ORTHOGONAL POLYNOMIALS. THE CASE OF LEGENDRE POLYNOMIALS

NOTES FOR “HILBERT SPACES” MATH 6580, FALL 2013

1. Legendre Polynomials: a Summary

Let P_n denotes the **monic Legendre** polynomial of degree n. Then

$$P_{n+1}(s) = sP_n(s) - \frac{n^2}{4n^2-1}\,P_{n-1}(s) \qquad \textbf{(Recursion Formula)}$$

$$\frac{d}{ds}\,(1-s^2)\,\frac{d}{ds}\,P_n + n(n+1)P_n = 0 \qquad \textbf{(Legendre's equation)}$$

$$P_n = \sum_{k=0}^{[n/2]} (-1)^k a_k\, s^{n-2k}\,, \qquad \text{with } a_0 = 1 \text{ and } \quad a_k = \frac{n!^2}{k!(n-k)!(n-2k)!}\,\frac{2(n-k)!}{(2n)!}$$

$$\|P_n\| = \left(\int_{-1}^{+1} P_n(s)^2 ds\right)^{1/2} = \sqrt{\frac{2}{2n+1}\,\frac{2^n n!^2}{(2n)!}} \qquad \textbf{(Norm)}$$

$$\frac{1}{\sqrt{1-2sx+x^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n(n!)^2}\,P_n(s)\,x^n \qquad \textbf{(Generating Function)}$$

$$P_n(1) = \frac{2^n(n!)^2}{(2n)!} \qquad \textbf{(Value at } s=1\textbf{)}$$

$$\int_{-1}^{+1} \frac{ds}{\sqrt{(1-2sx+x^2)(1-2sy+y^2)}} = \frac{1}{\sqrt{xy}}\;\ln\left(\frac{1+\sqrt{xy}}{1-\sqrt{xy}}\right)$$

2. The Stone-Weïerstrass Theorem

The Stone-Weïerstrass Theorem asserts that any complex valued continuous function on an interval $I = [a, b] \subset \mathbb{R}$ can be uniformaly approximated by a sequence of polynomials. The first proof goes back to a work of Karl Weïerstrass published in 1885 [3]. It was considerably generalized by by Marshall H. Stone in a paper published 1937 [1] and the proof was simplified by Stone in a subsequent paper published in 1948 [2]. In these notes we will not give the proof of Stone, but an explicit construction of an approximation using the *Bernstein polynomials*.

2.1. The Bernstein Approximation.

The Bernstein polynomials are defined as

$$B_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}\,, \qquad 0 \leq k \leq n\,. \tag{1}$$

The main property of these polynomials which will be used here is the following

$$(i)\ \sum_{k=0}^{n} B_{n,k}(t) = 1\,, \qquad (ii)\ B_{n,k}(t) \geq 0\,, \qquad \text{if} \qquad 0 \leq t \leq 1\,. \tag{2}$$

Let now $f : t \in [0, 1] \mapsto f(t) \in \mathbb{C}$ be a continuous function. Then its n-th *Bernstein approximation* is defined by

$$B_n f(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) B_{n,k}(t)\,.$$

Then, $B_n f$ is also a polynomial of degree n. The main result is the following version of the Stone-Weïerstrass Theorem

Theorem 1 (Stone-Weïerstrass)**.** *For any continuous function $f : [0, 1] \to \mathbb{C}$*

$$\lim_{n\to\infty} \sup_{0\leq t\leq 1} |f(t) - B_n f(t)| = 0\,.$$

The proof will require several steps including several estimates. Each of these steps will be described through lemmas.

Lemma 1. *The binomial coefficients are bounded as*

$$\binom{n}{k} \leq \left[\left(\frac{k}{n}\right)^k \left(1-\frac{k}{n}\right)^{n-k}\right]^{-1}\,, \qquad 0 \leq k \leq n\,.$$

Proof: Thanks to eq. (2), each Bernstein polynomial is bounded by $0 \leq B_{n,k}(t) \leq 1$ for $t \in [0, 1]$. Replacing t by k/n gives immediately the result. It is easy to check that $t = k/n$ gives the maximal value of $B_{n,k}(t)$, so that this estimate is optimal. □

Lemma 2. *The following inequality holds for $t \in [0, 1]$ and for $0 \leq k \leq n$ in integers*

$$0 \leq B_{n,k}(t) \leq e^{-2n(k/n-t)^2}\,.$$

Proof: Setting $\sigma = k/n$, the result of Lemma 1 leads to

$$B_{n,k}(t) \leq \left[\left(\frac{t}{\sigma}\right)^{\sigma} \left(\frac{1-t}{1-\sigma}\right)^{1-\sigma} \right]^n = e^{n\psi_t(\sigma)} ,$$

where $\psi_t(\sigma)$ denotes the logarithm of the expression in the brackets, namely

$$\psi_t(\sigma) = \sigma \, (\ln(t) - \ln(\sigma)) + (1-\sigma) \, (\ln(1-t) - \ln(1-\sigma)) \, .$$

In particular ψ_t vanishes at $\sigma = t$. Moreover, its derivative is given by

$$\partial_\sigma \psi_t(\sigma) = \ln(t) - \ln(\sigma) - \ln(1-t) + \ln(1-\sigma) \, .$$

It follows that $\partial_\sigma \psi_t$ also vanishes at $\sigma = t$. The second derivative is given by

$$\partial_\sigma^2 \psi_t(\sigma) = -\frac{1}{\sigma(1-\sigma)} \leq -4 \, , \qquad\qquad 0 \leq \sigma \leq 1 \, .$$

Using the Taylor expansion to second order gives

$$\psi_t(\sigma) = \int_t^\sigma (\tau - t) \, \partial_\sigma^2 \psi_t(\tau) \, d\tau = \int_\sigma^t (t-\tau) \, \partial_\sigma^2 \psi_t(\tau) \, d\tau \leq -2(\sigma - t)^2 \, .$$

This inequality gives the result by exponentiation. □

Lemma 3. *For any $\delta > 0$ small enough*

$$\sum_{|t-k/n|>\delta} B_{n,k}(t) \leq \sqrt{\frac{e\pi}{4}} \, n\delta \, e^{-n\delta^2/2} \, .$$

