CONTINUITY OF THE SPECTRUM OF A FIELD OF SELF-ADJOINT OPERATORS

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Abstract. Given a family of self-adjoint operators $(A_t)_{t\in T}$ indexed by a parameter t in some topological space T, necessary and sufficient conditions are given for the spectrum $\sigma(A_t)$ to be Fell-continuous *w.r.t* t. Equivalently the boundaries and the gap edges are continuous in t. If (T, d) is a complete metric space with metric d, these conditions are extended to guaranty Hölder's continuity of the spectral boundaries and the spectral gap edges. As a Corollary, an upper bound is provided on the size of closing gaps.

1. Introduction

Given a family of self-adjoint operators $(A_t)_{t\in T}$ indexed by a parameter t in some topological space T, what conditions are needed for this family to insure that the spectrum $\sigma(A_t)$ varies continuously with t ? This work is providing an answer using analysis.

This problem is usually solved by using continuous fields of Banach spaces, in particular of Hilbert spaces and C^*-algebras. This concept, initially proposed by Tomiyama [20, 21], was further developed by Dixmier and Douady [9]. The main source of reference is the book by Dixmier [10]. Unlike fiber bundles, a continuous field of Banach spaces may have pairwise non isomorphic fibers. The field of irrational rotation algebras over the interval $[0, 1]$ is a typical example [18]. Nevertheless one of the most powerful consequences of the continuity is that a continuous field of normal elements of a field of C^*-algebras admits a spectrum varying continuously with the parameter. The main question addressed in this work is to understand why, and to provide a proof without the machinery of continuous fields of C^*-algebras.

1.1. An Example: the Almost Mathieu Operator.

To illustrate the main difficulty, let H_α, where $\alpha \in [0, 1] = T$, be the Almost Mathieu model acting on $\mathcal{H} = \ell^2(\mathbb{Z})$ as follows

$$H_\alpha\psi(n) = \psi(n+1) + \psi(n-1) + 2\mu\cos 2\pi(n\alpha+\theta)\,, \qquad \psi \in \ell^2(\mathbb{Z})\,,\;\; n \in \mathbb{Z}. \tag{1}$$

In this definition θ is a fixed parameter and $\mu > 0$. It is clear that H_α is strongly continuous in α. On the other hand, if $\alpha \neq \beta$ are irrational and rationally independent, it is easy to check that $\|H_\alpha - H_\beta\| = 2\mu$, so that this family is not norm continuous. However, it has been shown [1, 6] that the norm of $\|H_\alpha\|$ is (Lipschitz) continuous in α and it was deduced from it that the gap edges of the spectrum of H_α where also (Lipschitz) continuous as long as the gap does not closed. Near the points where a gap closes, the gap edge are only Hölder continuous of exponent $1/2$ [17, 13].

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1.2. **Main Results.** The general formulation of the problem requires the following assumptions:

F1) T is a topological space.

F2) For each $t \in T$, let $\mathcal{H}_t$ be a Hilbert space. The family $\mathcal{H} = (\mathcal{H}_t)_{t\in T}$ is called a field of Hilbert spaces.

F3) For each $t \in T$ let A_t be a linear, self-adjoint operator on $\mathcal{H}_t$. The family $A = (A_t)_{t\in T}$ is called a field of self-adjoint operators.

Definition 1. *Under the assumptions [F1-F3] above, a field A of self-adjoint, bounded operators is called p2-continuous if the maps $\Phi_p : t \in T \mapsto \|p(A_t)\|$ are continuous whenever p is a polynomial in one variable with real coefficients and degree at most 2.*

In order to describe the continuity of the spectrum, the Fell topology [11] will be used (see Section 2.2). Equivalently this topology can be expressed in terms of the Hausdorff metric. The first main result is the following.

Theorem 1. *Let $A = (A_t)_{t\in T}$ be a family of bounded operators satisfying the assumptions [F1-F3]. Then the spectrum $\sigma(A_t)$ is a Fell-continuous function of t if and only if the spectral edges are continuous functions of t, if and only if the field A is p2-continuous.*

The previous results is valid only for fields of bounded self-adjoint operators. What about unbounded ones?

Definition 2. *Let $A = (A_t)_{t\in T}$ be a field of (not necessarily bounded) self-adjoint operators over T. It will be called R-continuous if the norm of its resolvent is continuous: more precisely if, for every $z \in \mathbb{C} \setminus \mathbb{R}$, the map $t \in T \mapsto \|(z - A_t)^{-1}\| \in [0, \infty)$ is continuous.*

Theorem 2. *If $A = (A_t)_{t\in T}$ is a field of self-adjoint operators. Then it is R-continuous if and only if its spectrum is Fell-continuous, if and only of its gap edges are continuous.*

Such results can be made more quantitative if a metric is introduced to describe the topology of T. Let (X, d_X) and (Y, d_Y) be complete metric spaces. For $\alpha > 0$, a function $f : X \to Y$ is called α-Hölder whenever

$$\mathrm{Hol}^{\alpha}(f) = \sup_{x\neq y} \frac{d_Y(f(x), f(y))}{d_X(x, y)^{\alpha}} < \infty \,.$$

A family $\mathcal{F}$ of α-Hölder functions from $X \to Y$ is called *uniformly* α-Hölder if $\mathrm{Hol}^{\alpha}(\mathcal{F}) = \sup_{f\in\mathcal{F}} \mathrm{Hol}^{\alpha}(f) < \infty$. Given $M > 0$ let $\mathcal{P}_2(M)$ be the set of polynomials of the form $p(z) = p_0 + p_1 z + p_2 z^2$ with $p_i \in \mathbb{R}$ and $\|p\|_1 = |p_0| + |p_1| + |p_2| \leq M$.

Definition 3. *Let [F1-F3] hold with (T, d) a complete metric space. The field A of self-adjoint bounded operators is called p2-α-Hölder continuous whenever, for all $M > 0$, the maps $\Phi_p : t \in T \mapsto \|p(A_t)\|$ are uniformly α-Hölder for $p \in \mathcal{P}_2(M)$.*

Theorem 1 indicates that a quantitative control on the norms provides an estimate on the behavior of the spectrum and conversely. This result can be expressed as follows.

