GEORGIA TECH SCHOOL OF MATHEMATICS

MATH 1512

HONOR CALCULUS II

Solutions to the Final Exam

December 9th, 2004

1. **Answer :**

$$\lim_{n\to\infty}(1+\frac{1}{n})^n \;=\; e \;=\; 2.718\cdots$$

Indeed, taking the log of the l.h.s. *gives* $n\ln(1+1/n) = n(1/n + O(1n^2)) \to 1$ *as* $n\to\infty$.

$$\lim_{x\to 0} e^{-1/x} \;=\; \left\{ \begin{array}{l} 0 \;\text{ if } x\to 0^+ \\ +\infty \;\text{ if } x\to 0^- \end{array} \right\}$$

$$\lim_{x\to 0}\cos(1/x+\pi/3) \;=\; \text{undefined}$$

Indeed, as $x\to 0$ $1/x\to\infty$ *so that the* cos *oscillates between* -1 *and* $+1$ *indefinitely.*

2. **Answer :**

$$\frac{1}{1-x^2} \;=\; 1+x^2+x^4+\cdots+x^{2m}+\frac{x^{2m+2}}{1-x^2}$$

The proof comes from setting $x^2=y$ *and from*

$$\frac{1}{1-y} = 1+\frac{y}{1-y} \tag{1}$$

Replacing the $1/(1-y)$ *by eq. (1) in the* r.h.s. *of eq. (1) gives*

$$\frac{1}{1-y} = 1+y+\frac{y^2}{1-y} = 1+y+y^2+\frac{y^3}{1-y} = \cdots = 1+y+\cdots+\frac{y^{m+1}}{1-y}$$

3. **Answer :**

$$\frac{1}{(1-x)^{1/3}} \;=\; 1+\frac{x}{3}+\frac{1\cdot 4}{3^2\cdot 2}x^2+\cdots+\frac{1\cdot 4\cdot 7\cdots(3n-2)}{3^n n!}x^n+\cdots$$

The Taylor series *of* $(1+x)^\alpha$ *is given by*

$$(1+x)^\alpha \;=\; 1+\alpha x+\alpha(\alpha-1)\frac{x^2}{2}+\cdots+\alpha(\alpha-1)(\alpha-n+1)\frac{x^n}{n!}+\cdots$$

If x *is replaced by* $-x$ *and* α *by* $-1/3$ *this gives the answer (Note : all signs cancel out)*

4. Are the following series convergent or not ?

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (x \in \mathbb{R}) \qquad \text{YES } \textit{by ratio test}$$

$$\sum_{n=0}^{\infty} (.9999)^n \qquad \text{YES } \textit{by ratio test, since } .9999 < 1$$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \qquad \text{NO } \textit{by integral test}$$

The integral $\int_2^A dx/x \ln x$ *can be computed by setting* $u = \ln x$. *It becomes* $\int_{\ln 2}^{\ln A} du/u = \ln\ln A - \ln\ln 2$ *which diverges as* $A \to \infty$

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \qquad \text{YES } \textit{by integral test}$$

The integral $\int_2^A dx/x \ln^2 x$ *can also be computed by setting* $u = \ln x$. *It becomes* $\int_{\ln 2}^{\ln A} du/u^2 = 1/\ln 2 - 1/\ln A$ *which converges as* $A \to \infty$

$$\sum_{n=2}^{\infty} \frac{1}{(\ln^2 n)^n} \qquad \text{YES } \textit{by comparison or by ratio test}$$

Here two types of arguments can be used :
(i) by comparison : for n *large enough,* $\ln n \geq 2$ *so that the series is bounded from above by* $1/2^n$ *which is a convergent harmonic series ;*
(ii) by ratio test :

$$\frac{x_{n+1}}{x_n} = \frac{(\ln n)^{2n}}{(\ln(n+1))^{2n+2}} = \left(\frac{\ln n}{\ln(n+1)}\right)^{2n} \frac{1}{\ln^2(n+1)} = O(1)\frac{1}{\ln^2(n+1)} \to 0$$