Proof: (i) Thanks to Lemma 2, it follows that

$$\sum_{|t-k/n|>\delta} B_{n,k}(t) \leq 2n \sum_{k/n-t>\delta} \frac{1}{n} \, e^{-2n(t-k/n)^2} \, .$$

The $r.h.s$ is a Riemann sum that can be bounded from above by an integral, using the fact that the function $g : x \mapsto e^{-2nx^2}$ is monotone decreasing over the interval $[\delta, +\infty)$ using

$$\sum_{k/n>t+\delta} \frac{1}{n} \; g\left(\frac{k}{n} - t\right) \leq \int_{\delta-1/n} g(x) dx \, .$$

Hence, as soon as $\delta > 2/n$, this gives

$$\sum_{k/n-t>\delta} \frac{1}{n} \, e^{-2n(t-k/n)^2} \leq \int_{\delta/2}^\infty e^{-2nx^2} dx \, .$$

The Gaussian integral can be estimated by choosing $0 < u < 1$ and

$$\int_A^\infty e^{-ax^2} dx = \int_A^\infty e^{-uax^2} e^{-(1-u)ax^2} dx \leq e^{-uaA^2} \int_0^\infty e^{-(1-u)ax^2} dx = e^{-uaA^2} \frac{1}{2} \sqrt{\frac{\pi}{(1-u)a}} \, .$$

Since u can be chosen anywhere in $(0,1)$, it is worth optimizing by chosing $u = 1 - 1/2aA^2$ giving

$$\int_A^\infty e^{-ax^2} dx \leq \sqrt{\frac{e\pi}{4}}\; a\; e^{-aA^2} .$$

Replacing A by $\delta/2$ and a by $2n$ gives the result. □

Proof of Theorem 1: Let $f : [0,1] \to \mathbb{C}$ be a continuous function. Since the interval $[0,1]$ is bounded, it is compact. Therefore f is *uniformly continuous*. Namely for all $\epsilon > 0$ there is $\delta > 0$ such that if s, t are in $[0,1]$ and satisfy $|s-t| \leq \delta$ then $|f(s) - f(t)| \leq \epsilon/2$. Using eq. (2 i), it follows that

$$f(t) - B_n f(t) = \sum_{k=0}^{n} \left(f(t) - f\left(\frac{k}{n}\right)\right) \; B_{n,k}(t) .$$

Let the finite sum over k be decomposed into $A = \sum_{|t-k/n|\leq\delta}(\;\cdot\;)$ and $B = \sum_{|t-k/n|>\delta}(\;\cdot\;)$. In the sum A, since $|t - k/n| \leq \delta$ it follows that $|f(t) - f(k/n)| \leq \epsilon/2$. Using eq. (2 i) again this gives $|A| \leq \epsilon/2$. The sum B is bounded by

$$\|f\|_\infty = \sup_{0\leq s\leq 1} |f(s)| , \qquad \Rightarrow \qquad |B| \leq 2\|f\|_\infty \sum_{|t-k/n|>\delta} B_{n,k}(t) .$$

Thanks to the Lemma 3, these two estimates give

$$|f(t) - B_n f(t)| \leq \frac{\epsilon}{2} + 2\|f\|_\infty \sqrt{e\pi}\; n\delta\; e^{-n\delta^2/2} .$$

Since $\lim_{n\to\infty} n\delta\; e^{-n\delta^2/2} = 0$, it follows that there is an integer N large enough so that, if $n \geq N$, then the second term is also bounded by $\epsilon/2$ leading to

$$n \geq N \qquad \Rightarrow \qquad |f(t) - B_n f(t)| \leq \epsilon .$$

This estimate is uniform *w.r.t.* t, so that

$$n \geq N \qquad \Rightarrow \qquad \|f - B_n f\|_\infty \leq \epsilon .$$

This implies $\lim_{n\to\infty} \|f - B_n f\|_\infty = 0$. □

2.2. **The Space $\mathcal{C}(0,1)$ and the Uniform Topology.** Let $\mathcal{C}(0,1)$ be the set of all continuous functions defined on $[0,1]$ with complex values. It is a complex vector space when endowed with the following operations: if $f, g \in \mathcal{C}(0,1)$ and if $\lambda \in \mathbb{C}$ is a scalar

$$f + g : s \in [0,1] \mapsto f(s) + g(s) \in \mathbb{C} , \qquad\qquad \lambda f : s \in [0,1] \mapsto \lambda f(s) \in \mathbb{C} .$$

The reader is invited to check that the sum of two continuous functions is still continuous, that the addition is commutative and associative, that the function $0(s) = 0\ \forall s \in [0,1]$ is a neutral element and that $-f$ is the opposite to f. Moreover the reader is invited to check that the scalar multiplication λf gives also a continuous function, that is also associative and distributive *w.r.t.* the addition.

The uniform norm is defined by

$$\|f\|_\infty = \sup_{s\in[0,1]} |f(s)| .$$

The reader is invited to check that it satisfies the axiom of norms namely

- $\|f\|_\infty \geq 0$, and $\|f\|_\infty = 0$ if and only if $f = 0$.

- $\|\lambda f\|_\infty = |\lambda|\, \|f\|_\infty$ if $\lambda \in \mathbb{C}$.
- **Triangle inequality:** $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

In addition, the reader is invited to show that the following inequality is equivalent to the triangle one

$$\Big| \|f\|_\infty - \|g\|_\infty \Big| \leq \|f-g\|_\infty \,.$$

Hence $\mathcal{C}(0,1)$ is a normed complex vector space. Moreover

Theorem 2. *The normed vector space $(\mathcal{C}(0,1), \|\cdot\|_\infty)$ is a Banach space, namely it is complete or, equivalently, any Cauchy sequence of continuous function converges to a continuous function.*

Proof: (i) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Therefore for $\epsilon > 0$ there is $N \in \mathbb{N}$ such that if $n, m > N$ then $\|f_n - f_m\|_\infty \leq \epsilon$. In particular $|f_n(s) - f_m(s)| \leq \epsilon$ for any $s \in [0,1]$. Since $f_n(s) \in \mathbb{C}$, it is a Cauchy sequence in the complex plane and therefore it converges. Let $f(s) = \lim_{n\to\infty} f_n(s)$. The function f is defined on $[0,1]$. By construction, then, if $n \geq N$, then $|f(s)-f_n(s)| = \lim_{m\to\infty} |f_m(s)-f_n(s)| \leq \epsilon$. In particular $\|f-f_n\|_\infty = \sup_{s\in[0,1]} |f(s)-f_n(s)| \leq \epsilon$ as well.

