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Semiclassical methods in solid state physics : two examples

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Résumé. — Ce travail est une revue de deux problèmes motivés par l'étude des modèles bidimensionnels pour la supraconductivité à haute température critique. La première partie concerne l'étude du spectre d'énergie pour des électrons de Bloch bidimensionnels soumis à un champ magnétique uniforme. Une analyse semi-classique permet d'en comprendre les propriétés qualitatives et quantitatives. La deuxième partie est un plaidoyer pour l'utilisation des méthodes du "Chaos Quantique" dans l'étude des systèmes de fermions fortement corrélés. La distribution des écarts de niveaux d'un modèle $t - J$ en deux dimensions, en fournit une illustration.

Abstract. — We present here a review of two problems motivated by 2D models for high T_c superconductivity. The first part concerns the energy spectrum of 2D Bloch electrons in a uniform magnetic field. A semiclassical analysis provides a qualitative as well as a quantitative understanding of this spectrum. In the second part we make the case for the application of "Quantum Chaos" to strongly correlated fermion systems. It is illustrated by the level spacing distribution for the $t - J$ model in two dimensions.

I met R. Rammal for the first time at the Paris meeting on Statistical Mechanics in January 1983. Pierre Moussa introduced us to each other. Some similarity between Rammal's work on the Sierpinski lattice [75] and some one dimensional almost periodic models studied by Bessis, Moussa and myself [15] was the reason why he wanted to make my acquaintance.

During the short discussion we had together at this meeting, it became immediately clear that we had two points in common : we had both got our degree in a French university, and both worked hard to understand the structure of the Hofstadter spectrum [55].

On the first point, we shared the hardship of being outsiders in the French research institutions heavily dominated by the smart but arrogant "Grandes Ecoles" alumni.

On the second, we were both fascinated by the complexity and the importance of magnetic field effects in Solid State Physics.

Bloch electrons in a magnetic field, superconductor or normal metal networks, the quantum Hall effect, electronic properties of quasicrystals, high T_c superconductors and quantum chaos

have been the focus of our frequent and continuing discussions over the years since then. Not to mention the subjects of daily life, professional matters, politics and private problems as well.

Scientific ambition, a strong need for affection and a wounded pride secretly underlied his life. Rammal died on a Friday in the middle of a scientific discussion after years of heartbreak for his country falling apart, and years of acute physical pain, despite the hope that medical technology gave him for the better.

Jean Bellissard
Toulouse, July 1992

1. Bloch electrons in a magnetic field.

Electronic motion in a crystal subject to a magnetic field has been one of the main topics in Solid State Physics. A magnetic field breaks the time reversal symmetry and reveals details of the microscopic structure of a crystal. The Hall effect, the de Haas van Alphen effect, electronic diamagnetism in metals, the Meissner effect, flux quantization of vortices in superconductors and more recently the magnetoresistance oscillations in slightly disordered metals, and the quantum Hall effect, are examples of the complexity a magnetic field introduces in homogeneous materials.

The first study of such problems goes back to Landau in 1930 [58] for the motion of electrons in a metal, in the effective mass approximation. It was soon followed by the work of Peierls [72] for Bloch electrons. But it took until the fifties, with a paper by Luttinger [64] to start a systematic study of the problem. The main remark coming out of these works is that magnetic translation operators [94] are generating the effective Hamiltonian describing the electronic motion in the independent electron approximation. The case of two dimensional systems in a perpendicular uniform magnetic field became increasingly important after the development of electronic devices such as MOSFETS or heterojunctions [2]. It is nowadays possible to get metallic thin films 200 Å thick. Many of the afore mentioned effects are actually enhanced in two dimensions.

More recently, the discovery of high T_c superconductors in copper-oxides [12] led several experts to propose that the main mechanism leading to Cooper pairs, lies probably in the planes of the copper atoms, reducing the problem to two dimensions. Moreover, the flux phases approach reduces the problem to the case of independent charged particles in a lattice and a magnetic field. Even though this last approximation is questioned nowadays, it led many physicists to go back to the question of the 2D electronic lattice motion in a uniform magnetic field.

