Georgia Tech

School of Mathematics
math 4280

Information Theory

Final Exam

May 5th, 2011, 120 minutes

First Name : ____

Last Name : ____

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1. **A Metric:** A function $\rho(x,y)$ defined on the space $\mathcal{X} \times \mathcal{X}$, is a *metric*, whenever it satisfies, for all x, y, z

   - $\rho(x,y) \geq 0$,
   - $\rho(x,y) = \rho(y,x)$,
   - $\rho(x,y) = 0$ if and only if $x = y$,
   - $\rho(x,y) + \rho(y,z) \geq \rho(x,y)$.

   (a) Let X, Y be two random variables with values in $\mathcal{X}$. Then show that $\rho(X,Y) = H(X|Y) + H(Y|X)$ satisfies the first, second and fourth properties above.

   (b) If two random variables are considered as equal whenever, there is a one-to-one mapping transforming one into the other, then show that the third property is also satisfied.

   (c) Verify that

$$\begin{aligned}
\rho(X,Y) &= H(X) + H(Y) - 2I(X;Y) \\
&= H(X,Y) - I(X;Y) \\
&= 2H(X,Y) - H(X) - H(Y)\,.
\end{aligned}$$

(Use this page for your solution)

2. **Maximum Entropy:** Find the probability distribution $p(x)$ of a random variable X, taking on integer values, that maximizes the entropy $H(X)$ subject to the constraint

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} np(n) = A$$

Compute this distribution as a function of A.

(Hint: two Lagrange multipliers are needed. Compute p and A in terms of them and inverse these formulae)

(Use this page for your solution)

3. **Asymptotic Equipartition:** Let $X_1, X_2, \cdots$ be *i.i.d.* drawn according to the probability distribution $p(x)$. Find

$$\lim_{n\to\infty} (p(X_1, \cdots, X_n))^{1/n}$$

(Hint: take the log and use the Law of Large Numbers)

4. **Huffman Coding:** Consider the random variable

$$X = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 0.49 & 0.26 & 0.12 & 0.04 & 0.04 & 0.03 & 0.02 \end{pmatrix}$$

(a) Find a binary Huffman code.

(b) Find the expected code length.

(c) One admits that the entropy of X is $H(X) = 2.0128$: is your result compatible with this value ? Explain why.

(Use this page for your calculations)

5. **Differential Entropy:** Let the variable X take values in the positive real line $[0, \infty)$:

   (a) The positive random variable X is called *exponential* if its distribution is given by $p(x) = C \exp\{-x/\mu\}$ where $C, \mu > 0$. Compute C and $\mathbb{E}(X)$ in terms of μ.

   (b) Compute the entropy $h(X)$ if X is an exponential random variable of mean μ.

   (c) Find the distribution of X which maximizes the differential entropy subject to the constraint $\mathbb{E}(X) \leq \lambda$.
   (Hint: it is advised to use the natural log in this calculation, instead of the log in base 2)

   (d) Let Z be an exponential noise, independent of X and with $\mathbb{E}(Z) = \mu$. Let $Y = X + Z$ be the outcome of the corresponding channel, where X is a positive random variable with the mean constraint $\mathbb{E}(X) \leq \lambda$. Show that the capacity of this channel is given by

$$C = \log\left(1 + \frac{\lambda}{\mu}\right)$$

(Use this page for your calculations)

6. **Rate Distorsion:** Consider a source X distributed uniformly over the set $\{1, 2, \cdots, m\}$. Find the rate distortion function for this source when the Hamming distortion d is chosen.

Reminder:

(a) *The Hamming distortion d is defined by*

$$d(x, \hat{x}) = \begin{cases} 0 & \text{if } x = \hat{x} \\ 1 & \text{if } x \neq \hat{x} \end{cases}$$

(b) *The rate distortion function $R(D)$ is defined as the minimum of the mutual information $I(X; \hat{X})$ over the conditional distribution $p(\hat{x}|x)$ subjected to the constraint $\sum_{x,\hat{x}} p(x)p(\hat{x}|x)d(\hat{x}, x) \leq D$.*

(Use this page for your calculations)