(ii) Using the triangle inequality it follows that $\big|\|f_n\|_\infty - \|f_m\|_\infty\big| \leq \|f_n - f_m\|_\infty$ so that the sequence $(\|f_n\|_\infty)_{n\in\mathbb{N}}$ of nonnegative real numbers is also Cauchy, therefore it converges. In particular

$$|f(s)| = \lim_{n\to\infty} |f_n(s)| \leq \lim_{n\to\infty} \|f_n\| < \infty \,.$$

Hence the function f is bounded, so that $\|f\|_\infty = \sup_{s\in[0,1]} |f(s)|$ is a well defined nonnegative real number.

(iii) It remains to show that f is continuous. Given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, then $\|f - f_n\|_\infty \leq \epsilon/3$. Consequently, choosing $n \geq N$, $|f(s) - f(t)| \leq |f(s) - f_n(s)| + |f_n(s) - f_n(t)| + |f_n(t) - f(t)| \leq 2\epsilon/3 + |f_n(s) - f_n(t)|$. Since f_n is continuous, it is uniformly continuous on $[0,1]$, and there is $\delta > 0$ such that is $|s-t| \leq \delta$ then $|f_n(s) - f_n(t)| \leq \epsilon/3$. Hence if $|s-t| \leq \delta$ then $|f(s) - f(t)| \leq \epsilon$ showing that f is indeed continuous. □

Let now $\mathbb{C}[X]$ be the set of polynomials in the indeterminate X with complex coefficients. Hence an element of $\mathbb{C}[X]$ is an expression of the form $P = p_0 + p_1 X + p_2 X^2 + \cdots + p_n X^n$, where the p_k's are complex numbers called the *coefficients* of P. The maximum integer n such that $p_n \neq 0$ is called the degree of P. Given $P \in \mathbb{C}[X]$ and $s \in [0,1]$ let $P(s)$ be the *value* of P at s, namely

$$P(s) = p_0 + p_1 s + p_2 s^2 + \cdots + p_n s^n \in \mathbb{C} \,.$$

Hence the evaluation of P on points of the interval $[0,1]$ gives a function defined on $[0,1]$ with complex values. Moreover, each monomial $X^n : s \in [0,1] \mapsto s^n \in [0,1] \subset \mathbb{C}$ is continuous, so that P defines an element of $\mathcal{C}(0,1)$. The Stone-Weïerstrass theorem can be rephrased as follows

Theorem 3 (Stone-Weïerstrass II)**.** *The set of evaluation of polynomials on $[0,1]$ is a dense linear subspace of $\mathcal{C}(0,1)$ for the uniform topology.*

Proof: Clearly the space $\mathbb{C}[X]$ is a linear space and the evaluation map is linear as well. Hence the set of evaluation of polynomials on $[0,1]$ is a linear subspace of $\mathcal{C}(0,1)$. Thanks to Theorem 1, any $f \in \mathcal{C}(0,1)$ can be uniformly approximated by its Bernstein approximation $B_n f(t)$. Since $B_n f(t)$ is a polynomial with respect to t, the result is proved. □

Corollary 1 (Stone-Weïerstrass III). *Let $-\infty < a < b < +\infty$ be two real numbers. Let $\mathcal{C}(a,b)$ be the set of continuous functions $f : [a,b] \to \mathbb{C}$. Then the set of evaluations of polynomials on $[a,b]$ is dense in $\mathcal{C}(a,b)$.*

Proof: The main reason is that the two normed spaces $\mathcal{C}(a,b)$ and $\mathcal{C}(0,1)$ are isomorphic. For indeed, if $s \in [a,b]$ then $s = tb + (1-t)a$ for some $t \in [0,1]$ given by $t = (s-a)/(b-a)$. Hence, if $g \in \mathcal{C}(a,b)$ let $\phi(g) : t \in [0,1] \mapsto g(ta + (1-t)b) \in \mathbb{C}$. Clearly $\phi(g)$ is a continuous function of t and the map $\phi : g \in \mathcal{C}(a,b) \mapsto \phi(g) \in \mathcal{C}(0,1)$ is linear and isometric, because $\|\phi(g)\|_\infty = \sup_{0\leq t\leq 1} |g(ta + (1-t)b)| = \sup_{a\leq s\leq b} |g(s)| = \|g\|_\infty$. Similarly, by setting, for $f \in \mathcal{C}(0,1)$, $\psi(f); s \in [a,b] \mapsto f((s-a)/(b-a))$, $\psi : \mathcal{C}(0,1) \to \mathcal{C}(a,b)$ is also linear and isometric. In addition, $\psi \circ \phi(g) = g$ and $\phi \circ \psi(f) = f$ as the reader is invited to check. Therefore $\psi = \phi^{-1}$. Since the image of a polynomial by ϕ or by ψ is a polynomial, the Theorem 3 shows that the conclusion holds. □

2.3. **The Space $L^2([a,b],\rho)$.** Let $-\infty < a < b < +\infty$ be two real numbers and let $\rho : [a,b] \to [0,+\infty)$ be a measurable function such that $\int_a^b \rho(s)ds < \infty$ and such that the set of point on which ρ vanishes has zero Lebesgue measure. Then an inner product is defined on $\mathcal{C}(a,b)$ by setting

$$(f,g) \stackrel{def}{=} \int_a^b f(s)\overline{g(s)}\,\rho(s)\,ds\,.$$

The reader is invited to check that $\mathcal{C}(a,b)$ becomes an inner product space, in particular in proving that $(f,f) = 0 \Rightarrow f = 0$. Let the corresponding norm be denoted by

$$\|f\|_2 = \|f\|_{L^2(\rho)} \stackrel{def}{=} (f,f)^{1/2}\,.$$

The two norms $\|\cdot\|_\infty$ and $\|\cdot\|_{L^2(\rho)}$ define two topologies. However, while $\mathcal{C}(a,b)$ is complete for the first one, it is not complete for the other one. Hence these two topologies *are NOT equivalent*. Nevertheless the following holds

Proposition 1. *Let $f \in \mathcal{C}(a,b)$. Then*

$$\|f\|_{L^2(\rho)} \leq \|f\|_\infty \left(\int_a^b \rho(s)\,ds\right)^{1/2}\,.$$

In particular, any uniformly convergent sequence is converging in the norm $\|\cdot\|_{L^2(\rho)}$.

