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Noncommutative Manifolds, the Instanton Algebra and Isospectral Deformations

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Abstract: We give new examples of noncommutative manifolds that are less standard than the NC-torus or Moyal deformations of $\mathbb{R}^n$. They arise naturally from basic considerations of noncommutative differential topology and have non-trivial global features.

The new examples include the instanton algebra and the NC-4-spheres S^4_θ. We construct the noncommutative algebras $\mathcal{A} = C^\infty(S^4_\theta)$ of functions on NC-spheres as solutions to the vanishing, $\mathrm{ch}_j(e) = 0$, $j < 2$, of the Chern character in the cyclic homology of $\mathcal{A}$ of an idempotent $e \in M_4(\mathcal{A})$, $e^2 = e$, $e = e^*$. We describe the universal noncommutative space obtained from this equation as a noncommutative Grassmannian as well as the corresponding notion of admissible morphisms. This space Gr contains the suspension of a NC-3-sphere S^3_θ distinct from quantum group deformations $\mathrm{SU}_q(2)$ of $\mathrm{SU}(2)$.

We then construct the noncommutative geometry of S^4_θ as given by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and check all axioms of noncommutative manifolds. In a previous paper it was shown that for any Riemannian metric $g_{\mu\nu}$ on S^4 whose volume form $\sqrt{g}\, d^4x$ is the same as the one for the round metric, the corresponding Dirac operator gives a solution to the following quartic equation,

$$\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle = \gamma_5,$$

where $\langle\ \rangle$ is the projection on the commutant of 4×4 matrices.

We shall show how to construct the Dirac operator D on the noncommutative 4-spheres S^4_θ so that the previous equation continues to hold without any change.

Finally, we show that any compact Riemannian spin manifold whose isometry group has rank $r \geq 2$ admits isospectral deformations to noncommutative geometries.

1. Introduction

It is important to have available examples of noncommutative manifolds that are less standard than the NC-torus [4,13] or the old Moyal deformation of $\mathbb{R}^n$ whose algebra is boring. This is particularly so in view of the upsurge of activity in the interaction between string theory and noncommutative geometry started in [11,22,25]. The new examples should arise naturally, have non-trivial global features (and also pass the test of noncommutative manifolds as defined in [8]).

This paper will provide and analyse such very natural new examples, including the instanton algebra and the NC-4-spheres S^4_θ, obtained from basic considerations of noncommutative differential topology.

We shall also show quite generally that any compact Riemannian spin manifold whose isometry group has rank $r \geq 2$ admits isospectral deformations to noncommutative geometries.

A noncommutative geometry is described by a spectral triple

$$(\mathcal{A}, \mathcal{H}, D), \tag{1.1}$$

where $\mathcal{A}$ is a noncommutative algebra with involution $*$, acting in the Hilbert space $\mathcal{H}$ while D is a self-adjoint operator with compact resolvent and such that,

$$[D, a] \text{ is bounded } \forall\, a \in \mathcal{A}. \tag{1.2}$$

The operator D plays in general the role of the Dirac operator [19] in ordinary Riemannian geometry. It specifies both the metric on the state space of $\mathcal{A}$ by

$$d(\varphi, \psi) = \operatorname{Sup}\{|\varphi(a) - \psi(a)|;\ \|[D, a]\| \leq 1\} \tag{1.3}$$

and the K-homology fundamental class (cf. [6]). What holds things together in this spectral point of view of NCG is the nontriviality of the pairing between the K-theory of the algebra $\mathcal{A}$ and the K-homology class of D, given in the even case by

$$[e] \in K_0(\mathcal{A}) \to \text{Index } D_e^+ \in \mathbb{Z}. \tag{1.4}$$

Here $[e]$ is the class of an idempotent

$$e \in M_r(\mathcal{A}),\ e^2 = e,\ e = e^* \tag{1.5}$$

in the algebra of $r \times r$ matrices over $\mathcal{A}$, and

$$D_e^+ = e\, D^+ e\ , \tag{1.6}$$

where $D^+ = D\,(\frac{1+\gamma}{2})$ is the restriction of D to the range $\mathcal{H}^+$ of $\frac{1+\gamma}{2}$ and γ is the $\mathbb{Z}/2$ grading of $\mathcal{H}$ in the even case; thus D is of the form,

$$D = \begin{bmatrix} 0 & D_+^* \\ D_+ & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.7}$$

The cornerstone of the general theory is an operator theoretic index formula [6,12,16] which expresses the above index pairing (1.4) by explicit *local* cyclic cocycles on the algebra $\mathcal{A}$. These local formulas become extremely simple in the special case where only the top component of the Chern character $\mathrm{Ch}(e)$ in cyclic homology fails to vanish. (This is easy to understand in the analogous simpler case of ordinary manifolds since the Atiyah–Singer index formula gives the integral of the product of the Chern character $\mathrm{Ch}(E)$, of the bundle E over the manifold M, by the index class; if the only component of $\mathrm{Ch}(E)$ is ch_n, $n = \frac{1}{2}\dim M$, only the 0-dimensional component of the index class is involved in the index formula.)

Under this assumption the index formula reduces indeed to the following,

$$\text{Index } D_e^+ = (-1)^m \int\!\!\!\!\!- \gamma \left(e - \frac{1}{2}\right) [D, e]^{2m} \, D^{-2m}, \tag{1.8}$$

provided the components $\mathrm{ch}_j(e)$ all vanish for $j < m$. Here γ is the $\mathbb{Z}/2$ grading of $\mathcal{H}$ as above, the resolvent of D is of order $\frac{1}{2m}$ (i.e. its characteristic values μ_k are $0(k^{-\frac{1}{2m}})$) and $\int\!\!\!\!\!-$ is the coefficient of the logarithmic divergency in the ordinary operator trace [15, 26].

We began in [10] to investigate the algebraic relations implied by the vanishing,

$$\mathrm{ch}_j(e) = 0 \quad j < m, \tag{1.9}$$

of the Chern character of e in the cyclic homology of $\mathcal{A}$. Note that this vanishing at the chain level is a much stronger condition than the vanishing of the usual Chern differential form.

For $m = 1$ (and $r = 2$ in (1.5)) we found commutative solutions with $\mathcal{A} = C^\infty(S^2)$ as the algebra generated by the matrix components,

$$e_{ij}, \; e = [e_{ij}] \in M_2(\mathcal{A}). \tag{1.10}$$

In fact, for $m = 1$ the commutativity is imposed by the relations $e^2 = e$, $e = e^*$ and $\mathrm{ch}_0(e) = 0$.

For $m = 2$ (and $r = 4$ in (1.5)) we also found commutative solutions with $\mathcal{A} = C^\infty(S^4)$ where S^4 appears as quaternionic projective space but the computations of [10] used an "Ansatz" and did not analyse the general solution. In particular this left open the possibility of a noncommutative solution for $m = 2$ (and $r = 4$). We shall show in this paper that such noncommutative solutions do exist and provide very natural examples of NC 4-spheres S^4_θ. We shall also describe the noncommutative space associated to (1.9) for $m = 2$ (and $r = 4$) as a noncommutative Grassmannian as well as the corresponding notion of admissible morphisms. This space Gr contains our NC 4-spheres S^4_θ as the suspension of a NC 3-sphere S^3_θ distinct from quantum group deformations $\mathrm{SU}_q(2)$ of $\mathrm{SU}(2)$.