Theorem 3. *Let [F1-F3] hold with (T, d) a complete metric space. Let $A = (A_t)_{t\in T}$ be a family of self-adjoint, bounded operators such that $\sup_{t\in T} \|A_t\| < \infty$.*

(i) *If A is a p2-α-Hölder continuous field then the spectrum $\sigma(A_t)$ is $\alpha/2$-Hölder continuous with respect to the Hausdorff metric.*

(ii) *If the spectrum $\sigma(A_t)$ is α-Hölder continuous with respect to the Hausdorff metric then A is a p2-α-Hölder continuous field.*

More precisely, the behaviour of the spectral gaps can be formulated as follows.

Theorem 4. *For a complete metric space (T, d) let $A = (A_t)_{t\in T}$ be a family of bounded operators satisfying the assumptions [F1-F3]. If A is p2-α-Hölder continuous then the spectral gap edges of A are α-Hölder continuous at t if this gap does not close.*

It may happen that at some point $t_0 \in T$ a spectral gap of A_t closes at the spectral value c. The precise definition of a closed gap is given in Section 2.2 Definition 5 . Such a closed gap will be called *isolated* if the distance of c from any gap in the spectrum of A_{t_0} is positive.

Theorem 5. *For a complete metric space (T, d) let $A = (A_t)_{t\in T}$ be a p2-α-Hölder continuous field of bounded, self-adjoint operators satisfying the assumptions [F1-F3]. If c is an isolated closed gap of A_{t_0} then the width of the gaps of A_t closing on c at $t = t_0$ are only $\alpha/2$-Hölder continuous at $t = t_0$.*

If the closed gap is not isolated, though it is possible to construct examples for which the width of the closing gap can vanish as slowly as wished (see Section 3.4, Example 1).

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2. Continuity

2.1. **The Core Argument.** The main argument can be phrased as follows. Let $t_0 \in T$ and let $(a, b) \subset \mathbb{R}$ be a gap in the spectrum of $A_0 = A_{t_0}$. Namely, $a, b \in \sigma(A_0)$, but $(a, b) \cap \sigma(A_0) = \emptyset$. In order to prove that b becomes a continuous function of t near t_0, let c be chosen close to b in the gap, that is $(a + b)/2 < c < b$. Then $(A_0 - c)^2$ admits $(b - c)^2$ as lowest point in its spectrum. If now $m > 0$ is large enough so that $(A_t - c)^2 < m^2$ for t near t_0, it follows that the norm of $m^2 - (A_0 - c)^2$ is precisely given by $m^2 - (b - c)^2$. By continuity of this norm in t, and undoing what was done above for t_0, but for t close enough from t_0, it follows that for any $b - c > \epsilon > 0$ there is a neighborhood U of t_0 such that for $t \in U$, $(c - \epsilon, c + \epsilon) \cap \sigma(A_t) = \emptyset$ and there is $b_t \in \sigma(A_t)$ such that $0 < b_t - b < \epsilon$. Such an idea can be expressed more technically as follows.

Lemma 1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded, self-adjoint, linear operator on a Hilbert space $\mathcal{H}$. Let p denotes the polynomial defined by $p(z) = m^2 - z^2$ with $m > \|A\|$. Then, if $B_r(x)$ denotes the open ball centered at x with radius r,*

(i) *the inequality $\|p(A)\| \leq m^2 - r^2$, with $m > r$, holds if and only if $B_r(0) \cap \sigma(A) = \emptyset$,*
(ii) *the inequality $\|p(A)\| > m^2 - r^2$, with $m > r$ holds if and only if $B_r(0) \cap \sigma(A) \neq \emptyset$.*

Proof: The two statement being equivalent, only (i) will be proved. Since A is self-adjoint, its spectrum is contained in the real line. Moreover, if p is any polynomial, $\sigma(p(A)) = p(\sigma(A))$. It follows that $p(\lambda) \geq m^2 - \|A\|^2 > 0$ for $\lambda \in \sigma(A)$.

Now $B_r(0) \cap \sigma(A) = \emptyset$ if and only if all $\lambda \in \sigma(A)$ satisfies $|\lambda| \geq r$, or, equivalently, $m^2 - \lambda^2 \leq m^2 - r^2$. This is equivalent to $\|p(A)\| = \sup_{\lambda\in\sigma(A)}\{m^2 - \lambda^2\} \leq m^2 - r^2$. □

2.2. **The Fell Topology.** In a seminal paper [11] (see also [8]), Fell introduced a topology on the space of closed subsets of a topological space X that he called the *Hausdorff Topology* and that is also called the *hit and miss* topology elsewhere [14, 16]. Whenever X is a metric space, the Hausdorff topology is induced by the Hausdorff metric [14, 8].

Let $\mathcal{C}(X)$ denote the set of all closed sets of X. For $K \subseteq X$ compact and for $\mathcal{F}$ a finite family of open sets of X let

$$\mathcal{U}(K, \mathcal{F}) := \{F \in \mathcal{C}(X)\,;\, K \cap F = \emptyset,\ F \cap O \neq \emptyset \text{ for all } O \in \mathcal{F}\}\,.$$

Then $W \subset \mathcal{C}(X)$ is open if it is a union of certain $\mathcal{U}(K, \mathcal{F})$'s. Fell established that $\mathcal{C}(X)$ is compact. But it might not be Hausdorff (separated). However, if, in addition, X is locally compact, then $\mathcal{C}(X)$ is compact and Hausdorff. The first example of application is the following

Proposition 1. *Let $X = (-L, R] \subset \mathbb{R}$ for some $0 \leq L \leq \infty$. Then the map $F \in \mathcal{C}(X) \mapsto \max\{F\} \in \mathbb{R}$ is Fell-continuous.*

Proof: Let $F_0 \in \mathcal{C}(X)$ with maximum λ. Since F_0 is closed, $\lambda \in F_0$. If $\lambda < R$, then for any $0 < \epsilon < R - \lambda$ let $K_\epsilon = [\lambda + \epsilon, R]$ and let $O_\epsilon = (\lambda - \epsilon, R]$. Then K_ϵ is compact in X and O_ϵ is open. If $F \in \mathcal{U}(K_\epsilon, \{O_\epsilon\})$, it follows that $F \cap K_\epsilon = \emptyset$ which implies $\max F < \lambda + \epsilon$, because F is closed. Moreover, since $F \cap O_\epsilon \neq \emptyset$, it follows that $\lambda - \epsilon < \max F$. Consequently, $F \in \mathcal{U}(K_\epsilon, \{O_\epsilon\})$ implies $|\max F - \max F_0| < \epsilon$, proving the continuity at F_0. If now $\lambda = R$, the same argument works with $K = \emptyset$. □