Proof: For indeed

$$\|f\|^2_{L^2(\rho)} = \int_a^b |f(s)|^2\,\rho(s)\,ds \leq \|f\|^2_\infty \int_a^b \rho(s)\,ds\,.$$

Taking the square root of both sides gives the result. Hence if $(f_n)_{n\in\mathbb{N}}$ is a uniformly convergent sequence, namely there is $f \in \mathcal{C}(a,b)$ such that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$, then $lim_{n\to\infty}\|f - f_n\|^2_{L^2(\rho)} = 0$. However the converse is not true in general. □

Definition 1. *The completion of $\mathcal{C}(a,b)$ with respect to the inner product topology defined by ρ is denoted $L^2([a,b],\rho)$.*

Like for $L^2([0,1])$, any element of the completion is an equivalent class of measurable functions h such that $\int_a^b |h(s)|^2\,\rho(s)\,ds < \infty$ modulo the addition of a function vanishing on a set of zero Lebesgue measure.

Theorem 4. *The polynomials on $[a, b]$ are dense in the Hilbert space $L^2([a, b], \rho)$.*

Proof: If $h \in L^2([a, b], \rho)$, by construction, for any $\epsilon > 0$ there is $f \in \mathcal{C}(a, b)$ such that $\|h - f\|_2 \leq \epsilon/2$. Thanks to Theorem 3 and its Corollary 1, it follows that there is a polynomial P such that $C \|f - P\|_\infty \leq \epsilon/2$ if C denotes the positive constant defined by

$$C^2 = \int_a^b \rho(s)\, ds\,.$$

Using Proposition 1, it follows that $\|h - P\|_2 \leq \epsilon$. Since ϵ can be chosen as small as wished, the conclusion holds. □

3. Orthogonal Polynomials

Thanks to the Theorem 4, it follows that, if X^n denotes the map $X^n : s \in [a, b] \mapsto s^n \in \mathbb{R} \subset \mathbb{C}$, the family $(X^n)_{n=0}^\infty$ is linearly independent and generates a dense linear subspace of $L^2([a, b], \rho)$. The purpose of this section is to apply the Gram-Schmidt procedure to this family. This will lead to an orthonormal family $(p_n)_{n=0}^\infty$ of polynomials. The properties of these polynomials will be described. *To simplify the notations, $\mathcal{H}$ will denote the space $L^2([a, b], \rho)$ and the notation $\|f\|_2$ will be used instead of $\|f\|_{L^2(\rho)}$.* All along this Section the following definition will be used

Definition 2. *A polynomial is called monic whenever its coefficient of highest degree is equal to 1. Hence this can be expressed as $P(s) = s^n + O(s^{n-1})$.*

3.1. **Construction of an Orthogonal Family.** Let the Gram-Schmidt procedure be applied to the family of monomials. By definition, monomials are linearly independent.

Step 0: Following the Gram-Schmidt procedure, let $P_0 = X^0$ be the constant function equal to 1. It is the unique monic polynomial of degree zero. Then $p_0 = P_0/C$ is a unit vector in $\mathcal{H}$. It is worth remarking that $\|P_0\|_2 = C$.

Step 1: As a warm up, let the step 1 on the Gram-Schmidt procedure be described in detail. Then, a vector P_1 will be obtained as a linear combination of X^0, X^1 so that $(P_1, P_0) = 0$. Hence if $P_1 = \alpha X + \beta P_0$, this gives

$$0 = (P_1, P_0) = \alpha(X, P_0) + \beta(P_0, P_0) = 0\,, \qquad \Rightarrow \qquad \beta = -\alpha \frac{(X, P_0)}{(P_0, P_0)}\,.$$

Consequently

$$P_1 = \alpha\,(X - aP_0)\,, \qquad a = \frac{(X, P_0)}{(P_0, P_0)}\,.$$

It follows that $P_1(s) = \alpha(s - a)$ is a polynomial of degree 1. If P_1 is normalized so as to be monic, it follows that p_1 is a unit vector if

$$p_1 = \frac{P_1}{\|P_1\|_2}\,, \qquad P_1(s) = s - a\,, \qquad a = \frac{1}{C^2} \int_a^b s\rho(s)\, ds\,.$$

Step 2: Just to understand the procedure, let the second step of the Gram-Schmidt procedure be described in detail as well. The linear space $\mathcal{P}_1$ generated by p_0, p_1 is the linear space generated by X^0, X^1 namely it is exactly the set of all polynomials of degree less that or equal to 1. By construction P_2 is a linear combination of X^2 and of p_0, p_1 namely of X^0, X^1, X^2. Thus it is a polynomial of degree 2 which is orthogonal to any polynomial of degree 1. By choosing the normalization of P_2 to make it monic, this leads to

$$P_2(s) = s^2 + as + b\,, \qquad\qquad (P_2, X) = (P_2, X^0) = 0\,.$$

At this point, it is convenient to introduce the n-th moment m_n defined by

$$m_n = \int_a^b s^n\,\rho(s)\,ds\,.$$

It will be important to remark that each moment is a real number and that the moments of even order are positive numbers. The previous orthogonality relation can be written as

$$\begin{aligned}
\int_a^b (s^2 + as + b)s\,\rho(s)\,ds &= m_3 + am_2 + bm_1 = 0\,,\\
\int_a^b (s^2 + as + b)\,\rho(s)\,ds &= m_2 + am_1 + bm_0 = 0\,.
\end{aligned}$$

As a preparation for the general step, this set of linear equation can be written in the following matrix form

$$\begin{bmatrix} m_2 & m_1 \\ m_1 & m_0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -m_3 \\ -m_2 \end{bmatrix}$$