Our next task will be to analyse the metrics (i.e. the operators D) on our solutions of Eq. (1.9). In [10] it was shown that for any Riemannian metric $g_{\mu\nu}$ on S^4 whose volume form $\sqrt{g}\, d^4x$ is the same as the one for the round metric, the corresponding Dirac operator gives a solution to the following quartic equation,

$$\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle = \gamma_5, \tag{1.11}$$

where $\langle\ \rangle$ is the projection on the commutant of 4×4 matrices (recall that $e \in M_4(\mathcal{A})$ is a 4×4 matrix).

We shall show in this paper how to construct the Dirac operator on the noncommutative 4-spheres S^4_θ so that Eq. (1.11) continues to hold without any change. Combining this Eq. (1.11) with the index formula gives a quantization of the volume,

$$\int\!\!\!\!\!- ds^4 \in \mathbb{N} \quad ds = D^{-1} \tag{1.12}$$

and fixes (in a given K-homology class for the operator D) the leading term of the spectral action [3],

$$\mathrm{Trace}\left(f\left(\frac{D}{\Lambda}\right)\right) = \frac{\Lambda^4}{2} \int\!\!\!\!- \, ds^4 + \cdots . \tag{1.13}$$

Since the next term is the Hilbert-Einstein action in the usual Riemannian case [3,18, 17], it is very natural to compare various solutions (commutative or not) of (1.11) using this action.

2. Components of the Chern Character and the Instanton Algebra

Let $\mathcal{A}$ be an algebra (over $\mathbb{C}$) and

$$e \in M_r(\mathcal{A}),\ e^2 = e \tag{2.1}$$

be an idempotent. The component $\mathrm{ch}_n(e)$ of the (reduced) Chern character of e is an element of

$$\mathcal{A} \otimes \underbrace{\overline{\mathcal{A}} \otimes \cdots \otimes \overline{\mathcal{A}}}_{2n}, \tag{2.2}$$

where $\overline{\mathcal{A}} = \mathcal{A}/\mathbb{C}1$ is the quotient of $\mathcal{A}$ by the scalar multiples of the unit 1. The formula for $\mathrm{ch}_n(e)$ is (with λ_n a normalization constant),

$$\mathrm{ch}_n(e) = \lambda_n \sum \left(e_{i_0 i_1} - \frac{1}{2}\,\delta_{i_0 i_1}\right) \otimes \widetilde{e}_{i_1 i_2} \otimes \widetilde{e}_{i_2 i_3} \cdots \otimes \widetilde{e}_{i_{2n} i_0}, \tag{2.3}$$

where δ_{ij} is the usual Kronecker symbol and only the class $\widetilde{e}_{i_k i_{k+1}} \in \overline{\mathcal{A}}$ is used in the formula. The crucial property of the components $\mathrm{ch}_n(e)$ is that they define a *cycle* in the (b, B) bicomplex of cyclic homology [5,20],

$$B\,\mathrm{ch}_n(e) = b\,\mathrm{ch}_{n+1}(e). \tag{2.4}$$

For any pair of integers m, r we let $\mathcal{A}_{m,r}$ be the universal algebra associated to the relations,

$$\mathrm{ch}_j(e) = 0 \quad \forall\, j < m. \tag{2.5}$$

More precisely we let $\mathcal{A}_{m,r}$ be generated by the r^2 elements e_{ij}; $i, j \in \{1, \ldots, r\}$ and we first impose the relations

$$e^2 = e, \quad e = [e_{ij}]. \tag{2.6}$$

An admissible homomorphism,

$$\rho : \mathcal{A}_{m,r} \to \mathcal{B}, \tag{2.7}$$

to an arbitrary algebra $\mathcal{B}$, is given by the $\rho(e_{ij}) \in \mathcal{B}$ which fulfill

$$\rho(e)^2 = \rho(e), \tag{2.8}$$

and $\mathrm{ch}_j(\rho(e)) = 0$ for $j < m$, thus

$$\sum \left(\rho(e_{i_0 i_1}) - \frac{1}{2}\,\delta_{i_0 i_1}\right) \otimes \widetilde{\rho(e_{i_1 i_2})} \otimes \cdots \otimes \widetilde{\rho(e_{i_{2j} i_0})} = 0, \tag{2.9}$$

where the symbol $\sim$ means that only the class in $\overline{\mathcal{B}}$ matters. We define $\mathcal{A}_{m,r}$ as the quotient of the algebra defined by (2.6) by the intersection of kernels of all admissible morphisms ρ.

Elements of the algebra $\mathcal{A}_{m,r}$ can be represented as polynomials in the generators e_{ij} and to prove that such a polynomial $P(e_{ij})$ is non-zero in $\mathcal{A}_{m,r}$ one must construct a solution to the above equations for which $P(e_{ij}) \neq 0$. To get a C^* algebra we endow $\mathcal{A}_{m,r}$ with the involution given by,

$$(e_{ij})^* = e_{ji} \tag{2.10}$$

which means that $e = e^*$ in $M_r(\mathcal{A})$. We define a norm by,

$$\|P\| = \mathrm{Sup}\ \|(\pi(P))\|, \tag{2.11}$$

where π ranges through all representations of the above equations in Hilbert space. Such a π is given by a Hilbert space $\mathcal{H}$ and a self-adjoint idempotent,

$$E \in M_r(\mathcal{L}(\mathcal{H})),\ E^2 = E,\ E = E^* \tag{2.12}$$

such that (2.9) holds for $\mathcal{B} = \mathcal{L}(\mathcal{H})$.

One checks that for any polynomial $P(e_{ij})$ the quantity (2.11), i.e. the supremum of the norms,

$$\|P(E_{ij})\| \tag{2.13}$$

is finite.

We let $A_{m,r}$ be the C^* algebra obtained as the completion of $\mathcal{A}_{m,r}$ for the above norm.

To get familiar with the (a priori noncommutative) spaces $\mathrm{Gr}_{m,r}$ such that,

$$A_{m,r} = C(\mathrm{Gr}_{m,r}) \tag{2.14}$$

we shall first recall from [10] what happens in the simplest case $m = 1$, $r = 2$.

One has $e = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix}$ and the condition (2.7) just means that

$$e_{11} + e_{22} = 1 \tag{2.15}$$

while (2.6) means that

$$\begin{aligned} e_{11}^2 + e_{12}\,e_{21} &= e_{11}, & e_{11}\,e_{12} + e_{12}e_{22} &= e_{12}, \\ e_{21}\,e_{11} + e_{22}\,e_{21} &= e_{21}, & e_{21}\,e_{12} + e_{22}^2 &= e_{22}. \end{aligned} \tag{2.16}$$

By (2.15) we get $e_{11} - e_{11}^2 = e_{22} - e_{22}^2$, so that (2.16) shows that $e_{12}\,e_{21} = e_{21}\,e_{12}$. We also see that e_{12} and e_{21} both commute with e_{11}. This shows that $\mathcal{A}_{1,2}$ is commutative and allows to check that $\mathrm{Gr}_{1,2} = S^2$ is the 2-sphere. Thus $\mathrm{Gr}_{1,2}$ is an ordinary commutative space.

Next, we move on to the case $m = 2$, $r = 4$. We shall show now that $\mathrm{Gr}_{2,4}$ is a non-commutative space. It differs from standard NC-Grassmannians and its very interesting structure will be analysed elsewhere. Note that the notion of admissible morphism is a non trivial piece of structure on $\mathrm{Gr}_{2,4}$ since the identity map is not admissible.