This is because $\ln(n+1) = \ln n + \ln(1 + 1/n) = \ln n + 1/n + O(1/n^2)$ *so that*

$$\frac{\ln n}{\ln(n+1)} = \frac{1}{1 + O(1/(n \ln n))} \quad \Rightarrow \quad \left(\frac{\ln n}{\ln(n+1)}\right)^{2n} = e^{-2nO(1/(n\ln n))} = O(1)$$

$$\sum_{n=3}^{\infty} (-1)^n \frac{1}{\ln\ln n} \qquad \text{YES } \textit{by Leibniz theorem}$$

The sequence $1/n\ln n$ *is decreasing, converges to zero, so that the alternate series converges simply.*

5. **The Euler constant :**

   (a) For $n < x < n+1$ then $1/(n+1) < 1/x < 1/n$. Therefore

$$\frac{1}{n+1} < \int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n < \frac{1}{n}$$

   (b) The previous inequality gives (using $\ln 1 = 0$)

$$\begin{aligned}\ln(n+1) &= (\ln(n+1) - \ln n) + (\ln n - \ln(n-1)) + \cdots + (\ln 2 - \ln 1) \\ &< \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1\end{aligned}$$

   Thus $\xi_n = 1 + 1/2 + \cdots + 1/n - \ln(n+1) > 0$.

   (c) From the definition of ξ_n it follows that $\xi_n - \xi_{n-1} = 1/n + \ln n - \ln(n+1)$. Using (5a) above this gives

$$0 < \xi_n - \xi_{n-1} < \frac{1}{n} - \frac{1}{n+1}$$

   (d) It follows that for $m > n$

$$\xi_m - \xi_n = (\xi_m - \xi_{m-1}) + (\xi_{m-1} - \xi_{m-2}) + \cdots + (\xi_{n+1} - \xi_n)$$

   Hence, thanks to (5c), $\xi_m - \xi_n > 0$ and

$$\begin{aligned}\xi_m - \xi_n &= (\xi_m - \xi_{m-1}) + (\xi_{m-1} - \xi_{m-2}) + \cdots + (\xi_{n+1} - \xi_n) \\ &< \left(\frac{1}{m} - \frac{1}{m+1}\right) + \left(\frac{1}{m-1} - \frac{1}{m}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \frac{1}{n+1} - \frac{1}{m+1} < \frac{1}{n+1}\end{aligned}$$

   (e) The inequality obtained in (5d) shows that $(\xi_n)_{n>0}$ is an increasing sequence such that (use (5d) with $n = 1$)

$$0 < \xi_1 < \xi_n < \xi_1 + \frac{1}{2}$$

   Hence the sequence is bounded and thus converges. Another argument is that, since $\lim_{n\to\infty} 1/(n+1) = 0$, $(\xi_n)_{n>0}$ is a Cauchy sequence, and thus converges to $C \in \mathbb{R}$. Since $\xi_1 = 1 - \ln 2 = 0.307\cdots$ it follows that $1/2 + \xi_1 = 0.807\cdots < 1$ so that the limit $C = \lim_{n\to\infty} \xi_n$ exists and satisfies $0 < 0.307\cdot \leq C \leq 0.807\cdot < 1$.

   C is called the Euler constant and $C = 0.577215665\cdots$.

   See `http://mathworld.wolfram.com/Euler-MascheroniConstant.html` *for more informations.*

6. Using the Gauss method find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Answer :

$$A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 & 1 & -2 \\ 2 & 2 & -1 & 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

Proof : *To find the inverse of A, it is sufficient to solve the system of linear equations $AB = I$, with solution (if it exists) $B = A^{-1}$, and which can be written as*

$$a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + a_{i4}b_{4j} = \delta_{ij} \qquad \forall i, j \in \{1, \cdots, n\} \tag{2}$$

In the previous expression a_{ij} stands for the (i, j)-cœfficient of A and δ_{ij} is the Kronecker symbol *defined by $\delta_{ii} = 1$ and $\delta_{ij} = 0$ when $i \neq j$. This gives 16 equations with 16 unknown given by the matrix elements $((b_{ij}))$ of B. To compute this inverse it is convenient to organize the computation as follows :*