Proposition 2. *Let X, Y be locally compact spaces and let $f : X \to Y$ be a continuous proper function. Then, the map $\widehat{f} : F \in \mathcal{C}(X) \mapsto f(F) \in \mathcal{C}(Y)$ is continuous.*

Proof: A continuous function is proper whenever the inverse image of any compact space is compact. In addition a function is closed if it sends closed sets into closed sets. The Closed Map Lemma [7, 15] asserts that any continuous proper function between locally compact Hausdorff spaces is also closed. Therefore, the function $\widehat{f}$ is well defined. Let now $K \subset Y$ be compact and let $\mathcal{F}$ be a finite family of open subsets of Y. Hence $\mathcal{U}(K, \mathcal{F})$ is an open set in $\mathcal{C}(Y)$. Let $W = \mathcal{U}(f^{-1}(K), f^{-1}(\mathcal{F}))$. Since f is continuous, $f^{-1}(O)$ is open for any $O \in \mathcal{F}$. Since f is proper, it follows that $f^{-1}(K)$ is compact. Hence W is an open set in $\mathcal{C}(X)$. Then $F \in W$ means $F \cap f^{-1}(K) = \emptyset$ while $F \cap f^{-1}(O) \neq \emptyset$ for all $O \in \mathcal{F}$. This impies that $f(F) \cap K = \emptyset$. For indeed otherwise, there would be $x \in F$ with $f(x) \in K$, implying that $x \in f^{-1}(K)$ a contradiction. In addition, if $O \in \mathcal{F}$ and if $x \in F \cap f^{-1}(O) \neq \emptyset$ then $f(x) \in f(F) \cap O$ showing that $f(F) \cap O \neq \emptyset$. Hence $\widehat{f}(F) \in \mathcal{U}(K, \mathcal{F})$, which implies $W \subset f^{-1}(\mathcal{U}(K, \mathcal{F}))$, namely $\widehat{f}$ is continuous. □

If (X, d) is a complete metric space it turns out that this topology on $\mathcal{C}(X)$ coincides with the topology induced by the *Hausdorff distance* [14, 8]. The Hausdorff distance d_H is defined as follows: if $x \in X$ and $A \subset X$, then $\operatorname{dist}(x, A) = \inf\{d(x, y)\,;\, y \in A\}$. Given two subsets $A, B \subset X$, $\delta(A, B)$ is defined by $\delta(A, B) = \max_{x \in A} \operatorname{dist}(x, B)$. Then the Hausdorff distance of the two sets A, B is simply $d_H(A, B) = \max\{\delta(A, B), \delta(B, A)\}$. In general d_H is only a pseudo-metric on the set of all subsets of X. However if restricted to the set $\mathcal{C}(X)$ it becomes a metric defining the Fell topology.

2.3. **Proof of Theorem 1.** Let $A = (A_t)_{t \in T}$ be a p2-continuous field of self-adjoint operators (see Definition 1). Let $t_0 \in T$. By continuity of the norm $\|A_t\|$ with respect to t, for any $m > \|A_{t_0}\|$ there is an open neighborhood U_0 of t_0 such that if $t \in U_0$ then $\|A_t\| < m$.

Lemma 2. *If $A = (A_t)_{t \in T}$ is a p2-continuous field of self-adjoint, bounded operators then the map $t \in T \mapsto \sigma(A_t) \subset \mathbb{R}$ is continuous in the Fell topology.*

Proof: (i) Let $K \subset \mathbb{R}$ be compact and $\mathcal{F}$ be a finite set of open subsets of $\mathbb{R}$ chosen so that $\sigma(A_{t_0}) \in \mathcal{U}(K, \mathcal{F})$. Since K and $\sigma(A_{t_0})$ are closed and since $K \cap \sigma(A_{t_0}) = \emptyset$ there exists for any given $x \in K$ a $r(x) > 0$ so that $B_{r(x)}(x) \cap \sigma(A_{t_0}) = \emptyset$. The family of (smaller) open balls $\{B_{r(x)/2}(x)\,;\, x \in K\}$ covers K. By compactness of K, there is a finite set $\{x_1, \cdots, x_l\} \subset K$ such that

$$K \subset \bigcup_{k=1}^{l} B_{r_k/2}(x_k)\,, \qquad B_{r_k}(x_k) \cap \sigma(A_{t_0}) = \emptyset\,, \qquad r_k := r(x_k),\;\; k \in \mathbb{N}\,.$$

Let now m be chosen so that $2\sup_{t\in U_0} \|A_t\| + \sup_{x\in K} |x| < m$. Using Lemma 1, the condition $B_{r_k}(x_k) \cap \sigma(A_{t_0}) = \emptyset$ is equivalent to $\|m^2 - (A_{t_0} - x_k)^2\| < m^2 - r_k^2$. The p2-continuity implies that there is an open neighborhood U_k of t_0 such that for $t \in U_k$ then $\|m^2 - (A_t - x_k)^2\| < m^2 - r_k^2/4$. If now $U = \bigcap_{k=1}^{l} U_k$ it follows from the previous bounds and from the Lemma 1, that, for all $k \in \{1, 2, \cdots, l\}$, $B_{r_k/2}(x_k) \cap \sigma(A_t) = \emptyset$ for $t \in U$ implying $K \cap \sigma(A_t) = \emptyset$.

(ii) Similarly, let $O \in \mathcal{F}$. Since $O \cap \sigma(A_{t_0}) \neq \emptyset$, it follows that for any $x \in O \cap \sigma(A_{t_0})$ there is an $r(x) = r$ such that $B_r(x) \subset O$ (since O is open in $\mathbb{R}$). Since $x \in \sigma(A_{t_0})$, then $|x| \leq \|A_{t_0}\|$ so that $\|A_{t_0} - x\| \leq 2\|A_{t_0}\| < m$. Consequently, since $B_{r/2}(x) \subset O$, the Lemma 1 implies that $\|m^2 - (A_{t_0} - x)^2\| > m^2 - r^2/4$. Using the p2-continuity, there is an open neighborhood V_O of t_0 in T such that for $t \in V_O$ the inequality $\|m^2 - (A_t - x)^2\| > m^2 - r^2$ holds. This leads to $O \cap \sigma(A_t) \supset B_r(x) \cap \sigma(A_t) \neq \emptyset$ for $t \in V_O$. As the family $\mathcal{F}$ is finite, the intersection $V = \bigcap_{O\in\mathcal{F}} V_O$ is open and contains t_0 and, using the part (i), $V \cap U$ are open neighborhoods of t_0 as well. Consequently, for $t \in U \cap V$ the spectrum of A_t satisfies $\sigma(A_t) \in \mathcal{U}(K, \mathcal{F})$. □

