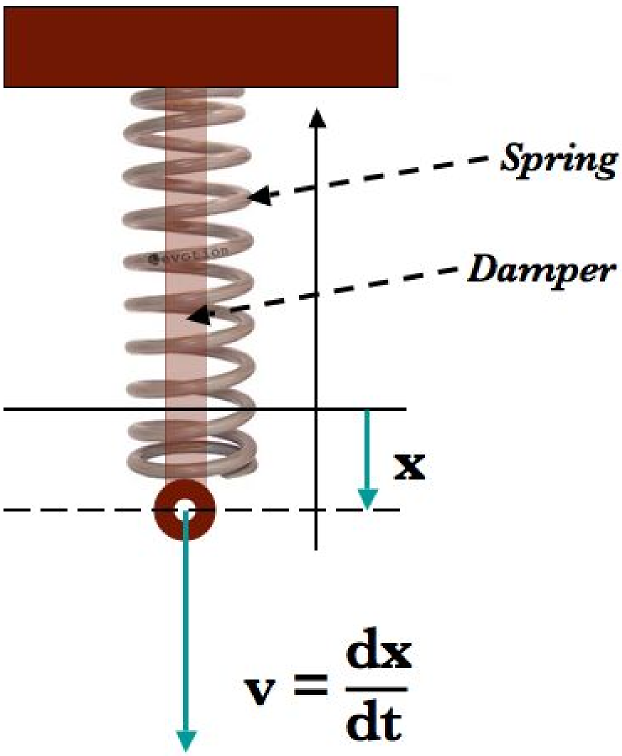
Georgia Tech School of Mathematics

Math 1502D

Calculus II

Final Exam : Solution

December 9th 2009, 1502D



A car suspension is represented by a spring attached to the body of the car on one side and on the axis of the wheel on the other side. The total force acting on the wheel side results from the superposition of the spring and the damper actions. The position at time t is given by $x(t)$ while the velocity is denoted by $v(t)$. For simplicity, the mass of the spring will be neglected, while the mass attached to the wheel will be supposed to be $m = 1$. The spring constant will be denoted by $k > 0$ while the friction coming from the damper will be denoted by $2f \geq 0$.

As a result, using Newton's law, the equation describing the motion of the wheel is (if t denotes the time)

$$\frac{d^2x}{dt^2} = -kx - 2fv$$

1. If $\mathbf{u}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$ write the equation of motion in the form of a linear differential equation $\mathbf{u}' = A\mathbf{u}$: give the explicit form of A and show that A is always invertible.

$$A = \begin{bmatrix} 0 & 1 \\ -k & -2f \end{bmatrix}$$

$$\det(A) = k \neq 0$$

2. (a) Compute the eigenvalues, μ_+, μ_- and the corresponding eigenvectors $\mathbf{u}_\pm$ of A with first coordinate equal to one.

$$\mu_\pm = -f \pm \sqrt{f^2 - k}$$

$$\mathbf{u}_\pm = \begin{bmatrix} 1 \\ \mu_\pm \end{bmatrix}$$

(b) Show that there is a critical value $f_c > 0$ of the friction so that if $f < f_c$ the two eigenvalues are non-real complex conjugate numbers, namely $\mu_+ = f + \imath\omega = \overline{\mu_-}$, while if $f > f_c$ the two eigenvalues are real and distinct : give the values of f_c and of ω

$$f_c = \sqrt{k}$$

$$f < f_c \quad \Rightarrow \quad \omega = \sqrt{k - f^2}$$

$$f > f_c \quad \Rightarrow$$

$$0 > \mu_+ = -f + \sqrt{f^2 - k} > \mu_- = -f - \sqrt{f^2 - k}$$

3. Whenever $f \neq f_c$, use the eigenvectors and the eigenvalues to write A as $A = UDU^{-1}$ where D is a diagonal matrix, while $U = [\mathbf{u}_+, \mathbf{u}_-]$: give the expression of U, U^{-1} and of D.

$$U = \begin{bmatrix} 1 & 1 \\ \mu_+ & \mu_- \end{bmatrix}$$

$$U^{-1} = \frac{1}{2\sqrt{f^2 - k}} \begin{bmatrix} -\mu_- & 1 \\ \mu_+ & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} \mu_+ & 0 \\ 0 & \mu_- \end{bmatrix}$$

4. It is assumed that $f < f_c$ and that $x(0) = 0$, $v(0) = v_0$ (*initial conditions*). It will be admitted that the solution for $\mathbf{u}(t)$ is $\mathbf{u}(t) = e^{tA}\mathbf{u}(0)$ and that $e^{tA} = Ue^{tD}U^{-1}$.