We can first reformulate the construction of [10] Sect. XI and get an admissible surjection,

$$A_{2,4} \xrightarrow{\sigma} C(S^4), \tag{2.17}$$

where S^4 appears naturally as quaternionic projective space, $S^4 = P_1(\mathbb{H})$. Let us recall from [10] that the equality,

$$E(x) = \begin{bmatrix} t & q \\ \bar{q} & 1-t \end{bmatrix} \in M_4(\mathbb{C}) \tag{2.18}$$

for $x = (q, t)$ given by a pair of a quaternion $q = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}$ and a real number t such that

$$q\,\bar{q} = t - t^2 \tag{2.19}$$

defines a map from the 4-sphere S^4 (the double of the 4-disk $|q| \leq 1$) to the Grassmannian of 2-dimensional projections $E = E^2 = E^*$ in $M_4(\mathbb{C})$ such that,

$$\mathrm{Trace}\,(F(x)\,F(y)\,F(z)) = 0 \quad \forall\, x, y, z \in S^4, \tag{2.20}$$

where $F(x) = 2\,E(x) - 1$ is the corresponding self-adjoint isometry.

The equality XI.54 of [10] is weaker than this statement but examining the proof one gets (2.20). To formulate the result for arbitrary even spheres S^{2m} we note first that using (2.4) the equality

$$\omega = \mathrm{ch}_m(e) \tag{2.21}$$

defines a *Hochschild cycle* $\rho(\omega) \in Z_{2m}(\mathcal{B})$ for any admissible morphism $\rho : \mathcal{A}_{m,r} \to \mathcal{B}$. We let $r = 2^m$ and construct an admissible surjection,

$$A_{m,2^m} \xrightarrow{\sigma} C(S^{2m}) \tag{2.22}$$

which is non trivial inasmuch as

$$\sigma(\omega) = v \tag{2.23}$$

is the volume form of the round oriented sphere.

To construct σ we let $\mathrm{Cl} = \mathrm{Cliff}\,(\mathbb{R}^{2m})$ be the Clifford algebra of the (oriented) Euclidean space $\mathbb{R}^{2m}$. We identify S^{2m} with the space of pairs (ξ, t), $\xi \in \mathbb{R}^{2m}$ and $t \in [-1, 1]$ such that $\|\xi\|^2 + t^2 = 1$. We then define a map from S^{2m} to the Grassmannian of self-adjoint idempotents in Cl by

$$E(\xi, t) = \frac{1}{2} + \frac{1}{2}\,(\gamma(\xi) + t\,\gamma), \tag{2.24}$$

where $\gamma(\xi)$ is the usual inclusion of $\mathbb{R}^{2m}$ in Cl such that

$$\gamma(\xi)^2 = \|\xi\|^2, \;\; \gamma(\xi) = \gamma(\xi)^* \tag{2.25}$$

and $\gamma \in \mathrm{Cl}$, $\gamma^* = \gamma$, $\gamma^2 = 1$ is the $\mathbb{Z}/2$ grading associated with the chosen orientation of $\mathbb{R}^{2m}$. One has $\gamma\,\gamma(\xi) = -\gamma(\xi)\gamma$ for any ξ which allows to check that $\gamma(\xi) + t\,\gamma$ is an involution and E a self-adjoint idempotent. Next, for $\ell < 2m$, ℓ *odd*,

$$\mathrm{Trace}\,((\gamma(\xi_1) + t_1\,\gamma) \dots (\gamma(\xi_\ell) + t_\ell\,\gamma)) = 0 \quad \forall\,\xi_j, t_j. \tag{2.26}$$

Indeed the coefficient of monomials in t of even degree is of the form Trace $(\gamma(\xi_1)\dots\gamma(\xi_{2k+1}))$ which vanishes by anticommutation with γ. The coefficient of monomials in t of odd degree is of the form Trace $(\gamma(\xi_1)\dots\gamma(\xi_{2k})\,\gamma)$, where $k < m$. It vanishes because γ is orthogonal to all the lower filtration of C. We thus get,

$$\mathrm{Trace}\left(\left(E(x_1) - \frac{1}{2}\right)\dots\left(E(x_\ell) - \frac{1}{2}\right)\right) = 0 \quad \forall\, x_1, \dots, x_\ell \in S^{2m} \tag{2.27}$$

provided ℓ is odd, $\ell < 2m$.

Hence E defines an admissible homomorphism $\sigma : A_{m,2^m} \to C(S^{2m})$ and one has, as in [10], the following result,

Theorem 1. a) *$E \in C^\infty(S^{2m}, M_r(\mathbb{C}))$ satisfies $E = E^2 = E^*$ and $\mathrm{ch}_j(E) = 0\ \forall\, j < m$.*

b) *The Hochschild cycle $\omega = \mathrm{ch}_m(E)$ is the volume form of the round sphere S^{2m}.*

c) *Let g be a Riemannian metric on S^{2m} with volume form $\sqrt{g}\,d^{2m}x = \omega$, then the corresponding Dirac operator D fulfills*

$$\left\langle \left(e - \frac{1}{2}\right)[D, e]^{2m} \right\rangle = \gamma,$$

where $e = E$ as above and $\langle\ \rangle$ is the projection on the commutant of $M_r(\mathbb{C})$.

We have identified $M_r(\mathbb{C})$ with the Clifford algebra $\mathrm{Cl} = \mathrm{Cliff}\,(\mathbb{R}^{2m})$, $r = 2^m$. This result shows in particular that $\mathrm{Gr}_{m,r}$, $r = 2^m$, contains S^{2m} in such a way that $\omega | S^{2m}$ is the volume form for the round metric. The proof is the same as in [10].

3. The Noncommutative 4-Sphere

Let us now move on to the inclusion $S^4_\theta \subset \mathrm{Gr}_{2,4}$, where S^4_θ is the noncommutative 4-sphere we are about to describe.

One should observe from the outset that the compact Lie group $SU(4)$ acts by automorphisms,

$$PSU(4) \subset \mathrm{Aut}\,(C^\infty \mathrm{Gr}_{2,4}) \tag{3.1}$$

by the following operation,

$$e \to U\,e\,U^*, \tag{3.2}$$

where $U \in SU(4)$ is viewed as a 4×4 matrix and $e = [e_{ij}]$ as above.

We shall now show that the algebra $C(\mathrm{Gr}_{2,4})$ is noncommutative by constructing explicit admissible surjections,

$$C(\mathrm{Gr}_{2,4}) \to C(S^4_\theta) \tag{3.3}$$

whose form is dictated by natural deformations of the 4-sphere similar in spirit to the standard deformation of $\mathbb{T}^2$ to $\mathbb{T}^2_\theta$.