Since X^0, X^1, X^2 are linearly independent, it follows that this set of equation as a unique solution, namely that the square matrix on the *l.h.s* is invertible, defining P_2 in a unique way. Since all moments are real numbers, it follows that the coefficients a, b defining $P - 2$ are real as well, so that P_2 is a polynomial with real coefficients. As in previous steps, p_2 will be defined as the corresponding unit vector, namely

$$p_2 = \frac{P_2}{\|P_2\|_2}\,.$$

Step n: Following the same procedure, P_n is therefore a monic polynomial of degree n which will be written as

$$P_n(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\,.$$

The orthogonality relations $(P_n, X^k) = 0$ for $0 \le k \le n-1$ lead to the equations

$$m_{n+k} + a_1 m_{n+k-1} + \cdots a_n m_k = 0\,, \qquad\qquad 0 \le k \le n-1\,.$$

Equivalently, these linear equations can be written in matrix form as follows

$$\begin{bmatrix} m_{2n-2} & m_{2n-3} & \cdots & m_{n-1} \\ m_{2n-3} & m_{2n-4} & \cdots & m_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n-1} & m_{n-2} & \cdots & m_0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} -m_{2n-1} \\ -m_{2n-2} \\ \vdots \\ -m_n \end{bmatrix}$$

Again, the square matrix on the left is real valued and invertible, leading to a unique solution for P_n as a monic polnomial of degree n with real coefficients. In much the same way, p_n denotes the corresponding unit vector

$$p_n = \frac{P_n}{\|P_n\|_2}\,.$$

Conclusion: the family $(P_n)_{n\in\mathbb{N}}$ is made of monic polynomials with $\deg(P_n) = n$ and

$$(P_n, P_m) = (P_m, P_n) = 0\,, \qquad \text{if} \qquad n \neq m\,.$$

Let $\mathcal{P}_n$ denote the linear subspace generated by $X^0, X, X^2, \cdots, X^n$. Then $\mathcal{P}_n$ is nothing but the subspace of polynomials of degree less than or equal to n and $P_0, P_1, \cdots, P_n$ represents an orthogonal basis in $\mathcal{P}_n$. In particular, P_n is orthogonal to any polynomial of degree less than n. Moreover, *this family is unique with this property.* Correspondingly the normalization of each of these monic polynomials leads to the orthonormal family $(p_n)_{n\in\mathbb{N}}$ which has similar properties. The orthogonal projection Π_n onto $\mathcal{P}_n$ can be written in either of the following two forms

$$\Pi_n f = \sum_{k=0}^{n} (f, p_k)\, p_k = \sum_{k=0}^{n} \frac{(f, P_k)}{(P_k, P_k)}\, P_k\,.$$

3.2. **Recursion Relation.** The construction made in the previous section gives an algorithm to built the family of orthogonal monic polynomials in terms of the moments. However, using the orthogonality property leads to interesting relations. The first important one is given by the following recursion formula

Theorem 5. *Let $(P_n)_{n=0}^{\infty}$ be the orthogonal family of monic polynomial defined by the Hilbert space $\mathcal{H} = L^2([a,b], \rho)$. Then there is a sequence $(a_n)_{n=0}^{\infty}$ of real numbers and a sequence $(b_n)_{n=1}^{\infty}$ of positive real numbers such that*

$$P_{n+1}(s) = (s - a_n)P_n(s) - b_n P_{n-1}(s)\,, \qquad n > 0\,,$$

with

$$a_n = \frac{(sP_n, P_n)}{\|P_n\|^2}\,, \qquad\qquad b_n = \frac{\|P_n\|^2}{\|P_{n-1}\|^2}\,.$$

Proof: The polynomial $sP_n(s) = s^{n+1} + O(s^n)$ is a monic polynomial of degree $(n+1)$. Hence, it can be uniquely written in the orthogonal basis of $\mathcal{P}_{n+1}$ as

$$sP_n = P_{n+1} + a_n P_n + b_n P_{n-1} + \sum_{k=0}^{n-2} c_k P_k\,.$$

In particular, for $k \leq n-2$

$$c_k(P_k, P_k) = (sP_n, P_k) = \int_a^b s\, P_n(s) P_k(s)\, \rho(s)\, ds = (P_n, sP_k)\,.$$

Since sP_k is a polynomial of degree $k+1 \leq n-1$ it is orthogonal to P_n so that $c_k = 0$. In a similar way

$$a_n(P_n, P_n) = (sP_n, P_n) = \int_a^b s\, |P_n(s)|^2\, \rho(s)\, ds \in \mathbb{R}\,.$$

Moreover,

$$b_n(P_{n-1}, P_{n-1}) = (sP_n, P_{n-1}) = (P_n, sP_{n-1})\,.$$

Since sP_{n-1} is a monic polynomial of degree n, it can be written as $sP_{n-1} = P_n + O(s^{n-1})$. Hence, since P_n is orthogonal to $\mathcal{P}_{n-1}$ it follows that $(Pn, sP_{n-1}) = (P_n, P_n)$. Therefore

$$b_n = \frac{\|P_n\|^2}{\|P_{n-1}\|^2} > 0\,.$$

This gives the result. □

3.3. **Symmetries.** The most common symmetry appearing in a space like $L^2([a,b],\rho)$ is the parity around the origin. This happens whenever $a = -b$ and $\rho(-s) = \rho(s)$. In this case the following holds

Proposition 2. *Let $[a,b] = [-b,b]$ be symmetric around the origin and let ρ be even, namely $\rho(-s) = \rho(s)$. Then*

$$P_n(-s) = (1)^n P_n(s)\,.$$

Proof: Writing $P_n(s) = s^n + O(s^{n-1})$ it follows that $Q_n(s) \stackrel{def}{=} (-1)^n P_n(-s)$ is monic. Moreover

$$(Q_n, X^k) = \int_{-b}^{b} (-1)^n P_n(-s) s^k\, rho(s)\, ds = (-1)^k \int_{-b}^{b} P_n(s) s^k\, rho(s)\, ds = 0\,,$$

by proceeding to the change of variable $s \to -s$. Hence, thanks to the uniqueness of such a family (see Section 3.1, Conclusion), it follows that $Q_n = P_n$. □

The reader is invited to check the following two results.

Proposition 3. *Let $[a,b] = [-b,b]$ be symmetric around the origin and let rho be even, namely $\rho(-s) = \rho(s)$. Then the moments satisfy*

$$m_{2n+1} = 0\,, \qquad\qquad n \geq 0\,.$$

In addition

$$P_{2n}(s) = Q_n(s^2)\,, \qquad\qquad P_{2n+1}(s) = sR_n(s^2)|,,$$

where Q_n, R_n are monic polynomials of degree n.

