of its image?

   $$\dim(\mathrm{Ker}(A)) =$$

   $$\dim(\mathrm{Im}(A)) =$$

   (b) What is the relation between the previous two spaces and the corresponding space $\mathrm{Ker}(A^t), \mathrm{Im}(A^t)$ for its transposed?

   $$\mathrm{Ker}(A^t) =$$

   $$\mathrm{Im}(A^t) =$$

4. Compute the inverse and the determinant of A and check your result

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

(Give the result here and make your computation below)

$\det A =$

$A^{-1} =$

Check $A \cdot A^{-1} =$

(Use this page for your calculation)

5. Compute the distance between $\mathbf{b}$ and $\mathrm{Im}(A)$ if

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad\qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

(Hint : (i) the sum of the first two columns is equal to the third and conclude that the $\mathrm{Rank}(A) = 2$ *; (ii) replace* A *by a matrix* B *with maximal rank and same image ; (iii) compute the projection* P *on* $\mathrm{Im}(A) = \mathrm{Im}(B)$ *by using the method of least square solution ; (iv) compute the projection* $\mathbf{b}_0 = P\mathbf{b}$ *of* $\mathbf{b}$ *on* $\mathrm{Im}(A)$ *; (v) compute the distance between* $\mathbf{b}$ *and* $\mathbf{b}_0$*.)*

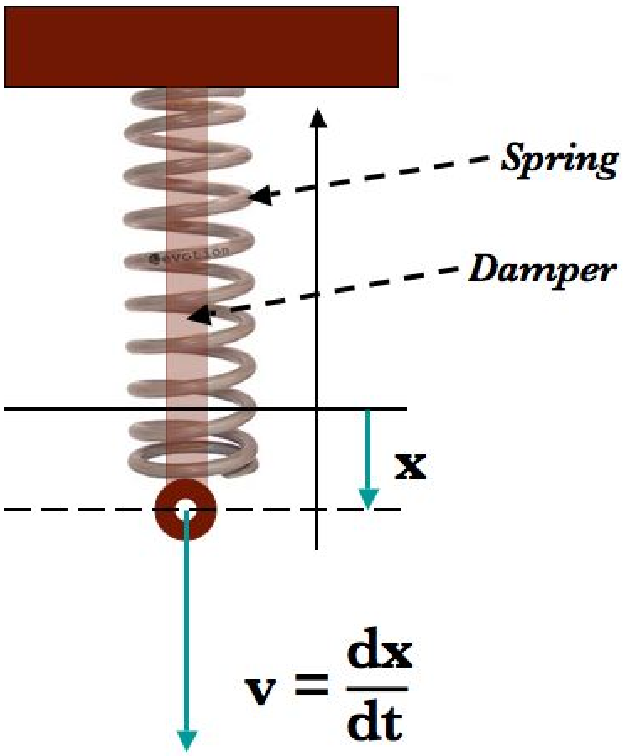
(Give the result here and make your computation below)

Distance =

(Use this page for your calculation)

III- Problem

(Estimated time 50 minutes)



A car suspension is represented by a spring attached to the body of the car on one side and on the axis of the wheel on the other side. The total force acting on the wheel side results from the superposition of the spring and the damper actions. The position at time t is given by $x(t)$ while the velocity is denoted by $v(t)$. For simplicity, the mass of the spring will be neglected, while the mass attached to the wheel will be supposed to be $m = 1$. The spring constant will be denoted by $k > 0$ while the friction coming from the damper will be denoted by $2f \geq 0$.

As a result, using Newton's law, the equation describing the motion of the wheel is (if t denotes the time)

$$\frac{dv}{dt} = -kx - 2fv\,,\ v = \frac{dx}{dt}$$

1. If $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$ write the equation of motion in the form of a linear differential equation $\mathbf{u}' = A\mathbf{u}$: give the explicit form of A and show that A is always invertible.

$$A =$$

$$\det(A) =$$

2. (a) Compute the eigenvalues, μ_+, μ_- and the corresponding eigenvectors $\mathbf{u}_\pm$ of A with first coordinate equal to one.

$$\mu_\pm =$$

$$\mathbf{u}_\pm =$$

(b) Show that there is a critical value $f_c > 0$ of the friction so that if $f < f_c$ the two eigenvalues are non-real complex conjugate numbers, namely $\mu_+ = f + \imath\omega = \overline{\mu_-}$, while if $f > f_c$ the two eigenvalues are real and distinct : give the values of f_c and of ω

$$f_c =$$

$$f < f_c \qquad \Rightarrow \qquad \omega =$$

$$f > f_c \qquad \Rightarrow \qquad \mu_+ > \mu_-$$

3. Whenever $f \neq f_c$, use the eigenvectors and the eigenvalues to write A as $A = VDV^{-1}$ where D is a diagonal matrix, while $V = [\mathbf{u}_+, \mathbf{u}_-]$: give the expression of V, V^{-1} and of D.

$$V \;=$$

$$V^{-1} \;=$$

$$D \;=$$

4. It is assumed that $f < f_c$ and that $x(0) = 0$, $v(0) = v_0$ (*initial conditions*). It will be admitted that the solution for $\mathbf{u}(t)$ is $\mathbf{u}(t) = e^{tA}\mathbf{u}(0)$ and that $e^{tA} = Ve^{tD}V^{-1}$.

   (a) Show, without calculation, that both coordinates x, v of $\mathbf{u}$ are linear combination of $e^{-ft+\imath t\omega}$ and $e^{-ft-\imath t\omega}$.

   (The grader will be especially in charge of rewarding sound arguments written in full plain English justifying the conclusion : these argument can be written below or on the next page)

   (b) Use the previous argument and the inital condition on x to show that $x(t) = ce^{-ft}\sin(\omega t)$ where c is some scalar that will not be computed here. The "de Moivre formula" $e^{\imath\theta} = \cos\theta + \imath\sin\theta$ can be used here.

   (The grader will be especially in charge of rewarding sound arguments written in full plain English justifying the conclusion : these argument can be written below or on the next page)

(c) From the previous result, compute $v(t)$ and, using the initial conditions again, deduce the expression of both $x(t)$ and $v(t)$, namely compute the scalar c previously introduced.

(The grader will be especially in charge of rewarding sound arguments written in full plain English justifying the conclusion : these argument can be written below or on the next page)

$$x(t) \ =$$

$$v(t) \ =$$

(d) Check that $v'(t) = -2fv(t) - kx(t)$

(Use this page for your calculations and reasonning)

5. This question concerns the case $f = f_c$:

   (a) Compute $x(t)$ in the limit $f \to f_c$.

   (Hint : l'Hôpital's rule might be used here)

$$x(t) \ =$$

   (b) Draw the graph of $x(t)$ as a function of time for $f < f_c$ and for $f = f_c$.

   (The grader will be especially in charge of rewarding graphs that are correct with complete information about the axis and the special points)

(Use this page for your calculations and reasonning)