Lemma 3. *Let $A = (A_t)_{t\in T}$ be a field of bounded self-adjoint operators. Let $f : \mathbb{R} \to \mathbb{C}$ be continuous. If the spectrum map $t \in T \mapsto \sigma(A_t)$ is Fell-continuous, then the norm-map $t \in T \mapsto \|f(A_t)\| \in \mathbb{R}_+$ is continuous.*

Remark 1. In particular, A is p2-continuous. □

Proof: As A_t is self-adjoint and bounded, $\sigma(A_t)$ is compact in $\mathbb{R}$ and $f(\sigma(A_t)) = \sigma(f(A_t)) \subset \mathbb{C}$. Moreover, $f(A_t)$ is a normal operator, so that the spectral theorem applies. The norm-map can be seen as the composition of the following continuous maps

$$t \mapsto \sigma(A_t) \stackrel{\widehat{f}}{\mapsto} \sigma(f(A_t)) \stackrel{|\cdot|}{\mapsto} |\sigma(f(A_t))| \stackrel{\max}{\mapsto} \|f(A_t)\|\,.$$

In this formula, the map $|\cdot|$ is nothing but $\widehat{g}$ if $g : z \in \mathbb{C} \mapsto |z| \in [0, \infty)$. The leftmost map is continuous by assumption. Thanks to Proposition 2, the second (with $X, Y = \mathbb{R}$ or $X, Y = \mathbb{C}$) and third maps on the left are continuous. At last Proposition 1 implies that the rightmost map is continuous. □

Definition 4. *Let $F \subset \mathbb{R}$ be closed. Any bounded connected component of its complement is called a gap of F. In particular, a gap is an open interval (a, b) with $a, b \in F$, $-\infty < a < b < +\infty$ and $(a, b) \cap F = \emptyset$.*

Definition 5. *Let T be a topological space and let $t \in T \mapsto F_t \in \mathcal{C}(\mathbb{R})$ be a Fell-continuous map. Given $t_0 \in T$, a closed gap of F_{t_0} is a point $c \in F_{t_0}$ such that*
(i) for all $\epsilon > 0$, the $(c - \epsilon, c) \cap F_{t_0} \neq \emptyset$ and $(c, c + \epsilon) \cap F_{t_0} \neq \emptyset$,
(ii) there is a non-empty set $U \subset T$ with $t_0 \in \overline{U} \setminus U$, and two functions $a : t \in U \mapsto a_t \in \mathbb{R}$ and $b : t \in U \mapsto b_t \in \mathbb{R}$ such that $a_t < b_t$, $\lim_{t\to t_0} a_t = \lim_{t\to t_0} b_t = c$ and the interval (a_t, b_t) is a gap of F_t.
A closed gap c of F_{t_0} is called isolated if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subset F_{t_0}$.

Lemma 4. *Let the map $t \in T \mapsto F_t \in \mathcal{C}(\mathbb{R})$ be Fell-continuous. Then its gap edges are continuous functions on T. In addition, if there is $R > 0$ and an open set $U \subset T$ such that for $t \in T$ the set F_t is contained in $[-R, R]$ then* $\max F_t$ *and* $\min F_t$ *are continuous in U.*

Proof: (i) Let $t_0 \in T$ and let (a, b) be a gap of F_{t_0}. Then, given $0 < \epsilon < (b - a)/2$, let $K_\epsilon = [a + \epsilon, b - \epsilon]$ be compact inside the gap. Similarly let $O_{\epsilon,+} = (b - \epsilon, b + \epsilon)$, $O_{\epsilon,-} = (a - \epsilon, a + \epsilon)$ and $\mathcal{F}_\epsilon = \{O_{\epsilon,+}, O_{\epsilon,-}\}$. Then, if $F \in \mathcal{U}(K_\epsilon, \mathcal{F}_\epsilon)$, let $\lambda_- = \max F \cap (-\infty, a + \epsilon)$ and $\lambda_+ = \min F \cap (b - \epsilon, +\infty)$. It follows that $|\lambda_- - a| < \epsilon$, $|\lambda_+ - b| < \epsilon$, and that (λ_-, λ_+) is a gap of F. Thanks to the continuity of the map $t \mapsto F_t$, there is an open neighborhood U of t_0 in T such that if $t \in U$ then $F_t \in \mathcal{U}(K_\epsilon, \mathcal{F}_\epsilon)$, proving the claim.

(ii) Let $t_0 \in T$ and let $c \in F_{t_0}$ be a closed gap. Then, given $\epsilon > 0$, let $\mathcal{F}_{c,\epsilon} = \{(c-\epsilon, c), (c, c+\epsilon)\}$. Hence $F_{t_0} \in \mathcal{U}(\emptyset, \mathcal{F}_{c,\epsilon})$. By Fell-continuity, it follows that there is an open neighborhood V of t_0 such that for $t \in V$, then $F_t \cap (c-\epsilon, c) \neq \emptyset$ and $F_t \cap (c, c+\epsilon) \neq \emptyset$. Let U be the set of $t \in V \setminus \{t_0\}$ such that F_t admits a gap (a_t, b_t) such that $c - \epsilon < a_t < b_t < c + \epsilon$. Since c is a closed gap, it follows that U is not empty and that $t_0 \in \overline{U} \setminus U$. Using the part (i) of the present proof, it follows that the two functions $a : t \in U \mapsto a_t \in \mathbb{R}$ and $b : t \in U \mapsto b_t \in \mathbb{R}$ are continuous in U. More generally, if $s \in \overline{U} \setminus U$, it follows that either F_s has no gap inside the closed interval $[c - \epsilon, c + \epsilon]$ or if it has a gap there, it must be one of the following forms: $(c - \epsilon, b_s)$ with $c - \epsilon < b_s < c + \epsilon$, $(a_s, c + \epsilon)$ with $c - \epsilon < a_s < c + \epsilon$ or $(c - \epsilon, c + \epsilon)$. If, in addition, $s \in V$, then F_s must have a non empty intersection with both $(c - \epsilon, c)$ and $(c, c + \epsilon)$, so that in the first case $b_s < c$, in the second case $a_s > c$ and the third case is excluded.