– *write A and the r.h.s. of 2 in a 4×8 rectangular matrix as*

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

– *then by adding the right multiple of the first row to each row below, cancel out the coefficients of the first column below the first row. Since only the second row has a nonzero cœfficient on the first column, only one such operation is necessary. This amounts to add row #1 to #2 and to normalize*

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right], \quad \Rightarrow \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

– *Next, keep canceling out the terms below the main diagonal of the* l.h.s.. *In our case its amounts to add row #2 to #3 and to normalize again :*

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right], \quad \Rightarrow \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

– *Proceeds in the same way with rows #3 and #4*

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{5}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right], \quad \Rightarrow \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right]$$

– *Next proceed upward to cancel out row by row the nondiagonal elements of the* l.h.s.. *This amounts to subtract $2/3\times$row#4 from row#3 (no normalization is needed anymore) then to subtract $1/2\times$row#3 from row#2*

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right], \quad \rightarrow \quad \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right]$$

– *At last, subtracting row#2 from row#1 gives*

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & 0 & \frac{2}{5} & \frac{2}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 0 & 0 & 1 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right], \quad \Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -2 & 1 & -2 \\ 2 & 2 & -1 & 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & 3 \end{bmatrix}$$

namely A^{-1} can be read on the r.h.s. *of the rectangular matrix. To make sure no mistake were made during the Gauss process, it is careful to check this answer namely*

$$AA^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \times \frac{1}{5}\begin{bmatrix} 3 & -2 & 1 & -2 \\ 2 & 2 & -1 & 2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 2 & 3 \end{bmatrix} = I$$

7. Given a system of 106 equations with 108 variables how many solution can one expect ?
   Answer : either no solution or an infinite number.

8. What are the *kernel* and the *image* of the linear map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by $T(x, y, z) = (2x + y + z, x + 2y + z, 3x + 3y + 2z)$? Compute their dimensions.
   Answer :

$$\text{Ker } T = \{X \in \mathbb{R}^3 \,;\, \exists a \in \mathbb{R}\,,\, X = a(1,1,-3)\} = \text{Span}\{(1,1,-3)\} \qquad \dim \text{Ker } T = 1$$

$$\text{Im } T = \{Y = (x', y', z') \in \mathbb{R}^3 \,;\, x' + y' = z'\} \qquad \dim \text{Im } T = 2$$

Remark : *the vector $(1, 1, -3)$ can be replaced by any nonzero multiple in the first equality.*

Proof : *By definition* Ker T *is the set of vectors $X = (x, y, z) \in \mathbb{R}^3$ such that $T(X) = 0$. To get such X's it is necessary and sufficient to solve the linear system*

$$2x + y + z = 0 \qquad x + 2y + z = 0 \qquad 3x + 3y + 2z = 0$$

The solutions are of the form $x = y = -z/3$. Therefore $X = a(1, 1, -3)$ for $a \in \mathbb{R}$, namely

$$\text{Ker } T = \{X \in \mathbb{R}^3 \,;\, \exists a \in \mathbb{R}\,,\, X = a(1,1,-3)\} = \text{Span}\{(1,1,-3)\}$$

In the previous equality, the vector $(1, 1, -3)$ can be replaced by any nonzero multiple , such as $(1/3, 1/3, -1)$. Consequently $\{(1, 1, -3)\}$ is a basis of Ker T *and thus*

$$\dim \text{Ker } T = 1$$

The image of T is the set of vectors Y of the form $Y = T(X)$ for some $X \in \mathbb{R}^3$. Since Ker $T \neq \{0\}$ *this image is not the full space because*

$$\dim \text{Ker } T + \dim \text{Im } T = \dim \mathbb{R}^3 = 3 \qquad \Rightarrow \qquad \dim \text{Im } T = 2$$