It has been seen that the interval $[a,b]$ can be transformed into $[0,1]$ by the map $s \mapsto t = (s-a)/(b-a)$. Correspondingly, $\rho(s)$ can be written as $\rho(tb + (1-t)a)$ while $ds = (b-a)dt$. Hence setting

$$\rho_0(t) = (b-a)\rho(tb + (1-t)a)\,,$$

and for $f \in L^2([a,b],\rho)$

$$Uf(t) = f(tb + (1-t)a)\,,$$

it follows that $U : L^2([a,b],\rho) \to L^2([0,1],\rho_0)$ is a linear unitary transformation since it is linear and satisfies

$$(Uf, Ug) \stackrel{def}{=} \int_0^1 Uf(t)\,\overline{Ug(t)}\,\rho_0(t)\,dt = \int_a^b f(s)\overline{g(s)}\,\rho(s)\,ds = (f,g)\,.$$

By construction any linear transformation change an orthonormal basis into another one and an orthogonal family into another one. Hence the family $(UP_n)_{n=0}^{\infty}$ is orthogonal and $(Up_n)_{n=0}^{\infty}$ is an orthonormal basis in $L^2([0,1],\rho_0)$. On the other hand, it is easy to check that, since the transformation $s \to t$ is a polynomial of degree one, the function UP_n is a polynomial of degree n as well such that

$$UP_n(t) = P_n(tb + (1-t)a) = (b-a)^n t^n + O(t^{n-1}) .$$

To make UP_n monic, it is necessary to divide it by $(b-a)^{-n}$, leading to

Proposition 4. *Let $\psi(s) = t = (s-a)/(b-a)$ and let $\rho_0(t) = \rho(tb + (1-t)a)(b-a)$. Then, if $(P_n)_{n=0}^{\infty}$ denotes the family of monic orthogonal polynomials in $\mathcal{H} = L^2([a,b],\rho)$, the unique orthogonal family of monic polynomials in the space $L^2([0,1],\rho_0)$ is given by*

$$Q_n(t) = \frac{P_n\big(tb + (1-t)a\big)}{(b-a)^n} .$$

Proof: The reader is invited to make the proof. □

4. An Example: the Legendre Polynomials

Legendre's polynomials correspond to the symmetric interval $[a,b] = [-1,+1]$ with $\rho(s) = 1$. They are usually denoted by P_n, but the normalization may change from reference to another. *In the present case, P_n will denote the monic Legendre polynomial of degree n.* Because of the symmetry, it follows that P_n has the parity of n. An elementary calculation gives the moments $(m_{2n+1} = 0)$ and the first polynomials as

$$m_{2n} = \frac{2}{2n+1} , \qquad P_0 = 1 , \; P_1 = s , \; P_2 = s^2 - \frac{1}{3} , \; P_3 = s(s^2 - \frac{3}{5}) .$$

4.1. **Differential Equation.** The first important result is the differential equation

Theorem 6. *The Legendre polynomials solve the differential equation*

$$\frac{d}{ds}(1-s^2)\frac{d}{ds}P_n + n(n+1)P_n = 0 . \qquad \textbf{(Legendre's equation)}$$

Proof: An elementary calculation shows that $T_n = \partial_s (1-s^2)\partial_s P_n$ is a polynomial of degree n such that

$$T_n = -n(n+1)s^n + O(s^{n-2}) ,$$

it follows easily from Proposition 3. It follows that the decomposition onto the monic orthogonal basis $(P_n)_{n=0}^{\infty}$ must have the form

$$T_n = -n(n+1)P_n + \sum_{k=0}^{n-1} c_k P_k .$$

The orthogonality gives $c_k \|P_k\|^2 = (T_n, P_k)$. The latter can be written as

$$(T_n, P_k) = \int_{-1}^{1} \frac{d}{ds}(1-s^2)\frac{d}{ds}P_n \, P_k(s)\, ds .$$

An integration by part gives

$$(T_n, P_k) = (1-s^2)\, P_k(s)\, \frac{d}{ds}P_n(s)\Big|_{s=-1}^{s=1} - \int_{-1}^{1} (1-s^2)\frac{d}{ds}P_n \, \frac{d}{ds}P_k(s)\, ds .$$

Since $(1-s^2)=0$ for $s=\pm 1$, the first term on the right hand side vanishes. Similarly, another integration by part leads to

$$(T_n,P_k)=\int_{-1}^{1} P_n\,\frac{d}{ds}\,(1-s^2)\,\frac{d}{ds}\,P_k(s)\,ds=(P_n,T_k)\,.$$

Since T_k is a polynomial of degree $k<n$, it follows that $(P_n,T_k)=0$ leading to $c_k=0$ whenever $0\leq k\leq n-1$. This gives the result. □

The previous Theorem allows to compute all the coefficients of the polynomial P_n as shown below

Proposition 5. *Setting $a_0=1$, the monic Legendre polynomials are given by*

$$P_n=\sum_{k=0}^{[n/2]}(-1)^k a_k\;s^{n-2k}\,,\qquad\text{with}\quad a_k=\frac{n!^2}{k!(n-k)!(n-2k)!}\,\frac{2(n-k)!}{(2n)!}\,. \tag{3}$$

Proof: The Legendre polynomial P_n being either odd or even, depending upon the parity of n can be written as

$$P_n(s)=\sum_{k=0}^{[n/2]}(-1)^k a_k\;s^{n-2k}$$

Its derivative is therefore given by

$$\frac{dP_n}{ds}=\sum_{k=0}^{[(n-1)/2]}(-1)^k(n-2k)\,a_k\;s^{n-1-2k}$$

Multiplying it by $-s^2$ can be reorganized as follows

$$\begin{aligned}-s^2\,\frac{dP_n}{ds} &= -ns^{n+1}+\sum_{k=1}^{[(n-1)/2]}(-1)^{k+1}(n-2k)\,a_k\;s^{n+1-2k}\\ &= -ns^{n+1}+\sum_{k=0}^{[(n-3)/2]}(-1)^{k}(n-2-2k)\,a_{k+1}\;s^{n-1-2k}\,.\end{aligned}$$