(iii) The continuity of $\max F_t$ and $\min F_t$ is a corollary of Proposition 1. □

Lemma 5. *Let $F : t \in T \mapsto F_t \in \mathcal{C}(\mathbb{R})$ be a map such that the gap edges, the maximum and the minimum of F_t are continuous functions of t. Then F is Fell-continuous.*

Proof: Let $t_0 \in T$. Let K a compact subset of $\mathbb{R}$, $\mathcal{F}$ a finite family of open subsets of $\mathbb{R}$, be chosen such that $F_{t_0} \in \mathcal{U}(K, \mathcal{F})$.

(i) The condition $F_{t_0} \cap K = \emptyset$ implies $\operatorname{dist}(K, F_{t_0}) = r > 0$. Therefore the open balls $B_r(x)$ centered at some point $x \in K$ do not intersect F_{t_0}. Moreover the smaller balls $B_{r/2}(x)$ cover K whenever $x \in K$. Therefore there is a finite subset $\{x_1, \cdots, x_l\} \subset K$ such that K is covered by the balls $B_{r/2}(x_k)$. Hence there is a gap of F_{t_0} of the form (a_k, b_k) with $a_k \leq x_k - r < x_k + r \leq b_k$: in this inequality a or b are allowed to be $-\infty$ or $+\infty$ respectively. Since the gap edges are continuous with respect to the variable t, there is U_k an open neighborhood of t_0 in T such that for $t \in U_k$, the closed set F_t admits a gap $(a_k(t), b_k(t))$ such that $|a_k - a_k(t)| < r/2$ and $|b_k - b_k(t)| < r/2$. Therefore $B_{r/2}(x_k) \cap F_t = \emptyset$. Taking $U = \bigcap_{k=1}^{l} U_k$ as a new open set containing t_0, it follows that $t \in U$ implies $F_t \cap K = \emptyset$.

(ii) Let now $O \in \mathcal{F}$. The condition $O \cap F_{t_0} \neq \emptyset$ implies that if $x \in O \cap F_{t_0}$ there is $r > 0$ such that $B_r(x) \subset O$. Thus there is V_O open containing t_0 such that for $t \in V_O$ then $B_r(x) \cap F_t \neq \emptyset$. If not, then for any $V \ni t_0$ open, there would be $t_V \in V$ such that $B_r(x) \cap F_{t_V} = \emptyset$. Therefore F_{t_V} would have a gap (a_V, b_V) such that $a_V \leq x - r < x + r \leq b_V$. Since the gap edges are continuous and since the net t_V converge to t_0, it follows that F_{t_0} would have a gap containing $B_r(x)$, which is impossible since $x \in F_{t_0}$. Since $\mathcal{F}$ is finite, it follows that $V = \bigcap_{O \in \mathcal{F}} V_O$ is an open set containing t_0. Therefore $t \in U \cap V$ implies $F_t \in \mathcal{U}(K, \mathcal{F})$. □

2.4. **Proof of Theorem 2.** The main remark is the following: let $z = x + \imath y$ with $y \neq 0$. Then, if A is a self-adjoint linear operator on some Hilbert space

$$\left\| \frac{1}{A-z} \right\| = \left\| \frac{1}{(A-x)^2 + y^2} \right\|^{1/2} .$$

If the spectrum of A admits a gap (a, b) and if $(a+b)/2 < x < b$, it follows from the spectral theorem that

$$\left\| \frac{1}{A-z} \right\| = \left(\frac{1}{(b-x)^2 + y^2} \right)^{1/2} .$$

Similarly if $a < x < (a+b)/2$ the same argument links the norm of the resolvent to a.

If now $(A_t)_{t\in T}$ is an R-continuous field of self-adjoint operators, the continuity of the norm of the resolvent implies that the gap edges are continuous, so that, thanks to Lemma 5, the spectrum is Fell-continuous as a function of t. Conversely, if the spectrum is Fell-continuous, Lemma 4 implies that the gap edges are continuous, so that, the previous identity implies the R-continuity of the field $(A_t)_{t\in T}$.

3. Hölder Continuity

The previous Section shows that a simple criterion permits to get continuity of the spectrum of a field of self-adjoint operators. However, in many cases, continuity is not sufficient because it is not quantitative enough. Namely if approximating a self-adjoint operator by a family, it is often necessary to control the speed of convergence. This can be done, for instance, if the topological space T is equipped with a metric. In some cases the topological space T can be very irregular, like a Cantor set or a fractal set. So a metric is really the minimal structure that can be considered. In a metric space (X, d), which will always be assumed to be complete, the natural set of function are the Lipschitz continuous functions. But it might be convenient to consider Hölder continuous functions instead.

3.1. **Metrics: a Reminder.** A metric space (X, d) is a set X equipped with a *metric* d, namely a map $d : X \times X \to [0, \infty)$ such that for any $x, y, z \in X$, (i) $d(x, y) = 0$ if and only if $x = y$, (ii) $d(x, y) = d(y, x)$, (iii) the *triangle inequality* holds $d(x, y) \leq d(x, z) + d(z, y)$. A metric is called an *ultrametric* whenever the triangle inequality is replaced by $d(x, y) \leq \max\{d(x, z), d(z, y)\}$. A subset U is called open whenever for any $x \in U$, there is $r > 0$ such that the open ball $B_r(x) = \{y \in X\,;\, d(x, y) < r\}$ is contained in U. A sequence $(x_n)_{n\in\mathbb{N}}$ of points in X is called Cauchy whenever for any $\epsilon > 0$, there is $N \in \mathbb{N}$ such that if $n, m \geq N$ then $d(x_n, x_m) < \epsilon$. Then (X, d) is complete if any Cauchy sequence converges.

If $0 < \alpha$ let d^α denotes the function $(x, y) \in X \times X \mapsto d(x, y)^\alpha$. Then if $\alpha \leq 1$, the inequality $(a + b)^\alpha \leq a^\alpha + b^\alpha$, whenever a, b are non negative, implies that d^α is a new metric. It defines the same topology, since the balls are the same. If d is an ultrametric, and if $\Phi : [0, \infty) \to [0, \infty)$ is monotone increasing such that $\Phi(0) = 0$, then the function $d_\Phi = \Phi \circ d$ is also an ultrametric defining the same family of balls.

Given $\epsilon > 0$ an ϵ-path γ joining x to y, denoted by $\gamma : x \stackrel{\epsilon}{\to} y$, is an ordered sequence $(x_0 = x, x_1, \cdots, x_{n-1}, x_n = y)$ such that $d(x_{k-1}, x_k) < \epsilon$ for $1 \leq k \leq n$. The length of γ is $\ell(\gamma) = \sum_{k=1}^n d(x_{k-1}, x_k)$. The topological space X is connected if and only if, given any pair $x, y \in X$ and any $\epsilon > 0$, there is an ϵ-path joining them. The metric d is called a *length metric* whenever $d(x, y)$ coincides with the minimal length of paths (for any $\epsilon > 0$) joining x to y [12].