Setting $Y = (x', y', z') = T(x, y, z)$ leads to

$$x' = 2x + y + z\,, \qquad y' = x + 2y + z \qquad z' = 3x + 3y + 2z$$

implying that $x' + y' = z'$. *Conversely, if* $x' + y' = z'$, *then some* $(x, y, z) \in \mathbb{R}^3$ *can be found (it might not be unique!!) such that* $x' = 2x + y + z$, $y' = x + 2y + z$, $z' = 3x + 3y + 2z$. *For indeed a Gauss process leads to the following solution* $x = (2x' - y')/3$, $y = (2y' - x')/3$ *and* $z = 0$. *This argument shows that*

$$\mathrm{Im}\ T \;=\; \{Y = (x', y', z') \in \mathbb{R}^3\,;\, x' + y' = z'\} \tag{3}$$

A basis of $\mathrm{Im}\ T$ *can be constructed by choosing, for instance,* $x = 1, y = 0$ *or* $x = 0, y = 1$, *namely*

$$Y_1 = (1, 0, 1) \qquad\qquad Y_2 = (0, 1, 1)$$

It is simple to check that both vectors are in $\mathrm{Im}\ T$ *(see 3) . Moreover they are linearly independent because if* $0 = b_1Y_1 + b_2Y_2$ *then* $b_1Y_1 + b_2Y_2 = (b_1, b_2, b_1 + b_2) = 0$ *implying* $b_1 = 0 = b_2$. *At last, any element of* $\mathrm{Im}\ T$ *can be written as a linear combination of* Y_1 *and* Y_2 *because* $Y \in \mathrm{Im}\ T \Leftrightarrow Y = (x', y', x' + y')$ *for some real numbers* x', y' *so that* $Y = x'Y_1 + y'Y_2$. *Since the basis* $\{Y_1, Y_2\}$ *of* $\mathrm{Im}\ T$ *has two vectors*

$$\dim \mathrm{Im}\ T \;=\; 2$$

9. For any three scalars a, b, c let $A = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. For any three scalars u, v, w let $B = \begin{bmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix}$.

   (a) Compute AB

$$AB \;=\; \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} 1 & a+u & b+v+aw \\ 0 & 1 & c+w \\ 0 & 0 & 1 \end{bmatrix} \tag{4}$$

   (b) Conclude that A is invertible and compute its inverse.

   For indeed, looking at (4) shows that $AB = I$ *admits a solution namely*

$$a + u = 0,\;\; c + w = 0,\;\; b + v + aw = 0, \qquad \Leftrightarrow \qquad u = -a,\;\; w = -c,\;\; v - ac - b$$

   Since $AB = I$ *is equivalent to* $B = A^{-1}$ *this gives*

$$A^{-1} \;=\; \begin{bmatrix} 1 & -a & ac - b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}$$

10. Let V be the space spanned by the monomials $\{X^2, X^5, X^8\}$ and let W be the space spanned by the monomials $\{X, X^5, X^9\}$. What are the space $V \cap W$ and $V + W$?

Answer :

$$V \cap W \;=\; \mathrm{Span}\{X^5\} \qquad\qquad V + W \;=\; \mathrm{Span}\{X, X^2, X^5, X^8, X^9\}$$

Remark : *In this question it is implicitly assumed that* V, W *are vector spaces over* $K = \mathbb{R}$ or $\mathbb{C}$. *Consequently they are both subspaces of the space* $K[X]$ *of polynomials with respect to the variable* X.

See `http://www.purplemath.com/modules/polydefs.htm` *for a definition of polynomials.*

See `http://www.edhelper.com/polynomials.htm` *for exercises on polynomials.*

Proof : *The elements of V are polynomials of the form*

$$p \in V \quad \Leftrightarrow \quad p(X) = p_2X^2 + p_5X^5 + p_8X^8 \quad \text{where } p_2, p_5, p_8 \in K$$

Similarly the elements of W are of the form

$$q \in W \quad \Leftrightarrow \quad q(X) = q_1X + q_5X^5 + q_9X^9 \quad \text{where } q_1, q_5, q_9 \in K$$

Therefore a polynomial belonging to the intersection $V \cap W$ must have both forms simultaneously, which is only possible if the terms of degree not equal to 5 vanish. Thus $r \in V \cap W \Leftrightarrow r(X) = r_5X^5$ for some $r_5 \in K$, namely $V \cap W$ is spanned *by the monomial X^5.*

In much the same way, elements of $V + W$ are polynomials $s \in K[X]$ that can be written as a sum of a polynomial $p \in V$ and of a polynomial $q \in W$. Thus s is a polynomial of the form

$$s \in V + W \quad \Leftrightarrow \quad s(X) = s_1X + s_2X^2 + s_5X^5 + s_8X^8 + s_9X^9 \quad \text{where } s_1, s_2, s_5, s_8 s_9 \in K$$

In other words $V + W$ is spanned by the monomials $\{X, X^2, X^5, X^8, X^9\}$.