We first determine the algebra generated by $M_4(\mathbb{C})$ and a projection $e = e^*$ such that $\langle e - \frac{1}{2} \rangle = 0$ as above and whose two by two matrix expression is of the form,

$$[e^{ij}] = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \tag{3.4}$$

where each q_{ij} is a 2×2 matrix of the form,

$$q = \begin{bmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{bmatrix}, \tag{3.5}$$

and $\lambda = \exp(2\pi i\theta)$ is a complex number of modulus one, different from -1 for convenience. Since $e = e^*$, both q_{11} and q_{22} are selfadjoint, moreover since $\langle e - \frac{1}{2} \rangle = 0$, we can find $t = t^*$ such that,

$$q_{11} = \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}, \; q_{22} = \begin{bmatrix} (1-t) & 0 \\ 0 & (1-t) \end{bmatrix}. \tag{3.6}$$

We let $q_{12} = \left[\begin{smallmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{smallmatrix} \right]$, we then get from $e = e^*$,

$$q_{21} = \begin{bmatrix} \alpha^* & -\bar{\lambda}\beta \\ \beta^* & \alpha \end{bmatrix}. \tag{3.7}$$

We thus see that the commutant $\mathcal{B}_\theta$ of $M_4(\mathbb{C})$ is generated by t, α, β and we first need to find the relations imposed by the equality $e^2 = e$.
In terms of

$$e = \begin{bmatrix} t & q \\ q^* & 1-t \end{bmatrix}, \tag{3.8}$$

the equation $e^2 = e$ means that $t^2 - t + qq^* = 0$, $t^2 - t + q^*q = 0$ and $[t, q] = 0$. This shows that t commutes with α, β, α^* and β^* and since $qq^* = q^*q$ is a diagonal matrix

$$\alpha\alpha^* = \alpha^*\alpha, \; \alpha\beta = \lambda\beta\alpha, \; \alpha^*\beta = \bar{\lambda}\beta\alpha^*, \; \beta\beta^* = \beta^*\beta \tag{3.9}$$

so that the C^* algebra $\mathcal{B}_\theta$ is not commutative for λ different from 1. The only further relation is, (besides $t = t^*$),

$$\alpha\alpha^* + \beta\beta^* + t^2 - t = 0. \tag{3.10}$$

We denote by S^4_θ the corresponding noncommutative space, so that $C(S^4_\theta) = \mathcal{B}_\theta$. It is by construction the suspension of the noncommutative 3-sphere S^3_θ whose coordinate algebra is generated by α and β as above and say the special value $t = 1/2$.

Had we taken the deformation parameter to be real, $\lambda = q \in \mathbb{R}$, the noncommutative 3-sphere S^3_q would coincide with the quantum group $SU_q(2)$. Similarly, had we taken the deformation parameter in S^4_θ to be real like in [14] we would have obtained a different deformation S^4_q of the commutative sphere S^4, whose algebra is different from the above one. More important, the two dimensional component $ch_1(e)$ of the Chern character would not vanish.

We shall now check that for the sphere S^4_θ the two dimensional component $ch_1(e)$ automatically vanishes as an element of the (normalized) (b,B)-bicomplex so that,

$$ch_n(e) = 0, \ n = 0, 1. \tag{3.11}$$

With $q = \begin{bmatrix} \alpha & \beta \\ -\lambda\beta^* & \alpha^* \end{bmatrix}$, we get,

$$\begin{aligned} ch_1(e) = \Big\langle \Big(t - \frac{1}{2}\Big) \, (dq\, dq^* - dq^*\, dq) \\ + \, q\,(dq^*\, dt - dt\, dq^*) + q^*\,(dt\, dq - dq\, dt) \Big\rangle, \end{aligned} \tag{3.12}$$

where the expectation in the right hand side is relative to $M_2(\mathbb{C})$ and we use the notation d instead of the tensor notation.

The diagonal elements of $\omega = dq\, dq^*$ are

$$\omega_{11} = d\alpha\, d\alpha^* + d\beta\, d\beta^*, \ \omega_{22} = d\beta^*\, d\beta + d\alpha^*\, d\alpha$$

while for $\omega' = dq^*\, dq$ we get,

$$\omega'_{11} = d\alpha^*\, d\alpha + d\beta\, d\beta^*, \ \omega'_{22} = d\beta^*\, d\beta + d\alpha\, d\alpha^*.$$

It follows that, since t is diagonal,

$$\Big\langle \Big(t - \frac{1}{2}\Big) \, (dq\, dq^* - dq^*\, dq) \Big\rangle = 0. \tag{3.13}$$

The diagonal elements of $q\, dq^*\, dt = \rho$ are

$$\rho_{11} = \alpha\, d\alpha^*\, dt + \beta\, d\beta^*\, dt, \ \rho_{22} = \beta^*\, d\beta\, dt + \alpha^*\, d\alpha\, dt$$

while for $\rho' = q^*\, dq\, dt$ they are

$$\rho'_{11} = \alpha^*\, d\alpha\, dt + \beta\, d\beta^*\, dt, \ \rho'_{22} = \beta^*\, d\beta\, dt + \alpha\, d\alpha^*\, dt.$$

Similarly for $\sigma = q\, dt\, dq^*$ and $\sigma' = q^*\, dt\, dq$ one gets the required cancellations so that,

$$ch_1(e) = 0. \tag{3.14}$$

We thus get,

Theorem 2. a) *$e \in C^\infty(S^4_\theta, M_4(\mathbb{C}))$ satisfies $e = e^2 = e^*$ and $\mathrm{ch}_j(e) = 0 \ \forall\, j < 2$.*
b) *$\mathrm{Gr}_{2,4}$ is a noncommutative space and $S^4_\theta \subset \mathrm{Gr}_{2,4}$.*

Since $ch_1(e) = 0$, it follows that $ch_2(e)$ is a Hochschild cycle which will play the role of the round volume form on S^4_θ and that we shall now compute. With the above notations one has,

$$ch_2(e) = \left\langle \begin{bmatrix} t - \frac{1}{2} & q \\ q^* & \frac{1}{2} - t \end{bmatrix} \left(\begin{bmatrix} dt & dq \\ dq^* & -dt \end{bmatrix} \right)^4 \right\rangle, \tag{3.15}$$

and the sum of the diagonal elements is

$$
\begin{aligned}
&\left(t-\frac{1}{2}\right)\,\Big((dt^2+dq\,dq^*)^2+(dt\,dq-dq\,dt)(dq^*\,dt-dt\,dq^*)\Big)\\
&\quad-\left(t-\frac{1}{2}\right)\,\Big((dq^*\,dt-dt\,dq^*)(dt\,dq-dq\,dt)+(dq^*\,dq+dt^2)^2\Big)\\
&\quad+q\Big((dq^*\,dt-dt\,dq^*)(dt^2+dq\,dq^*)+(dq^*\,dq+dt^2)(dq^*\,dt-dt\,dq^*)\Big)\\
&\quad+q^*\Big((dt^2+dq\,dq^*)(dt\,dq-dq\,dt)+(dt\,dq-dq\,dt)(dq^*\,dq+dt^2)\Big).
\end{aligned} \tag{3.16}
$$

Since t and dt are diagonal 2×2 matrices of operators and the same diagonal terms appear in $dq\,dq^*$ and $dq^*\,dq$, by the same argument by which we got the vanishing (3.13), the first two lines only contribute by,

$$
\left\langle\left(t-\frac{1}{2}\right)\,(dq\,dq^*\,dq\,dq^*-dq^*\,dq\,dq^*\,dq)\right\rangle. \tag{3.17}
$$

Similarly, the last two lines only contribute by

$$
\begin{aligned}
&\Big\langle q^*\,(dt\,dq\,dq^*\,dq-dq\,dt\,dq^*\,dq+dq\,dq^*\,dt\,dq-dq\,dq^*\,dq\,dt)\\
&-q\,(dt\,dq^*\,dq\,dq^*-dq^*\,dt\,dq\,dq^*+dq^*\,dq\,dt\,dq^*-dq^*\,dq\,dq^*\,dt)\Big\rangle.
\end{aligned} \tag{3.18}
$$