If we denote by U and V the two magnetic translations describing such a system, they are unitary operators fulfilling the following commutation rules :

$$UV = \mathrm{e}^{2i\pi\alpha} VU \ . \tag{1}$$

Here, $\alpha = \phi/\phi_0$ is the ratio between the magnetic flux ϕ through the unit cell and the flux quantum $\phi_0 = h/e$ (h is Planck's constant and e the electron charge).

To describe the electronic motion in a crystal, let us consider first the case for which $\alpha = 0$ corresponding to the absence of a magnetic field. Only those energy levels near the Fermi energy participate in conduction, so that we can restrict ourselves to a finite set of bands crossing the Fermi surface. For simplicity, let us restrict ourselves to the case of only one such band. Let $E(\mathbf{k})$ ($\mathbf{k} = (k_1, k_2)$ being the quasimomentum) be the corresponding band function. By Bloch theory, $E(\mathbf{k})$ is periodic with respect to the reciprocal lattice. Peierls then proposed

to replace $\mathbf{k} = (k_1, k_2)$ by the dimensionless operator $\mathbf{K} = (K_1, K_2)$, where

$$K_\mu = \frac{a_\mu}{\hbar}\left(i\hbar\frac{\partial}{\partial x_\mu} - eA_\mu\right) = \text{const}\,(P_\mu - eA_\mu) \ ,$$

in the expression of E to get the effective hamiltonian. Here $\mathbf{A} = (A_1, A_2)$ is the vector potential, a_μ ($\mu = 1, 2$) are the lattice spacings in each direction.

Remembering that e^{iK_1} and e^{iK_2} satisfy the commutation relation (1), the Peierls Hamiltonian can be written as a convergent power series in the operators U and V.

The rigorous justification of this Peierls substitution has been done in [13, 85] and leads to corrections to the Peierls Hamiltonian. But whatever the corrections, the effective Hamiltonian always admits a representation in the form :

$$H_{\text{eff}} = \sum_{(m_1,m_2)\in\mathbf{Z}^2} h_{m_1,m_2}(\alpha)U^{m_1}V^{m_2}e^{-i\pi\alpha m_1 m_2} \ , \tag{2}$$

where $h_{m_1,m_2}(\alpha)$'s are smooth functions of α. This series usually converges absolutely.

The simplest case consists in looking at a square lattice for which we ignore the higher order terms in (2). It gives rise to the so-called "Harper model" namely [49] :

$$H_{\text{Harper}} = U + U^{-1} + V + V^{-1} \tag{3}$$

The analogy between (1) and the canonical commutation relations (where $\hbar = h/2\pi$),

$$[Q, P] = i\hbar \ , \tag{4}$$

between the position Q and the momentum P operators, is realized if we replace U by e^{iP} , V by e^{iQ} and $2\pi\alpha$ by $\hbar$. Therefore α plays the role of an effective Planck constant in (1). The main difference is that $\hbar$ is a fixed parameter once and for all, whereas α is proportional to the magnetic field and can be varied like a tunable parameter.

We thus have realized a concrete system in which one can test the semiclassical methods. In a crystal with lattice spacing of order of 1Å, and a strong realistic magnetic field of order of 10 T we get α of the order of 10^{-5}, which is quite small and justifies a semiclassical approach. However if we build artificially a network with lattice spacing of the order of 1μm, in a magnetic field of few Gauss, we can get $\alpha \approx 1$. Such networks were indeed built by the Grenoble group [68, 69] by mean of superconductors. The de Gennes [34] and Alexander [1] theory for such networks permits to relate in a simple way the groundstate energy of the Harper Hamiltonian (3) to the normal metal-superconductor transition curve in the temperature-magnetic field phase space. In this problem however the electron charge e must be replaced by the Cooper pair charge $2e$, in ϕ_0.