11. Let $\mathbb{R}[X]$ be the real vector space of polynomials in X with real cœfficients. Let $T : \mathbb{R}[X] \mapsto \mathbb{R}[X]$ be the linear operator defined by $T(p) = Xp' - 7p$ where p' denote the first derivative of $p \in \mathbb{R}[X]$.
    (a) Compute $T(X^n)$ for all $n \in \mathbb{N}$
    (b) Compute the kernel of T.
    (c) Compute the image of T.

Answer :

$$T(X^n) = (n-7)X^n \qquad \text{Ker } T = \text{Span}\{X^7\}$$
$$\text{Im } T = \{q \in \mathbb{R}[X]\,;\ q(X) = \sum_{k \geq 0} q_k X^k\ ,\ q_7 = 0\}$$

Proof :

(a) $T(X^n) = X dX^n/dX - 7X^n = (n-7)X^n$ *for all* $n \in \mathbb{N}$.

(b) By definition, the kernel of T is the set of polynomials $p \in \mathbb{R}[X]$ solutions of the equation $T(p) = 0$. Writing p as

$$p(X) = p_0 + p_1X + p_2X^2 + \cdots + p_nX^n + \cdots + p_NX^N$$

where N is its degree *and $p_0, p_1, \cdots, p_N$ are its* cœfficients *(namely real numbers here), leads to*

$$\begin{aligned} T(p) &= p_0T(1) + p_1T(X) + p_2T(X^2) + \cdots + p_nT(X^n) + \cdots + p_NT(X^N) \\ &= -7p_0 - 6p_1X - 5p_2X^2 + \cdots (n-7)p_nX^n + \cdots + (N-7)p_NX^N \end{aligned} \tag{5}$$

Consequently $p \in \text{Ker } T$ if and only if the r.h.s *of the last equation vanishes identically, namely if and only if $(n-7)p_n = 0$ for all n's. If $n \neq 7$ this implies $p_n = 0$, whereas for $n = 7$ the terms $(7-7)p_7 = 0$ whatever the value of p_7. Hence $p \in \text{Ker } T$ if and only $p = p_7X^7$. This means that* Ker T *is exactly the subspace spanned by X^7.*

(c) By definition, a polynomial q belongs to the image of T if and only if it can be written as $q = T(p)$ for some polynomial $p \in \mathbb{R}[X]$. Thanks to eq. (5) such a p exists if and only if it satisfies the equation $(n-7)p_n = q_n$. Thus, if $n \neq 7$ this gives $p_n = q_n/(n-7)$ while if $n = 7$ there is a solution if and only if $q_7 = 0$. Hence Im T *is exactly the set of polynomials with real cœfficients with cœfficient of degree 7 vanishing.*

12. **Problem :** Let a, b, c, d, e be scalars in $K = \mathbb{R}$ or $\mathbb{C}$. The goal of this problem is to compute the determinant of the matrix

$$\det A_5 \;=\; \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \\ a^2 & b^2 & c^2 & d^2 & e^2 \\ a^3 & b^3 & c^3 & d^3 & e^3 \\ a^4 & b^4 & c^4 & d^4 & e^4 \end{vmatrix} \;=\; G(a,b,c,d,e)$$

It will be convenient to use the following vectors

$$X(a) \;=\; \begin{bmatrix} 1 \\ a \\ a^2 \\ a^3 \\ a^4 \end{bmatrix}$$

Let $D(X^1, X^2, X^3, X^4, X^5)$ be the 5-linear K-valued totally antisymmetric map such that $D(E^1, E^2, E^3, E^4, E^5) = 1$ if $\{E^1, E^2, E^3, E^4, E^5\}$ denotes the canonical basis of K^5.