The direct computation gives $ch_2(e)$ as a sum of five components

$$
ch_2(e)=(t-\frac{1}{2})\,\Gamma_t+\alpha\,\Gamma_\alpha+\alpha^*\,\Gamma_{\alpha^*}+\beta\,\Gamma_\beta+\beta^*\,\Gamma_{\beta^*}\,, \tag{3.19}
$$

with the operators $\Gamma_t, \Gamma_\alpha, \Gamma_{\alpha^*}, \Gamma_\beta, \Gamma_{\beta^*}$ explicitly given by

$$
\begin{aligned}
\Gamma_t=\;&(d\alpha\,d\alpha^*-d\alpha^*\,d\alpha)(d\beta\,d\beta^*-d\beta^*\,d\beta)\\
&+(d\beta\,d\beta^*-d\beta^*\,d\beta)(d\alpha\,d\alpha^*-d\alpha^*\,d\alpha)\\
&+(d\alpha\,d\beta-\lambda\,d\beta\,d\alpha)(d\beta^*\,d\alpha^*-\bar\lambda\,d\alpha^*\,d\beta^*)\\
&+(d\beta^*\,d\alpha^*-\bar\lambda\,d\alpha^*\,d\beta^*)(d\alpha d\beta-\lambda\,d\beta\,d\alpha)\\
&+(d\alpha^*\,d\beta-\bar\lambda\,d\beta\,d\alpha^*)(\lambda\,d\alpha\,d\beta^*\,-d\beta^*\,d\alpha)\\
&+(\lambda\,d\alpha\,d\beta^*\,-d\beta^*\,d\alpha)(d\alpha^*\,d\beta-\bar\lambda\,d\beta\,d\alpha^*)\,;
\end{aligned} \tag{3.20}
$$

$$
\begin{aligned}
\Gamma_\alpha=\;&(dt\,d\alpha^*-d\alpha^*\,dt)(d\beta^*\,d\beta-d\beta\,d\beta^*)\\
&+(d\beta^*\,d\beta-d\beta\,d\beta^*)(dt\,d\alpha^*-d\alpha^*\,dt)\\
&+(d\beta\,dt-dt\,d\beta)(d\beta^*\,d\alpha^*-\bar\lambda\,d\alpha^*\,d\beta^*)\\
&+\lambda\,(d\beta^*\,d\alpha^*-\bar\lambda\,d\alpha^*\,d\beta^*)(d\beta\,dt-dt\,d\beta)\\
&+(d\alpha^*\,d\beta-\bar\lambda\,d\beta\,d\alpha^*)(d\beta^*\,dt-dt\,d\beta^*)\\
&+\;\lambda\,(d\beta^*\,dt-dt\,d\beta^*)(d\alpha^*\,d\beta-\bar\lambda\,d\beta\,d\alpha^*);
\end{aligned} \tag{3.21}
$$

$$
\begin{aligned}
\Gamma_{\alpha^*} = &(dt\, d\alpha - d\alpha\, dt)(d\beta\, d\beta^* - d\beta^*\, d\beta) \\
&+ (d\beta\, d\beta^* - d\beta^*\, d\beta)(dt\, d\alpha - d\alpha\, dt) \\
&+ (d\alpha\, d\beta - \lambda\, d\beta\, d\alpha)(dt\, d\beta^* - d\beta^*\, dt) \\
&+ \bar{\lambda}\,(dt\, d\beta^* - d\beta^*\, dt)(d\alpha\, d\beta - \lambda\, d\beta\, d\alpha) \\
&+ (dt\, d\beta - d\beta\, dt)(d\beta^*\, d\alpha - \lambda\, d\alpha\, d\beta^*) \\
&+ \bar{\lambda}\,(d\beta^*\, d\alpha - \lambda\, d\alpha\, d\beta^*)(dt\, d\beta - d\beta\, dt);
\end{aligned}
\tag{3.22}
$$

$$
\begin{aligned}
\Gamma_{\beta} = &(dt\, d\beta^* - d\beta^*\, dt)(d\alpha^*\, d\alpha - d\alpha\, d\alpha^*) \\
&+ (d\alpha^*\, d\alpha - d\alpha\, d\alpha^*)(dt\, d\beta^* - d\beta^*\, dt) \\
&+ \lambda\,(dt\, d\alpha - d\alpha\, dt)(d\beta^*\, d\alpha^* - \bar{\lambda}\, d\alpha^*\, d\beta^*) \\
&+ (d\beta^*\, d\alpha^* - \bar{\lambda}\, d\alpha^*\, d\beta^*)(dt\, d\alpha - d\alpha\, dt) \\
&+ \bar{\lambda}\,(d\alpha^*\, dt - dt\, d\alpha^*)(d\beta^*\, d\alpha - \lambda\, d\alpha\, d\beta^*) \\
&+ (d\beta^*\, d\alpha - \lambda\, d\alpha\, d\beta^*)(d\alpha^*\, dt - dt\, d\alpha^*);
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
\Gamma_{\beta^*} = &(dt\, d\beta - d\beta\, dt)(d\alpha\, d\alpha^* - d\alpha^*\, d\alpha) \\
&+ (d\alpha\, d\alpha^* - d\alpha^*\, d\alpha)(dt\, d\beta - d\beta\, dt) \\
&+ (d\alpha^*\, dt - dt\, d\alpha^*)(d\alpha\, d\beta - \lambda d\beta\, d\alpha) \\
&+ \bar{\lambda}\,(d\alpha\, d\beta - \lambda d\beta\, d\alpha)(d\alpha^*\, dt - dt\, d\alpha^*) \\
&+ (dt\, d\alpha - d\alpha\, dt)(d\alpha^*\, d\beta - \bar{\lambda}\, d\beta\, d\alpha^*) \\
&+ \lambda\,(d\alpha^*\, d\beta - \bar{\lambda}\, d\beta\, d\alpha^*)(dt\, d\alpha - d\alpha\, dt).
\end{aligned}
\tag{3.24}
$$

One can equivalently (in order to avoid any confusion with ordinary differentials) write the Hochschild cycle $c = ch_2(e)$ as

$$
c = \left(t - \frac{1}{2}\right) c_t + \alpha\, c_\alpha + \alpha^*\, c_{\alpha^*} + \beta\, c_\beta + \beta^*\, c_{\beta^*}, \tag{3.25}
$$

where the components $c_t, c_\alpha, c_{\alpha^*}, c_\beta, c_{\beta^*}$, which are elements in $\mathcal{B}_\theta \otimes \mathcal{B}_\theta \otimes \mathcal{B}_\theta \otimes \mathcal{B}_\theta$, have an expression of the same form as the corresponding operators in (3.20-3.24) with the symbol d substituted by the tensor product symbol $\otimes$. The vanishing of bc, which has six hundred terms, can be checked directly from the commutation relations (3.9). The cycle c is totally "λ-antisymmetric".

4. The Noncommutative Geometry of S^4_θ

The next step consists in finding the Dirac operator which gives a solution to the basic quartic equation (1.11). Let $\mathcal{A} = C^\infty(S^4_\theta)$ be the algebra of smooth functions on the noncommutative sphere S^4_θ. We shall now describe a spectral triple

$$
(\mathcal{A}, \mathcal{H}, D) \tag{4.1}
$$

which describes the geometry on S^4_θ corresponding to the round metric.