1.1 ALGEBRAIC APPROACH. — Another approach to (1) consists in considering the case for which $\alpha = p/q$ is a rational number. Then U^q and V are commuting. In an irreducible representation we can always represent U and V by mean of $q \times q$ matrices $e^{ik_1}u$ and $e^{ik_2}v$ where u and v are unitaries such that :

$$u^q = v^q = \mathbf{I} \qquad uv = e^{2i\pi p/q}vu \ . \tag{5}$$

Substituting in the expression (2) for the effective Hamiltonian, we get a $\mathbf{k}$-dependent $q \times q$ matrix which can be numerically diagonalized on a computer, giving rise to a family of q subbands.

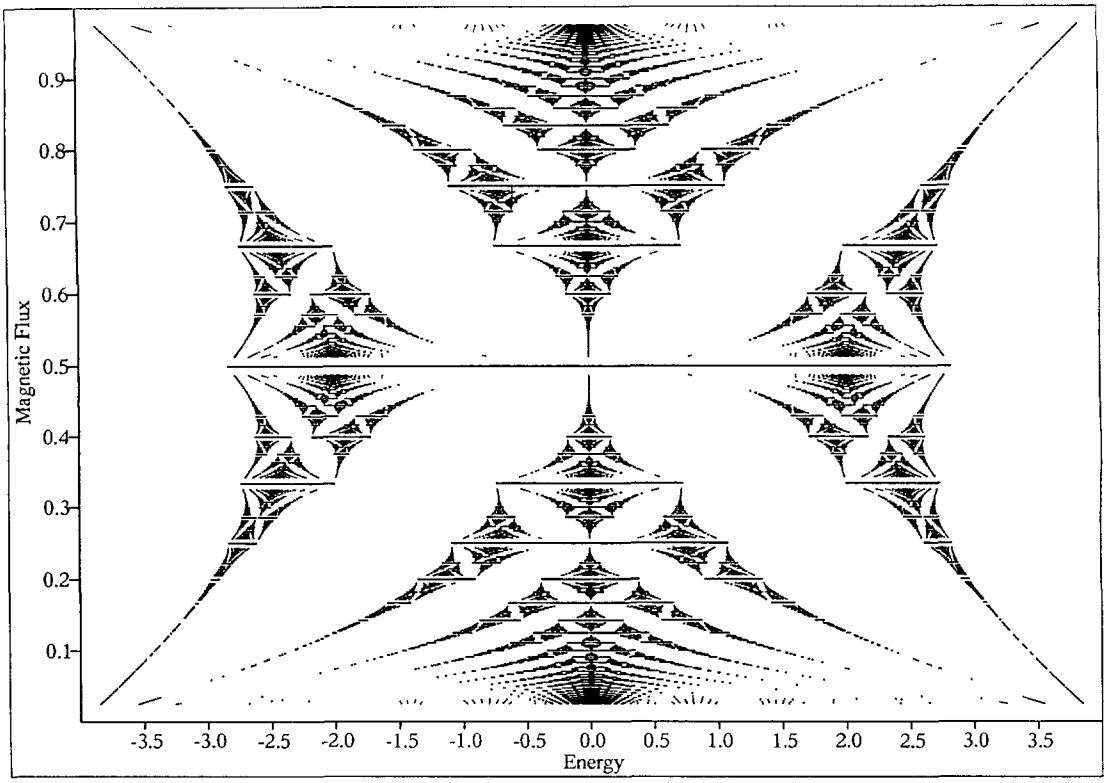

Fig.1. — Spectrum of Harper's model (Hofstadter's butterfly).

For Harper's model, this diagonalization is especially simple for the subband edges are obtained for $k_\mu = 0$ or π/q mod 2π. It gave rise to the famous Hofstadter's butterfly [55] (see Fig. 1). As one can see from figure 1, the eigenvalue spectrum as the magnetic flux varies, exhibits a fractal structure of high complexity.