Given (X, d_X) and (Y, d_Y) two metric spaces, and given $\alpha > 0$, a function $f : X \to Y$ is α-Hölder if there is $C > 0$ such that $d_Y(f(x), f(x')) \leq C d_X(x, x')^\alpha$ for any pair of points x, x' in X. It follows that α-Hölder functions are continuous. The Hölder constant is defined as

$$\mathrm{Hol}^\alpha(f) = \sup_{x \neq x'} \frac{d_Y(f(x), f(x'))}{d_X(x, x')^\alpha}$$

More generally this definition can be more local as follows [12]: if $r > 0$

$$\mathrm{Hol}^\alpha_r(f)(x) = \sup_{x'; 0 < d_X(x,x') < r} \frac{d_Y(f(x), f(x'))}{d_X(x, x')^\alpha}\,, \qquad \mathrm{Hol}^\alpha_r(f) = \sup_{x \in X} \mathrm{Hol}^\alpha_r(f)(x) \leq \mathrm{Hol}^\alpha(f)\,.$$

Clearly this quantity is a non decreasing function of r, so that the limit $r \to 0$ exists and is called the α-*dilation* of f at x:

$$\mathrm{dil}^\alpha(f)(x) = \lim_{r \downarrow 0} \mathrm{Hol}^\alpha_r(f)(x)\,, \qquad \mathrm{dil}^\alpha(f) = \sup_{x \in X} \mathrm{dil}^\alpha(f)(x) \leq \mathrm{Hol}^\alpha(f)\,.$$

For $\alpha = 1$ Hölder continuous functions are called *Lipschitz*, and Hol^1 is denoted by Lip. If d_X is a length metric it follows that $\mathrm{Lip}(f) = \mathrm{dil}(f)$ [12]. If d_X is a length metric and $\alpha > 1$ then $\mathrm{Hol}^\alpha(f) < \infty$ if and only if f is a constant function. However, there are spaces for which some non constant functions are α-Hölder continuous for some $\alpha > 1$. In particular if X is a Cantor set and d_X an ultrametric the characteristic function of any clopen set is α-Hölder for any $\alpha > 0$.

In this Section, the topology of T is induced by a metric d for which it is complete. The real line or the complex plane, or any of their subsets, will be endowed with the usual Euclidean metric. If $t_0 \in T$ the continuity of $\|A_t\|$ implies that there is an open subset U_0 containing t_0 such that $\sup_{t \in U_0} \|A_t\| < \infty$. Replacing T by U_0 if necessary, there is no loss of generality in assuming that $\sup_{t \in T} \|A_t\| = m < \infty$. Thanks to the Definition 3, given any $M > 0$ the following constant is finite

$$C_M = \sup\{\mathrm{Hol}^\alpha(\Phi_p)\,;\, \|p\|_1 \leq M\}.$$

3.2. **Proof of Theorem 3.** Equipped with the Hausdorff metric d_H the space of closed subsets $\mathcal{C}(X)$ is a metric space [8]. Note that the distance between two closed sets might be infinite.

Proposition 3. *Let (X, d_X) and (Y, d_Y) be two metric spaces and $f : X \to Y$ be a α-Hölder continuous proper function. Then, the map $\widehat{f} : F \in \mathcal{C}(X) \mapsto f(F) \in \mathcal{C}(Y)$ is α-Hölder continuous.*

Proof: As mentioned in Proposition 2 the function $\widehat{f}$ is well defined since f is continuous and proper. Let $K, L \in \mathcal{C}(X)$. A short computation leads to

$$\mathrm{dist}(f(x), f(L)) = \inf\{d_X(f(x), f(y))\,;\, y \in L\} \leq C \,\inf\{d_Y(x, y)^\alpha\,;\, y \in L\} = C\,\mathrm{dist}(x, L)^\alpha.$$

Maximizing over x and exchanging the roles of K and L, leads to the claim. □

Lemma 6. *Let $A = (A_t)_{t \in T}$ be p2-α-Hölder continuous field of self-adjoint, bounded operators such that $m := \sup_{t \in T} \|A_t\| < \infty$. Then the spectrum $\sigma(A_t)$ is $\alpha/2$-Hölder continuous with Hölder constant less than $\sqrt{C_{4m^2+2}}$.*

Proof: Let $s, t \in T$. According to the definition of the Hausdorff metric it suffices to show $\mathrm{dist}(\lambda, \sigma(A_t)) \leq \sqrt{C_{4m^2+2}}\; d(s, t)^{\frac{\alpha}{2}}$ for all $\lambda \in \sigma(A_s)$. Without loss of generality suppose that $\lambda \in \sigma(A_s) \backslash \sigma(A_t)$. Since $\lambda \in \sigma(A_s)$, it follows that $\|4m^2 - (A_s - \lambda)^2\| = 4m^2$ and $(A_t - \lambda)^2 \leq 4m^2$.

The polynomial $p(z) = 4m^2-(z-\lambda)^2$ has a norm $\|p\|_1 = 1+2|\lambda|+4m^2-\lambda^2 = 4m^2+2-(1-|\lambda|)^2 \leq 4m^2+2$. Since $\lambda \notin \sigma(A_t)$ the norm $\|p(A_t)\|$ is exactly $4m^2 - \mathrm{dist}(\lambda,\sigma(A_t))^2$. Consequently,

$$\mathrm{dist}(\lambda,\sigma(A_t))^2 = \big|\|p(A_t)\| - \|p(A_s)\|\big| \leq C_{4m^2+2}\, d(s,t)^\alpha \,.$$

□

For a self-adjoint, bounded operator A the maximum $\max\{|\sigma(A)|\}$ is exactly $\|A\|$. Let $A = (A_t)_{t\in T}$ be a field of self-adjoint, bounded operators such that $m := \sup_{t\in T}\|A_t\| < \infty$. Then, $\sigma(A_t)$ is a subset of the compact subset $[-m,m]$ for all $t \in T$.