In order to do that we first need to find good coordinates on S^4_θ in terms of which the operator D will be easily expressed. We choose to parametrize α, β and t as follows:

$$\alpha = \frac{u}{2}\,\cos\varphi\cos\psi, \quad \beta = \frac{v}{2}\,\sin\varphi\cos\psi, \quad t = \frac{1}{2} + \frac{1}{2}\,\sin\psi. \tag{4.2}$$

Here φ and ψ are ordinary angles with domain

$$0 \le \varphi \le \frac{\pi}{2}, \quad -\frac{\pi}{2} \le \psi \le \frac{\pi}{2}, \tag{4.3}$$

while u and v are the usual unitary generators of the algebra $C^\infty(\mathbb{T}^2_\theta)$ of smooth functions on the noncommutative 2-torus. Thus the presentation of their relations is

$$uv = \lambda vu, \quad uu^* = u^*u = 1, \quad vv^* = v^*v = 1. \tag{4.4}$$

One checks that α, β, t given by (2) satisfy the basic presentation of the generators of $C^\infty(S^4_\theta)$ which thus appears as a *subalgebra* of the algebra generated (and then closed under smooth calculus) by $e^{i\varphi}$, $e^{i\psi}$, u and v.

For $\theta = 0$ one readily computes the round metric,

$$G = 4\,(d\alpha\, d\overline{\alpha} + d\beta\, d\overline{\beta} + dt^2) \tag{4.5}$$

and in terms of the coordinates, φ, ψ, u, v one gets,

$$G = \cos^2\varphi\cos^2\psi\, du\, d\overline{u} + \sin^2\varphi\cos^2\psi\, dv\, d\overline{v} + \cos^2\psi\, d\varphi^2 + d\psi^2. \tag{4.6}$$

Up to normalization, its volume form is given by

$$\sin\varphi\cos\varphi\,(\cos\psi)^3\,\overline{u}\, du \wedge \overline{v}\, dv \wedge d\psi \wedge d\varphi. \tag{4.7}$$

In terms of these rectangular coordinates we get the following simple expression for the Dirac operator,

$$\begin{aligned} D = {} & (\cos\varphi\cos\psi)^{-1}\, u\, \frac{\partial}{\partial u}\,\gamma_1 + (\sin\varphi\cos\psi)^{-1}\, v\, \frac{\partial}{\partial v}\,\gamma_2 \\ & + \frac{1}{\cos\psi}\,\sqrt{-1}\left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\,\mathrm{cotg}\,\varphi - \frac{1}{2}\mathrm{tg}\,\varphi\right)\gamma_3 + \sqrt{-1}\left(\frac{\partial}{\partial\psi} - \frac{3}{2}\,\mathrm{tg}\,\psi\right)\gamma_4. \end{aligned} \tag{4.8}$$

Here γ_μ are the usual Dirac 4×4 matrices with

$$\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu},\ \gamma_\mu^* = \gamma_\mu. \tag{4.9}$$

It is now easy to move on to the noncommutative case, the only tricky point is that there are nontrivial boundary conditions for the operator D, which are in particular antiperiodic in the arguments of both u and v. We shall just leave them unchanged in the NC case, the only thing which changes is the algebra and the way it acts in the Hilbert space as we shall explain in more detail in the next section. The formula for the operator D is now,

$$\begin{aligned} D = {} & (\cos\varphi\cos\psi)^{-1}\,\delta_1\,\gamma_1 + (\sin\varphi\cos\psi)^{-1}\,\delta_2\,\gamma_2 \\ & + \frac{1}{\cos\psi}\,\sqrt{-1}\left(\frac{\partial}{\partial\varphi} + \frac{1}{2}\,\mathrm{cotg}\,\varphi - \frac{1}{2}\,\mathrm{tg}\,\varphi\right)\gamma_3 + \sqrt{-1}\left(\frac{\partial}{\partial\psi} - \frac{3}{2}\,\mathrm{tg}\,\psi\right)\gamma_4, \end{aligned} \tag{4.10}$$

where the γ_μ are the usual Dirac matrices and where δ_1 and δ_2 are the standard derivations of the NC torus so that

$$\begin{aligned} \delta_1(u) = u, \quad \delta_1(v) = 0, \\ \delta_2(u) = 0, \quad \delta_2(v) = v. \end{aligned} \tag{4.11}$$

One can then check that the corresponding metric is the right one (the round metric).

In order to compute the operator $\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle$ (in the tensor product by $M_4(\mathbb{C})$) we need the commutators of D with the generators of $C^\infty(S^4_\theta)$. They are given by the following simple expressions:

$$\begin{aligned}
[D, \alpha] &= \frac{u}{2} \left\{ \gamma_1 - \sqrt{-1} \sin(\phi)\, \gamma_3 - \sqrt{-1} \cos\phi) \sin(\psi)\, \gamma_4 \right\}, \\
[D, \alpha^*] &= -\frac{u^*}{2} \left\{ \gamma_1 + \sqrt{-1} \sin(\phi)\, \gamma_3 + \sqrt{-1} \cos(\phi) \sin(\psi)\, \gamma_4 \right\}, \\
[D, \beta] &= \frac{v}{2} \left\{ \gamma_2 + \sqrt{-1} \cos(\phi)\, \gamma_3 - \sqrt{-1} \sin(\phi) \sin(\psi)\, \gamma_4 \right\}, \\
[D, \beta^*] &= -\frac{v^*}{2} \left\{ \gamma_2 - \sqrt{-1} \cos(\phi)\, \gamma_3 + \sqrt{-1} \sin(\phi) \sin(\psi)\, \gamma_4 \right\}, \\
[D, t] &= \frac{\sqrt{-1}}{2} \cos(\psi)\, \gamma_4.
\end{aligned} \tag{4.12}$$

We check in particular that they are all bounded operators and hence that for any $f \in C^\infty(S^4_\theta)$ the commutator $[D, f]$ is bounded. Then, a long but straightforward calculation shows that the operator $\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle$ is a multiple of $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$. One first checks that it is equal to $\pi(c)$, where c is the Hochschild cycle in (III.3.25) and π is the canonical map from the Hochschild chains to operators given by

$$\pi(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0[D, a_1] \ldots [D, a_n]. \tag{4.13}$$

One can then check the various conditions which in the commutative case suffice to characterize Riemannian geometry [8,9].

Theorem 3. a) *The spectral triple* $(C^\infty(S^4_\theta), \mathcal{H}, D)$ *fulfills all axioms of noncommutative manifolds.*

b) *Let* $e \in C^\infty(S^4_\theta, M_4(\mathbb{C}))$ *be the canonical idempotent given in (III.3.8). The Dirac operator* D *fulfills*

$$\left\langle \left(e - \frac{1}{2}\right) [D, e]^4 \right\rangle = \gamma,$$

where $\langle\ \rangle$ *is the projection on the commutant of* $M_4(\mathbb{C})$ *and* γ *is the grading operator.*

The real structure [7] is given by the charge conjugation operator J, which involves in the noncommutative case the Tomita-Takesaki antilinear involution. The order one condition,

$$[[D, a], b^0] = 0 \quad \forall\, a, b \in C^\infty(S^4_\theta), \tag{4.14}$$

where $b^0 = J b^* J^{-1}$ follows easily from the derivation rules for the δ_j.

As we shall mention in the next section, Poincaré duality continues to hold.

5. Isospectral Deformations

We shall show in this section how to extend Theorem 3 of the previous section to arbitrary metrics on the sphere S^4 which are invariant under rotation of u and v and have the same volume form as the one of the round metric. We shall in fact describe a very general construction of isospectral deformations of noncommutative geometries which implies in particular that any compact spin Riemannian manifold M whose isometry group has rank ≥ 2 admits a natural one-parameter isospectral deformation to noncommutative geometries M_θ. The deformation of the algebra will be performed along the lines of [23].