(a) Show that $G(a,b,c,d,e) = D\left(X(a), X(b), X(c), X(d), X(e)\right)$.

The determinant of a 5×5 *matrix* A *is the scalar* $\det A$ *such that, for any family* $\{X^1, X^2, X^3, X^4, X^5\}$ *of vectors in* K^5

$$D(AX^1, AX^2, AX^3, AX^4, AX^5) = \det A \;\; D(X^1, X^2, X^3, X^4, X^5)$$

It is indeed a (remarkable) Theorem that this number DOES NOT depend upon the choice of the five vectors $\{X^1, X^2, X^3, X^4, X^5\}$ *nor does it depend upon the choice of* D *as long as* D *is* 5*-linear and totally antisymmetric!! If* D *is normalized by the condition* $D(E^1, E^2, E^3, E^4, E^5) = 1$*, it follows from the previous formula that*

$$D(AE^1, AE^2, AE^3, AE^4, AE^5) = \det A$$

The vector $X^i = AE^i$ *is exactly the ith-column of the matrix* A*. Applied to* $A = A_5$ *above this gives*

$$A_5E^1 = X(a)\,, \quad A_5E^2 = X(b)\,, \quad A_5E^3 = X(c)\,, \quad A_5E^4 = X(d)\,, \quad A_5E^5 = X(e)\,.$$

so that

$$G(a,b,c,d,e) = \det A_5 = D\left(X(a), X(b), X(c), X(d), X(e)\right)$$

(b) Conclude then that $G(a,b,c,d,e)$ is a polynomial in each of the (a,b,c,d,e) of degree at most 4.

The multilinearity of $D(X^1, X^2, X^3, X^4, X^5)$ *in each vector* X^i *shows that it can be decomposed into a sum of terms, each of which being a product of the form* $\pm c_1c_2c_3c_4c_5D(E^1, E^2, E^3, E^4, E^5) = \pm c_1c_2c_3c_4c_5$ *(by normalization) where* c_i *is one of the coordinates of* X^i*. Since* $X(a)$ *have coordinates given by powers of* a *of degree less than or equal to* 4*, it follows that* $c_1c_2c_3c_4c_5 = a^{k_1}b^{k_2}c^{k_3}d^{k_4}e^{k_5}$ *where the* k_i*'s are integers between* 0 *and* 4*. Hence,* G *is a polynomial with respect to each of the* 5*-variables* a, b, c, d, e *of degree at most* 4*.*

(c) Conclude also that $G(a,b,c,d,e)$ changes sign if two letters are exchanged.

Exchanging two letters in $G(a,b,c,d,e)$ *is equivalent to exchanging the corresponding columns of* A_5*, and therefore, due to the total antisymmetry of* D*, it results in a change of sign.*

(d) Conclude also that $G(a,b,c,d,e) = 0$ if two of the (a,b,c,d,e) are equal.

If two of the numbers a,b,c,d,e are equal, $G(a,b,c,d,e)$ does not change by exchanging them, but at the same time, by antisymmetry, it changes sign. Thus $G(a,b,c,d,e) = -G(a,b,c,d,e)$ when that happens, namely $G(a,b,c,d,e) = 0$.

(e) Show then that, if (b,c,d,e) are considered as parameters, the polynomial $P(a) = G(a,b,c,d,e)$ admits (b,c,d,e) as roots and thus $P(a) = R(b-a)(c-a)(d-a)(e-a)$ where R depends only on (b,c,d,e).

Thanks to the previous result (12b), $P(a)$ is a polynomial of degree at most 4 in a, with cœfficients depending polynomialy on b,c,d,e. Moreover (12d) shows that $P(a)$ vanishes for $a = b,c,d,e$. Therefore b,c,d,e are roots of a, so that P can be decomposed into prime factor as $P(a) = R(b-a)(c-a)(d-a)(e-a)$ where R is also a polynomial in a,b,c,d,e. But since P has degree at most 4 w.r.t. a, and since $(b-a)(c-a)(d-a)(e-a)$ is also a polynomial od degree 4 in a, it follows that R does not depend on a.