   (a) Show, without calculation, that both coordinates x, v of $\mathbf{u}$ are linear combination of $e^{-ft+\imath t\omega}$ and $e^{-ft-\imath t\omega}$.

   Since the eigenvalues of A are the diagonal elements of D, the diagonal elements of e^{tD} are $e^{t\mu_\pm}$. Since $f < f_c$ it follows that these eigenvalues are $e^{-tf\pm\imath t\omega}$. On the other hand, the two components of $e^tA\mathbf{u}(0)$, namely $x(t)$ and $v(t)$, are both linear combinations of the elements of e^{tD}, namely of $e^{-tf\pm\imath t\omega}$.

   (b) Use the previous argument and the inital condition on x to show that $x(t)$ is proportional to $e^{-ft}\sin(\omega t)$. The "de Moivre formula" $e^{\imath\theta} = \cos\theta + \imath\sin\theta$ can be used here.

   It follows from the previous remark, that there are two scalars $a_\pm$ such that

$$x(t) = a_+e^{-tf+\imath t\omega} + a_-e^{-tf-\imath t\omega}$$

   Since $x(0) = 0$, it follows, by replacing t by 0 in this formula, that $a_- = -a_+$ so that, using the De Moivre formula,

$$x(t) = a_+e^{-tf}\left(e^{\imath t\omega} - e^{\imath t\omega}\right) = 2\imath a_+e^{-tf}\sin\omega t$$

   Changing notations, this gives

$$x(t) = c\,e^{-tf}\sin\omega t \tag{1}$$

   where c is a scalar. Since $x(t)$ is a real number for all time, $c \in \mathbb{R}$.

(c) From the previous result, compute $v(t)$ and, using the initial conditions again, deduce the expression of both $x(t)$ and $v(t)$.

The eq. (1) gives for the velocity

$$v(t) = \frac{dx}{dt} = c\, e^{-ft}\,(\omega \cos \omega t - f \sin \omega t)$$

In particular, replacing t by 0 gives $v(0) = v_0 = c\omega$ so that

$$x(t) \;=\; v_0\, \frac{\sin \omega t}{\omega}\; e^{-ft} \qquad (2)$$

$$v(t) \;=\; v_0\, e^{-ft} \left(\cos \omega t - \frac{f\, \sin \omega t}{\omega} \right) \qquad (3)$$

(d) Check that $v'(t) = -2fv(t) - kx(t)$

Differentiating v *w.r.t.* time gives

$$v'(t) = v_0\; e^{-ft} \left(-f \cos \omega t + \frac{f^2\, \sin \omega t}{\omega} - f \cos \omega t - \omega\, \sin \omega t \right)$$

It should be remarked that the coefficient of $\sin \omega t$ in the bracket is $(f^2 - \omega^2)/\omega = (2f^2 - k)/\omega$. This gives

$$v'(t) = v_0\; e^{-ft}(-k)\, \frac{\sin \omega t}{\omega} - 2fv_0\; e^{-ft} \left(\cos \omega t - \frac{f\, \sin \omega t}{\omega} \right)$$

which is $v'(t) == -k\, x(t) - 2f\, v(t)$.

5. This question concerns the case $f = f_c$:

(a) Compute $x(t)$ in the limit $f \to f_c$.

If $f \to f_c$, it follows that $\omega \to 0$. Therefore, thanks to eq. (2) giving the value of $x(t)$ as a function of ω,

$$\lim_{\omega \to 0} x(t) = \lim_{\omega \to 0} v_0 \frac{\sin \omega t}{\omega} e^{-ft}$$

Thanks to l'Hôpital rule, it follows that, if $f = f_c = \sqrt{k}$

$$x(t) \;=\; v_0\, t\, e^{-f_c t} = \; v_0\, t\, e^{-\sqrt{k}\, t}$$

(b) Draw the graph of $x(t)$ as a function of time for $f < f_c$ and for $f = f_c$.