Lemma 7. *Let $A = (A_t)_{t\in T}$ be a field of self-adjoint, bounded operators such that $m := \sup_{t\in T}\|A_t\| < \infty$. If the spectrum $\sigma(A_t)$ is α-Hölder continuous with Hölder constant C then A is a p2-α-Hölder continuous field.*

Proof: Let $K := [-m,m] \subseteq \mathbb{R}$ be the compact subset such that $\sigma(A_t) \subset K$ for all $t \in T$. Any polynomial $p(z) = p_0 + p_1 z + p_2 z^2$ restricted to K is Lipschitz continuous, i.e. $\alpha = 1$, with Lipschitz constant $2\,m\,|p_1| + |p_2|$. Recall that the norm-map can be seen as the composition of the following maps

$$t \mapsto \sigma(A_t) \overset{\widehat{p}}{\mapsto} \sigma(p(A_t)) \overset{|\cdot|}{\mapsto} |\sigma(p(A_t))| \overset{\max}{\mapsto} \|p(A_t)\| \,.$$

The absolute value $|\cdot| : [-m,m] \to [0,m]$ and the $\max : \mathcal{K}(\mathbb{R}) \to \mathbb{R}$ are both Lipschitz continuous with Lipschitz constant 1, so that

$$\big|\|p(A_t)\| - \|p(A_s)\|\big| \leq (2\,m\,|p_1| + |p_2|)\, d_H(\sigma(A_s),\sigma(A_t)) \leq (2\,m\,|p_1| + |p_2|)\, C\, d(s,t)^\alpha \,,$$

for all $s,t \in T$ by using Proposition 3. Hence the number $C_M = \sup\{\mathrm{Hol}^\alpha(\Phi_p)\,;\, \|p\|_1 \leq M\}$ is finite proving that A is a p2-α-Hölder continuous field. □

3.3. **Proof of Theorem 4.**

Lemma 8. *Let $A = (A_t)_{t\in T}$ be p2-α-Hölder continuous such that $m := \sup_{t\in T}\|A_t\| < \infty$. Then its spectrum edges are α-Hölder with Hölder constant less than C_{1+m}.*

Proof: If $a_t = \inf\sigma(A_t)$ and $b_t = \sup\sigma(A_t)$ then the largest of them $|a_t| \vee |b_t|$ coincides with $\|A_t\|$ while the smallest $c = |a_t| \wedge |b_t|$ satisfies $m - c = \|m - A_t\|$ leading immediately to the result. □

Lemma 9. *Let $A = (A_t)_{t\in T}$ be p2-α-Hölder continuous. Then the gap edges of an open gap $\mathfrak{g}$ are α-Hölder with α-dilation less than $3C_{(4m^2+2)}/|\mathfrak{g}|$ if $|\mathfrak{g}|$ denotes the gap width.*

Proof: Let $t_0 \in T$. Let $\mathfrak{g} = (a,b)$ be a gap of $\sigma(A_{t_0})$, so that $-\infty < a < b < +\infty$ and $|\mathfrak{g}| = b - a$. Let $\mathfrak{g}$ be subdivided into six intervals of equal length r and let c be the point located at 2/3 of $\mathfrak{g}$ close to b. That is $r = (b-a)/6$ and $c = a + 4r = b - 2r$. Thanks to Theorem 2, there is an open neighborhood U of t_0 in T, such that for $t \in U$, the spectrum of A_t has a gap (a_t, b_t) with $|a_t - a| < r$ and $|b_t - b| < r$. This implies (i) $b_t - a_t = b_t - b + b - a + a - a_t > 6r - 2r = 4r > 0$, (ii) $b_t - c = b_t - b + b - c$ so that $r < b_t - c < 3r$, (iii) $c - a_t = c - a + a - a_t > 4r - r = 3r$. Consequently c is closer to b_t than to a_t. Hence, if $t \in U$, the infimum of the spectrum of $(A_t - c)^2$ is exactly $(b_t - c)^2$. Since c belongs to the convex hull of $\sigma(A_t)$, it follows that $|c| \leq \|A_t\| \leq m$ so that $(b_t - c)^2 \leq (A_t - c)^2 \leq 4m^2$. Hence $4m^2 - (b_t - c)^2 = \|4m^2 - (A_t - c)^2\|$ whenever $t \in U$. The polynomial $p(z) = 4m^2 - (z - c)^2$ has a norm $\|p\|_1 = 1 + 2|c| + 4m^2 - c^2 = 2 + 4m^2 - (1 - |c|)^2 \leq 4m^2 + 2$. Hence, if $s,t \in U$, this gives

$|\|p(A_t)\| - \|p(A_s)\|| = |(b_t-c)^2 - (b_s-c)^2| = |b_t - b_s||b_t+b_s-2c|$. Thanks to the inequality (i) above, this gives $|b_t+b_s-2c| > 2r$. Hence

$$s,t \in U \qquad \Rightarrow \qquad |b_s - b_t| \leq \frac{3C_{(4m^2+2)}}{(b-a)}\; d(s,t)^\alpha \,.$$

Since U is unknown but can be taken arbitrarily small, this estimate gives an upper bound on the α-dilation at t_0 of the gap edge independently of which gap edges is considered. Changing c into $a+2r = b-4r$ will replace b_t by a_t, so that the same argument leads to the same estimate for the dilation of the lower gap edge. □

3.4. **Proof of Theorem 5.**

Lemma 10. *Let $A=(A_t)_{t\in T}$ be p2-α-Hölder continuous. Then if c is a closed isolated spectral gap of A_{t_0}, then there is $m>0$ such that any gap (a_t,b_t) of A_t closing at c satisfies*

$$b_t - a_t \leq 2\;\sqrt{C_{(4m^2+2)}}\; d(t,t_0)^{\alpha/2} \,. \tag{2}$$

Proof: In what follows, let F_t be the spectrum of A_t. For simplicity let F_0 denotes the spectrum of A_{t_0}. Let $c \in F_{t_0}$ be an isolated closed gap. Namely, thanks to Definition 5, and since c is isolated, the following assumptions hold
(i) there is $\delta > 0$, such that $(c-\delta, c+\delta) \subset F_{t_0}$,
(ii) there is a non-empty set U such that $t_0 \in \overline{U}\setminus U$ and for all $t\in U$, the set F_t admits a gap (a_t,b_t) such that $\lim_{t\to t_0} a_t = c = \lim_{t\to t_0} b_t$.
It follows from Theorem 1, that the gap edges a_t and b_t are continuous function of $t\in U$.