We let $(\mathcal{A}, \mathcal{H}, D)$ be the canonical spectral triple associated with a compact Riemannian spin manifold M. We recall that $\mathcal{A} = C^\infty(M)$ is the algebra of smooth functions on M, $\mathcal{H} = L^2(M, S)$ is the Hilbert space of spinors and D is the Dirac operator. We let J be the charge conjugation operator which is an antilinear isometry of $\mathcal{H}$.

Let us assume that the group $\mathrm{Isom}(M)$ of isometries of M has rank $r \geq 2$. Then, we have an inclusion

$$\mathbb{T}^2 \subset \mathrm{Isom}(M), \tag{5.1}$$

with $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the usual torus, and we let $U(s)$, $s \in \mathbb{T}^2$ be the corresponding (projective) unitary representation in $\mathcal{H} = L^2(M, S)$ so that by construction

$$U(s)\, D = D\, U(s), \quad U(s)\, J = J\, U(s). \tag{5.2}$$

Also,

$$U(s)\, a\, U(s)^{-1} = \alpha_s(a), \quad \forall\, a \in \mathcal{A}, \tag{5.3}$$

where $\alpha_s \in \mathrm{Aut}(\mathcal{A})$ is the action by isometries on the algebra of functions on M.

We let $p = (p_1, p_2)$ be the generator of the two-parameters group $U(s)$ so that

$$U(s) = \exp(i(s_1 p_1 + s_2 p_2)). \tag{5.4}$$

The operators p_1 and p_2 commute with D but anticommute with J. Both p_1 and p_2 have half-integral spectrum,

$$\mathrm{Spec}(2\, p_j) \subset \mathbb{Z}, \;\; j = 1, 2. \tag{5.5}$$

One defines a bigrading of the algebra of bounded operators in $\mathcal{H}$ with the operator T declared to be of bidegree (n_1, n_2) when,

$$\alpha_s(T) = \exp(i(s_1 n_1 + s_2 n_2))\, T, \quad \forall\, s \in \mathbb{T}^2, \tag{5.6}$$

where $\alpha_s(T) = U(s)\, T\, U(s)^{-1}$ as in (5.3). Any operator T of class C^∞ relative to α_s (i.e. such that the map $s \to \alpha_s(T)$ is of class C^∞ for the norm topology) can be uniquely written as a doubly infinite norm convergent sum of homogeneous elements,

$$T = \sum_{n_1, n_2} \widehat{T}_{n_1, n_2}, \tag{5.7}$$

with $\widehat{T}_{n_1, n_2}$ of bidegree (n_1, n_2), and where the sequence of norms $||\widehat{T}_{n_1, n_2}||$ is of rapid decay in (n_1, n_2).

Let $\lambda = \exp(2\pi i\theta)$. For any operator T in $\mathcal{H}$ of class C^∞ we define its left twist $l(T)$ by

$$l(T) = \sum_{n_1,n_2} \widehat{T}_{n_1,n_2}\, \lambda^{n_2 p_1}, \tag{5.8}$$

and its right twist $r(T)$ by

$$r(T) = \sum_{n_1,n_2} \lambda^{n_1 p_2}\, \widehat{T}_{n_1,n_2}. \tag{5.9}$$

Since $|\lambda| = 1$ and p_1, p_2 are self-adjoint, both series converge in norm. The construction involves in the case of half-integral spin the choice of a square root of λ. One has,

Lemma 4. a) *Let x be a homogeneous operator of bidegree (n_1, n_2) and y be a homogeneous operator of bidegree (n'_1, n'_2). Then,*

$$l(x)\, r(y) - r(y)\, l(x) = (x\, y\, - y\, x)\, \lambda^{n'_1 n_2 + n'_1 n'_2} \lambda^{n_2 p_1 + n'_1 p_2}. \tag{5.10}$$

In particular, $[l(x), r(y)] = 0$ if $[x, y] = 0$.
b) *Let x and y be homogeneous operators as before and define*

$$x * y = \lambda^{n'_1 n_2}\, xy; \tag{5.11}$$

then $l(x)l(y) = l(x * y)$.

To check a) and b) one simply uses the following commutation rule which is fulfilled for any homogeneous operator T of bidegree (m, n),

$$\lambda^{ap_1+bp_2}\, T = \lambda^{am+bn}\, T\, \lambda^{ap_1+bp_2}, \quad \forall a, b \in \mathbb{Z}. \tag{5.12}$$

One has then

$$l(x)\, r(y) = x\, \lambda^{n_2 p_1}\, \lambda^{n'_1 p_2}\, y\; = x\, y\, \lambda^{n'_1 n_2 + n'_1 n'_2}\; \lambda^{n_2 p_1 + n'_1 p_2} \tag{5.13}$$

and

$$r(y)\, l(x) = \; \lambda^{n'_1 p_2}\, y\, x\, \lambda^{n_2 p_1} = y\, x\, \lambda^{n'_1 n_2 + n'_1 n'_2}\; \lambda^{n_2 p_1 + n'_1 p_2} \tag{5.14}$$

which gives a). One checks b) in a similar way.
The product $*$ defined in (5.11) extends by linearity to an associative product on the linear space of smooth operators and could be called a $*$-product. One could also define a deformed “right product”. If x is homogeneous of bidegree (n_1, n_2) and y is homogeneous of bidegree (n'_1, n'_2) the product is defined by

$$x *_r y = \lambda^{-n'_1 n_2}\, xy. \tag{5.15}$$

Then, along the lines of the previous lemma one shows that $r(x)r(y) = r(x *_r y)$.

Next, we twist the antiunitary isometry J by

$$\widetilde{J} = J\, \lambda^{-p_1 p_2}. \tag{5.16}$$

One has $\widetilde{J} = \lambda^{p_1 p_2}\, J$ and hence

$$\widetilde{J}^2 = J^2. \tag{5.17}$$

Lemma 5. *For x homogeneous of bidegree (n_1, n_2) one has that*

$$\widetilde{J}\, l(x)\, \widetilde{J}^{-1} = r(J\, x\, J^{-1}). \tag{5.18}$$

For the proof one needs to check that

$$\widetilde{J}\, l(x) = r(J\, x\, J^{-1})\, \widetilde{J}. \tag{5.19}$$

One has

$$\lambda^{-p_1 p_2}\, x = x\, \lambda^{-(p_1+n_1)(p_2+n_2)} = x\, \lambda^{-n_1 n_2}\, \lambda^{-(p_1 n_2 + n_1 p_2)}\, \lambda^{-p_1 p_2}. \tag{5.20}$$

Then

$$\widetilde{J}\, l(x) = J\, \lambda^{-p_1 p_2}\, x\, \lambda^{n_2 p_1} = J\, x\, \lambda^{-n_1 n_2}\, \lambda^{-n_1 p_2}\, \lambda^{-p_1 p_2}, \tag{5.21}$$

while

$$r(J\, x\, J^{-1})\, \widetilde{J} = \lambda^{-n_1 p_2}\, J\, x\, J^{-1}\, J\, \lambda^{-p_1 p_2} = J\, x\, \lambda^{-n_1(p_2+n_2)}\, \lambda^{-p_1 p_2}. \tag{5.22}$$

Thus one gets the required equality.

We can now define a new spectral triple where both $\mathcal{H}$ and the operator D are unchanged while the algebra $\mathcal{A}$ and the involution J are modified to $l(\mathcal{A})$ and $\widetilde{J}$ respectively. By Lemma 4 b) one checks that $l(\mathcal{A})$ is still an algebra.