(f) By using the same argument with (b,c,d,e) instead of a, conclude that
$G(a,b,c,d,e) = g_5(b-a)(c-a)(d-a)(e-a)(c-b)(d-b)(e-b)(d-c)(e-c)(e-d)$
where g_5 is some scalar.

Thanks to the previous result (12d), R is a polynomial of degree at most 4 in b, vanishing for $b = c,d,e$, with cœfficients depending polynomialy on c,d,e. The same argument shows that $R = S(c-b)(d-b)(e-b)$ where S is a polynomial in b,c,d,e. Thus $G(a,b,c,d,e) = S(c-b)(d-b)(e-b)(b-a)(c-a)(d-a)(e-a)$. The product in the r.h.s. is already a polynomial of degree 4 in b, so that S cannot depend on b either. Again due to (12d), S is a polynomial of degree at most 4 in c, which vanishes for $c = d,e$ Thus $S = T(d-c)(e-c)$. Hence $G(a,b,c,d,e) = T\,(d-c)(e-c)(c-b)(d-b)(e-b)(b-a)(c-a)(d-a)(e-a)$. Again, since the product of the r.h.s. is a polynomial of degree 4 in c also, T cannot depend on c. Hence T is a polynomial of degree at most 4 in each of d,e vanishing when $e = d$ so that $T = U\,(e-d)$ where U is a polynomial of degree at most 4 in e,d. Hence again $G(a,b,c,d,e) = U\,(e-d)(d-c)(e-c)(c-b)(d-b)(e-b)(b-a)(c-a)(d-a)(e-a)$. But since the product has degree 4 in all variables, U cannot depend on any of the five numbers a,b,c,d,e. Thus $g_5 = U$ is a scalar.

(g) If $a = 0$, use Cramer's rule to show that $G(0,b,c,d,e) = bcde \cdot \det A_4$ where A_4 is obtained from A_5 by removing the last row and the first column. Conclude that $\det A_4 = g_5(c-b)(d-b)(e-b)(d-c)(e-c)(e-d)$.

To compute the scalar $g_5 \in K$, let $a = 0$. Then from the previous formula proved in (12f) it follows that

$$G(0,b,c,d,e) = g_5 bcde \cdot (e-d)(d-c)(e-c)(c-b)(d-b)(e-b)$$

On the other hand

$$G(0,b,c,d,e) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & b & c & d & e \\ 0 & b^2 & c^2 & d^2 & e^2 \\ 0 & b^3 & c^3 & d^3 & e^3 \\ 0 & b^4 & c^4 & d^4 & e^4 \end{vmatrix}$$

Cramer's rule, applied to the first column, leads to

$$G(0,b,c,d,e) = \begin{vmatrix} b & c & d & e \\ b^2 & c^2 & d^2 & e^2 \\ b^3 & c^3 & d^3 & e^3 \\ b^4 & c^4 & d^4 & e^4 \end{vmatrix} = bcde \begin{vmatrix} 1 & 1 & 1 & 1 \\ b & c & d & e \\ b^2 & c^2 & d^2 & e^2 \\ b^3 & c^3 & d^3 & e^3 \end{vmatrix} = bcde \det A_4$$

Hence

$$\det A_4 \;=\; g_5 \cdot (e-d)(d-c)(e-c)(c-b)(d-b)(e-b)$$

(h) Proceeding recursively, show that $g_5 = 1$.

Let A_3, A_2 be defined by

$$A_3 \;=\; \begin{bmatrix} 1 & 1 & 1 \\ c & d & e \\ c^2 & d^2 & e^2 \end{bmatrix} \qquad\qquad A_2 \;=\; \begin{bmatrix} 1 & 1 \\ d & e \end{bmatrix} \tag{6}$$

The same argument as before applied to A_4 instead of A_5, then to A_3 instead of A_4, leads to

$$\det A_3 \;=\; g_5 \cdot (e-d)(d-c)(e-c) \qquad\qquad \det A_2 \;=\; g_5 \cdot (e-d)$$

But the formula (6) shows that $\det A_2 = e - d$ *so that* $g_5 = 1$.