By definition, there is an open neighborhood $V\subset U$ of t_0 in T such that, whenever $t\in V\cap U$, then A_t admits a spectral gap (a_t,b_t) with $a_t<b_t$ such that $\max\{|a_t-c|,|a_t-c|\} < \delta$. It follows that $c-\delta < a_t < b_t < c+\delta$. Choosing $\lambda = (a_t+b_t)/2$, it follows that $\lambda\in F_0$ and the distance of λ to F_t is exactly $(b_t-a_t)/2$. Consequently

$$\|4m^2 - (A_{t_0}-\lambda)^2\| = 4m^2\,, \qquad\qquad \|4m^2-(A_t-\lambda)^2\| = 4m^2 - \frac{(b_t-a_t)^2}{4}\,.$$

Since the polynomial $p(z) = 4m^2-(z-\lambda)^2$ satisfies $\|p\|_1 \leq 4m^2+2$ (see the proof of Lemma 6), it follows that

$$(b_t-a_t)^2 \;\leq 4C_{(4m^2+2)}\; d(t,t_0)^\alpha\,, \qquad t\in V.$$

□

The gap closing condition is observed in many models. First, in the Almost-Mathieu model H_α (see eq. 1), the spectrum is a finite union of interval when $\alpha\in\mathbb{Q}$ since this is a periodic Hamiltonian. So any closed gap is isolated. Since it is p2-Lipschitz thanks to [6], the gap width can be at most 1/2-Hölder. A semi-classical calculation [17] confirms this prediction.
Another situation where gaps are closing is provided by a small perturbation of the Laplacian on $\mathbb{Z}$. Namely, for $\lambda\geq 0$ let H_λ be defined on $\ell^2(\mathbb{Z})$ by

$$H_\lambda\psi(n) = \psi(n+1)+\psi(n-1)+\lambda V(n)\psi(n)\,,$$

where $V(n)$ takes on finitely many values. For $\lambda=0$ the spectrum of H_λ is the interval $[-2,+2]$. The prediction provided by the *Gap Labeling Theorem* [4, 5], shows that for certain potentials V, gap may open as λ increases from $\lambda=0$. Explicit calculations made on several examples,

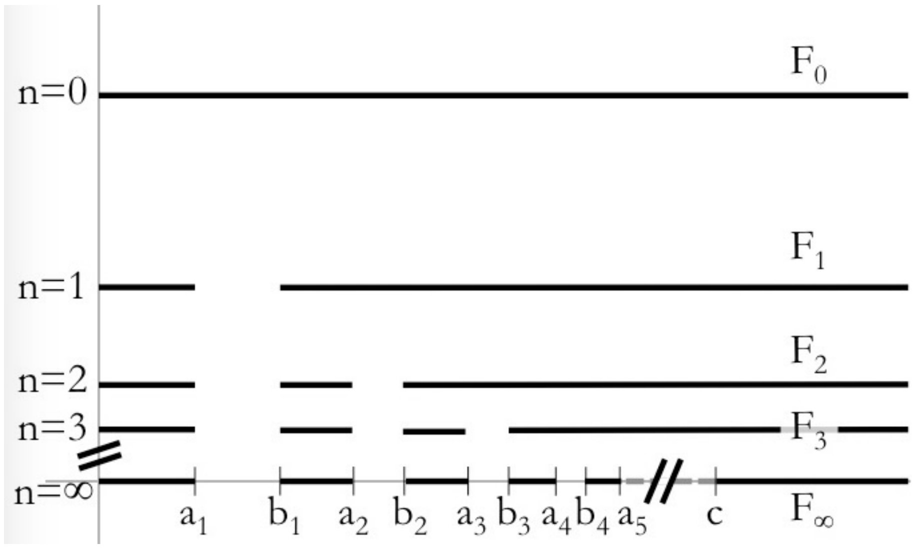

FIGURE 1. Example of a slow closing gap

such as the Fibonacci sequence [19], Thue-Morse sequence [2], the period doubling sequence [3], confirms that the gap opening cannot be faster than $O(\lambda^{1/2})$.

If the condition that c is isolated is relaxed, the following example (see Fig. 1) shows that the gap closing cannot be bounded in general.

Example 1 (A counter example). Let $(a_n, b_n)_{n\in\mathbb{N}}$ be a double sequence of real numbers such that $0 < a_n < b_n < a_{n+1} < c$ and $\sup_{n\in\mathbb{N}} a_n = c$ (see Fig. 1). Let $F_0 = [0, m]$ with $m > c$. Then the sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of F_0 is defined inductively by $F_{n+1} = F_n \setminus (a_{n+1}, b_{n+1})$. Then F_n can be seen as the spectrum of a bounded self-adjoint operator A_n. Clearly F_n converges in the Fell topology to $F_\infty = \bigcap_{n\in\mathbb{N}} F_n$ and c is a closed gap of F_∞. Here $0 \le A_n \le m$ for all n. If p is a polynomial of degree two with real coefficients, then $p(A_n)$ admits $\hat{p}(F_n)$ as its spectrum. Moreover, $p(z)$ can be written as $r + q(z-h)^2$ where h denotes its critical point. Depending on the sign of the coefficients q, r the maximum can be (i) either $p(0)$, (ii) $p(m)$, (iii) r if $h \in F_\infty$, (iv) $p(a_l)$ or $p(b_l)$ whenever $a_l < h < b_l$ and $n \ge l$. Hence $\|p(A_n)\|$ is eventually constant as $n \to \infty$. It follows that if $T = \mathbb{N} \cup \{\infty\}$ is endowed with any metric such that $d(n,\infty) \to 0$ as $n \to \infty$, the field A is p2-α-Hölder for any $\alpha > 0$. Let now consider the case where $|b_{n+1} - a_{n+1}| < |b_n - a_n|$ for all n's and let the ultra-metric d defined on $\mathbb{N} \cup \{\infty\}$ as

$$d(n,m) = d(m,n) = e^{-\kappa^m}\,, \qquad\qquad \text{if } m < n\,,$$

and $d(n,n) = 0$. Then $d_H(F_n, F_\infty) = |b_{n+1} - a_{n+1}|/2$. Let the sequences $(a_n), (b_n)$ be chosen such that there is $C > 0$ such that

$$|b_{n+1} - a_{n+1}| = 2d_H(F_n, F_\infty) = Cd(n,\infty)^{\alpha/2}$$

It follows that $|b_n - a_n| = Ce^{-(\alpha/2)\kappa^{n-1}} = Cd(n,\infty)^{\alpha/(2\kappa)}$. Hence the gaps width are Hölder in n but with an exponent $\alpha/(2\kappa) < \alpha/2$. □

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