Since D is of bidegree $(0, 0)$ one has,

$$[D,\ l(a)] = l([D,\ a]) \tag{5.23}$$

which is enough to check that $[D, x]$ is bounded for any $x \in l(\mathcal{A})$. For $x, y \in l(\mathcal{A})$ one checks that

$$[x,\ y^0] = 0, \quad y^0 = \widetilde{J}\, y^*\, \widetilde{J}^{-1}. \tag{5.24}$$

Indeed, one can assume that x and y are homogeneous and use Lemma 5 together with Lemma 4 a). Combining Eq. (5.24) with Eq. (5.23) one then checks the order one condition

$$\big[[D,\ x], y^0\big] = 0 \quad \forall\, x, y \in l(\mathcal{A}). \tag{5.25}$$

As a first corollary of the previous construction we thus get

Theorem 6. *Let M be a compact spin Riemannian manifold whose isometry group has rank ≥ 2. Then M admits a natural one-parameter isospectral deformation to noncommutative geometries M_θ.*

The deformed spectral triple is given by $(l(\mathcal{A}), \mathcal{H}, D)$ with $\mathcal{H} = L^2(M, S)$ the Hilbert space of spinors, D the Dirac operator and $l(\mathcal{A})$ is really the algebra of smooth functions on M with product deformed to the $*$-product defined in (5.11). Moreover, the real structure is given by the twisted involution $\widetilde{J}$ defined in (5.16). One checks using the results of [24] and [8] that Poincaré duality continues to hold for the deformed spectral triple. We showed in [8] that the Dirac operator for the Levi-Civita connection minimizes the action functional $⨍ D^{2-n}$ (where n is the dimension of M) among operators of the form $D + T$ which ϵ commute with J and have the same commutators as D with any $a \in \mathcal{A}$ (so that T belongs to the commutant of $\mathcal{A}$). It is important to check that this continues to hold in the deformed case. This is easy to see since we can also assume invariance under the action $U(s)\, T\, U(s)^{-1} = \alpha_s(T)$ so that the space of available perturbations T is smaller in the deformed case.

The above construction also allows us to extend Theorem 3 of the previous section to arbitrary metrics on the sphere S^4 which are invariant under rotation of u and v and have as volume form $\sqrt{g}dx$ the round one.

In [22] Nekrasov and Schwarz showed that Yang–Mills gauge theory on noncommutative $\mathbb{R}^4$ gives a conceptual understanding of the nonzero B-field desingularization of the moduli space of instantons obtained by perturbing the ADHM equations [1]. In [25], Seiberg and Witten exhibited the unexpected relation between the standard gauge theory and the noncommutative one. The above work raises the specific question for NC-spheres S^4_θ whether one can implement such a Seiberg–Witten relation as an isospectral one. It also suggests to extend the above isospectral deformations (Theorem 6) to more general compatible Poisson structures on a given spin Riemmannian manifold.

6. Final Remarks

We shall end this paper with several important remarks,

The odd case. First there are formulas for the *odd* Chern character in cyclic homology, similar to those of Sect. 2 above. Given an invertible element $u \in GL_r(\mathcal{A})$, the component $\mathrm{ch}_{n+\frac{1}{2}}(u)$ of its Chern character is as above an element of

$$\mathcal{A} \otimes \underbrace{\overline{\mathcal{A}} \otimes \cdots \otimes \overline{\mathcal{A}}}_{2n-1}, \tag{6.1}$$

where $\overline{\mathcal{A}} = \mathcal{A}/\mathbb{C}1$ is the quotient of $\mathcal{A}$ by the scalar multiples of the unit 1.

The formula for $\mathrm{ch}_{n+\frac{1}{2}}(u)$ is (with λ_n a normalization constant),

$$\begin{aligned} \mathrm{ch}_{n+\frac{1}{2}}(u) = \lambda_n \Big\{ \sum u_{i_0 i_1} \otimes u^{-1}_{i_1 i_2} \otimes u_{i_2 i_3} \cdots \otimes u^{-1}_{i_{2n-1} i_0} \\ - \sum u^{-1}_{i_0 i_1} \otimes u_{i_1 i_2} \otimes u^{-1}_{i_2 i_3} \cdots \otimes u_{i_{2n-1} i_0} \Big\}. \end{aligned} \tag{6.2}$$

As in the even case, the crucial property of the components $\mathrm{ch}_{n+\frac{1}{2}}(u)$ is that they define a *cycle* in the (b, B) bicomplex of cyclic homology,

$$B\, \mathrm{ch}_{n-\frac{1}{2}}(u) = b\, \mathrm{ch}_{n+\frac{1}{2}}(u). \tag{6.3}$$

For any pair of integers m, r we can define the odd analogues $\mathcal{B}_{m,r}$ as generated by the r^2 elements u_{ij}; $i, j \in \{1, \ldots, r\}$ and we impose as above the relations

$$u\, u^* = u^*\, u = 1, \quad u = [u_{ij}], \tag{6.4}$$

and

$$\mathrm{ch}_{j+\frac{1}{2}}(\rho(u)) = 0 \quad \forall\, j < m. \tag{6.5}$$

One can prove as an exercise that the suspension of the corresponding NC spaces are contained in the $Gr_{m,2r}$.

The Dirac operator and quantum groups. There exist formulas for q-analogues of the Dirac operator on quantum groups, (cf. [2,21]); let us call Q these "naive" Dirac operators. Now the fundamental equation to define the thought for true Dirac operator D which we used above implicitly on the deformed 3-sphere (after suspension to the 4-sphere and for deformation parameters which are complex of modulus one) is,

$$[D]_{q^2} = Q, \tag{6.6}$$

where the symbol $[x]_q$ has the usual meaning in q-analogues,

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{6.7}$$

The main point is that it is only by virtue of this equation that the commutators $[D, a]$ will be bounded, and they will be so not only for the natural action of the algebra $\mathcal{A}$ of functions on $SU_q(2)$ on the Hilbert space of spinors but also for the natural action of the opposite algebra $\mathcal{A}^o$; this is easy to prove in Fourier. But it is not true that $[Q, a]$ is bounded, for $a \in \mathcal{A}$, due to the unbounded nature of the bimodule defining the q-analogue of the differential calculus.

Yang–Mills theory. One can develop the Yang–Mills theory on S^4_θ since we now have all the required structure, namely the algebra, the calculus and the "vector bundle" e (naturally endowed, in addition, with a preferred connection ∇). One can check that the basic results of [6] apply. In particular Theorem 4, p 561 of [6] gives a basic inequality showing that the Yang–Mills action, $YM(\nabla) = ⨍ \theta^2\, ds^4$, (where $\theta = \nabla^2$ is the curvature, and $ds = D^{-1}$) has a strictly positive lower bound given by the topological invariant $⨍ \gamma(e - \frac{1}{2})[D, e]^4\, ds^4 = 1$. The next step is thus to extend the results of [1] on the classification of Yang–Mills connections to this situation. This was done in [13] for the noncommutative torus and in [22] for noncommutative $\mathbb{R}^4$. Note however that in the noncommutative case the NC-sphere S^4_θ *is not isomorphic* to the one-point compactification of noncommutative $\mathbb{R}^4$ used there. In particular, and in contrast to what happens for noncommutative $\mathbb{R}^4$, even the measure theory of S^4_θ is very sensitive to the irrationality of the parameter θ.

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