Adding it to the derivative gives

$$(1-s^2)\frac{dP_n}{ds}=-ns^{n+1}+\sum_{k=0}^{[(n-3)/2]}(-1)^k\big[(n-2k)\,a_k+(n-2-2k)\,a_{k+1}\big]\,s^{n-1-2k}+R\,,$$

where $R=(-1)^{m-1}2a_{m-1}s$ if $n=2m$ and $R=(-1)^m a_m$ if $n=2m+1$. Deriving another time gives

$$\frac{d}{ds}(1-s^2)\frac{dP_n}{ds}=-n(n+1)s^n+\sum_{k=0}^{[(n-3)/2]}(-1)^k(n-1-2k)\big[(n-2k)\,a_k+(n-2-2k)\,a_{k+1}\big]\,s^{n-2-2k}+R'$$

This can be written in the following form

$$\begin{aligned}\frac{d}{ds}(1-s^2)\frac{dP_n}{ds} &= -n(n+1)s^n + \\ &\sum_{k=1}^{[(n-1)/2]} (-1)^{k-1}(n+1-2k)\big[(n+2-2k)\,a_{k-1} + (n-2k)\,a_k\big]\,s^{n-2k} + R'\end{aligned}$$

$$\frac{d}{ds}(1-s^2)\frac{dP_n}{ds} = -n(n+1)s^n + \sum_{k=1}^{[(n-1)/2]} (-1)^{k-1}(n+1-2k)\big[(n+2-2k)\,a_{k-1} + (n-2k)\,a_k\big]\,s^{n-2k} + R'$$

Adding $n(n+1)P_n$ leads to the following recursion formula

$$(n+1-2k)\big[(n+2-2k)\,a_{k-1} + (n-2k)\,a_k\big] = n(n+1)a_k\,.$$

Hence

$$a_k = \frac{(n+1-2k)(n+2-2k)}{n(n+1)-(n-2k)(n+1-2k)}\;a_{k-1} = \frac{\mathcal{N}_k}{\mathcal{D}_k}\,.$$

Since $a_0 = 1$ this gives

$$a_k = \prod_{j=1}^{k} \frac{(n+2-2j)(n+1-2j)}{n(n+1)-(n-2j)(n+1-2j)}\,.$$

The numerator can be written as

$$\mathcal{N}_k = n(n-1)\cdots(n+2-k)(n+1-k) = \frac{n!}{(n-2k)!}\,.$$

The denominator involves the following formula $n(n+1)-m(m+1) = (n-m)(n+m+1)$ with $m = n-2k$. Thus

$$\mathcal{D}_k = \prod_{j=1}^{k} 2k(2n-(2k-1)) = 2^k k!\prod_{j=1}^{k}(2n-(2j-1))\,.$$

The previous expression involves the following product

$$\prod_{j=1}^{k}(2n-(2j-1)) = \frac{\prod_{j=1}^{n}(2j-1)}{\prod_{j=1}^{n-k}(2j-1)}\,.$$

But

$$\prod_{j=1}^{n}(2j-1) = \frac{\prod_{j=1}^{n}2j(2j-1)}{2^n n!} = \frac{(2n)!}{2^n n!}\,.$$

Inserting in the expression of $\mathcal{D}_k$ leads to

$$\mathcal{D}_k = 2^k k!\frac{(2n)!}{2^n n!}\,\frac{2^{n-k}(n-k)!}{(2(n-k))!} = \frac{(n-k)!}{n!}\;\frac{(2n)!}{2(n-k)!}\,.$$

Finally this gives

$$a_k = \frac{n!^2}{k!(n-k)!(n-2k)!}\,\frac{2(n-k)!}{(2n)!}\,.$$

□

4.2. **Recursion Formula and Norm.** The Proposition 5 leads to the following result

Proposition 6. *The monic Legendre polynomials satisfy the following recursion formula*

$$P_{n+1}(s) = sP_n(s) - \frac{n^2}{4n^2-1}\,P_{n-1}(s)\,. \tag{4}$$

Proof: The Theorem 5 and the symmetry around $s = 0$ imply the existence of a sequence b_n of positive numbers such that

$$P_{n+1}(s) = sP_n(s) - b_n P_{n-1}\,.$$

Since each polynomial is monic, it follows that $b_n = a_1(n+1) - a_1(n)$, where $a_1(n)$ denotes the coefficient of s^{n-2} in P_n. Thanks to Proposition 5,

$$a_1(n) = \frac{n(n-1)}{2(2n-1)}\,, \qquad \Rightarrow \qquad a_1(n+1) = \frac{n(n+1)}{2(2n+1)}\,.$$

It follows immediately that

$$b_n = \left(\frac{n(n+1)}{2(2n+1)} - \frac{n(n-1)}{2(2n-1)}\right) = \frac{n^2}{4n^2-1}\,,$$

□

Proposition 7. *The Hilbert norm of the monic Legendre polynomials is given by*

$$\|P_n\| = \left(\int_{-1}^{+1} P_n(s)^2 ds\right)^{1/2} = \sqrt{\frac{2}{2n+1}\,\frac{2^n n!^2}{(2n)!}} \tag{5}$$

Proof: The formula for b_n given in Theorem 5 leads to

$$\|P_n\| = \prod_{k=1}^{n} b_k^{1/2}\,\|P_0\|\,.$$

Since $\|P_0\|^2 = \int_{-1}^{+1} ds = 2$ this gives

$$\|P_n\| = \sqrt{2}\frac{1\cdot 2\cdots(n-1)\cdot n}{\sqrt{(1\cdot 3)(3\cdot 5)\cdots((2n-3)\cdot(2n-1))((2n-1)\cdot(2n+1))}}\,.$$

Hence the prefactor $\sqrt{2/(2n+1)}$ can be extracted and the product of the first n odd numbers is given by

$$1\cdot 3\cdots(2n-1) = \frac{(2n)!}{2^n n!}\,,$$

leading to the formula. □

4.3. **Generating Functional.** Another tool of calculation is provided by the generating functional. Typically it is a function given as a power series by

$$G(s,x) = \sum_{n=0}^{\infty} P_n(s)\, x^n\,.$$

It is worth remarking, however, that the normalization of the polynomials can be chosen in another way if this is convenient. This leads to an expression of the form

$$G(s,x) = \sum_{n=0}^{\infty} c_n\, P_n(s)\, x^n\,, \tag{6}$$

where the constant c_n can be chosen at will, if this gives G an expression easy to compute. Whatever the choice of these constants, though, G satisfies the following partial differential equation

Proposition 8. *For any choice of the constant c_n the generating functional G satisfies the following equation*

$$\left\{\partial_s(1-s^2)\partial_s + x\partial_x^2\, x\right\} G = 0\,.$$

Conversely any solution of this equation that can be expanded in powers of x and s has the form given in eq. (6)

Proof: (i) By definition

$$\partial_s(1-s^2)\partial_s\, G = \sum_{n=0}^{\infty} c_n\, \frac{d}{ds}(1-s^2)\frac{dP_n}{ds}\, x^n\,.$$

Thanks to Theorem 6, this is nothing but

$$\partial_s(1-s^2)\partial_s\, G = -\sum_{n=0}^{\infty} c_n\, n(n+1)P_n\, x^n\,.$$

Then it is enough to remark that $x\partial_x^2\, x^{n+1} = n(n+1)x^n$ to get the result.
(ii) Conversely, the same argument shows that if $G(s,x) = \sum_{n=0}^{\infty} Q_n(s)\, x^n$ for some function $Q_n(s)$, then each Q_n is a solution of

$$\frac{d}{ds}(1-s^2)\frac{dQ_n}{ds} + n(n+1)Q_n(s) = 0\,.$$

The calculation made in the proof of Proposition 5 shows that Q_n must be proportional to P_n.
□

A special generating functional was found in the past, given as follows

Proposition 9. *The function*

$$G(s,x) = \frac{1}{\sqrt{1-2sx+x^2}}\,,$$

is a solution of

$$\left\{\partial_s(1-s^2)\partial_s + x\partial_x^2\, x\right\} G = 0\,.$$

Proof: Differentiating G *w.r.t* s or x gives

$$\partial_s G = xG^3\,, \qquad\qquad \partial_x G = (s-x)G^3$$

Hence

$$\begin{aligned}\partial_s(1-s^2)\,xG^3 &= -2sxG^3 + 3x^2(1-s^2)G^5\,,\\ x\partial_x^2 xG &= x(2s-3x)G^3 + 3x^2(s-x)^2G^5\end{aligned}$$

Adding these two term and remarking that $G^3 = (1-2sx+x^2)G^5$ gives

$$\left\{\partial_s(1-s^2)\partial_s + x\partial_x^2\,x\right\}G = \left[-3x^2(1-2sx+x^2)+3x^2(1-s^2+(s-x)^2)\right]G^5 = 0\,.$$

□

Thanks to Proposition 8, it follows that there are constant c_n such that

$$\frac{1}{\sqrt{1-2sx+x^2}} = \sum_{n=0}^{\infty} c_n\,P_n(s)\,x^n\,. \tag{7}$$

In order to compute the c_n's, it is worth expanding the *l.h.s.* in power series. Setting $u = 2sx-x^2$ the *l.h.s.* is given by

$$\frac{1}{\sqrt{1-u}} = \sum_{l=0}^{\infty}\frac{(2l)!}{2^{2l}\,(l!)^2}\;u^l\,.$$

This series converges absolutely for $|u|<1$. Then

$$u^l = \sum_{k=0}^{l}\left(\begin{array}{c} l\\ k\end{array}\right)\,2^k s^k(-1)^{l-k}\,x^{2l-k}\,.$$

Inserting into the infinite series, and using the variables k and $m = l-k$ leads to the double sum

$$G = \sum_{k,m\geq 0}\frac{2(m+k)!}{2^{2m+k}(m+k)!m!k!}\;(-1)^m\,s^k\,x^{2m+k}\,.$$

Setting $n = 2m+k$ permits to group the terms proportional to x^n. This gives

$$G = \sum_{n=0}^{\infty} x^n \sum_{m=0}^{[n/2]}\frac{2(n-m)!}{2^n(n-m)!m!(n-2m)!}\;(-1)^m\,s^{n-2m}$$

The coefficient of x^n is indeed a polynomial Q_n of degree n, having the parity of n. Its coefficient of highest degree is

$$Q_n = \frac{(2n)!}{2^n(n!)^2}\;s^n + O(s^{n-2})\,.$$

On the other hand Q_n is proportional to P_n which is monic. Consequently this coefficient is exactly the constant c_n that was expected. To summarize

Theorem 7. *The following formula holds for the monic Legendre polynomials*

$$\frac{1}{\sqrt{1-2sx+x^2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^n (n!)^2}\, P_n(s)\, x^n \,. \tag{8}$$

Thanks to this formula, it becomes possible to get few more results such as

Proposition 10. *The monic Legendre polynomials satisfy*

$$P_n(1) = \frac{2^n (n!)^2}{(2n)!} \tag{9}$$

Proof: By setting $s = 1$ in eq. (8) the *l.h.s.* becomes $(1-x)^{-1} = \sum_{n\geq 0} x^n$, giving the result. □

Another consequence is the following

Proposition 11. *The following formula holds*

$$\int_{-1}^{+1} \frac{ds}{\sqrt{(1-2sx+x^2)(1-2sy+y^2)}} = \frac{1}{\sqrt{xy}}\, \ln\left(\frac{1+\sqrt{xy}}{1-\sqrt{xy}}\right) \tag{10}$$

Proof: The eq. (8) can be written in term of the normalized polynomials $p_n = P_n/\|P_n\|$ as follows

$$\frac{1}{\sqrt{1-2sx+x^2}} = \sum_{n=0}^{\infty} \sqrt{\frac{2}{2n+1}}\, p_n(s)\, x^n \,, \qquad\qquad p_n = \frac{P_n}{\|P_n\|} \,. \tag{11}$$

Hence, the orthonormality of the family $(p_n)_{n=0}^{\infty}$ leads to

$$\int_{-1}^{+1} \frac{ds}{\sqrt{(1-2sx+x^2)(1-2sy+y^2)}} = 2\sum_{n=0}^{\infty} \frac{(xy)^n}{2n+1} = \frac{1}{\sqrt{xy}}\, \ln\left(\frac{1+\sqrt{xy}}{1-\sqrt{xy}}\right)$$

□

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