The first mathematical difficulty, in this numerical approach, is to show that whenever the flux α becomes irrational, one can approximate the spectrum by mean of the rational one. This can be done using C^*-algebra techniques (see [71]).

A C^*-algebra $\mathcal{A}$ is a complex vector space, with a bilinear product, and an antilinear involution $a \in \mathcal{A} \mapsto a^* \in \mathcal{A}$ satisfying $(ab)^* = b^*a^*$. Moreover $\mathcal{A}$ has a norm for which it is a Banach space (namely each Cauchy sequence converges) and satisfies :

$$||a||^2 = ||a^*a|| \,. \tag{6}$$

A unit $\mathbf{I}$ is an element of $\mathcal{A}$ such that $a\mathbf{I} = \mathbf{I}a = a$, $\forall a \in \mathcal{A}$. Therefore from (6) $||\mathbf{I}|| = 1$. A C^*-algebra has not necessarily any unit, but if some unit exists it is unique. One can always enlarge a C^*-algebra by adding a unit by brut force. A $*$-homomorphism $\rho : \mathcal{A} \mapsto \mathcal{B}$ between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, is a linear map such that $\rho(ab) = \rho(a)\rho(b)$, $\rho(a^*) = \rho(a)^*$.

This structure is actually very natural from two points of view.

First of all, the topology generated by this norm is purely algebraic, because (6) tells us that $||a||^2$ is the spectral radius of a^*a (a purely algebraic concept). This is confirmed by the fact that if ρ is a $*$-isomorphism from $\mathcal{A}$ to $\mathcal{B}$ then ρ is norm preserving. Thus the norm is an

algebraic invariant.
Moreover, the antilinear involution permits to define positive elements, namely those elements of $\mathcal{A}$ which can be written in the form a^*a for some $a \in \mathcal{A}$. This positivity property is essential in Quantum Mechanics in that it permits to define energy and groundstates for instance. Then, if $a \in \mathcal{A}$, $||a||$ is the smallest non negative number "c" such that $0 \leq a^*a \leq c\mathbf{I}$.

It can be shown [78] that there is a unique largest C^*-algebra $\mathcal{A}_\alpha$ (up to isomorphism) generated by two "abstract" unitaries U and V satisfying (1). $\mathcal{A}_\alpha$ contains a unit. It is called the "'rotation algebra" and had been defined and studied by Rieffel [78]. Moreover, given any compact subset I of $\mathbf{R}$, one can show that there is a C^*-algebra $\mathcal{A}_I$ generated by elements of the form (2) where the h_{m_1,m_2} are continuous functions of α on I and vanish all but for a finite number of them. $\mathcal{A}_I$ has always a unit. In a certain sense $\mathcal{A}_I$ can be seen as $\cup_{\alpha\in I}\mathcal{A}_\alpha$. Actually for $\alpha \in I$ there is a natural $*$-homomorphism ρ_α . $\mathcal{A}_I \mapsto \mathcal{A}_\alpha$ which consists in evaluating all the h_{m_1,m_2}'s at α.

Given an element a in a C^*-algebra $\mathcal{A}_I$ with a unit $\mathbf{I}$, its spectrum $\mathrm{Sp}_{\mathcal{A}}(a)$ is the set of complex numbers $z \in \mathbf{C}$ such that $z\mathbf{I} - a$ has no inverse element in $\mathcal{A}$.
The main result justifying the numerical calculation is :

Theorem 1 *Let $H = H^*$ a self adjoint element of $\mathcal{A}_I$, where I is a compact subset of $\mathbf{R}$. Then the spectrum $\Sigma(\alpha) = Sp_{\mathcal{A}_\alpha}(\rho_\alpha(H))$ is continuous with respect to α namely its gap edges are continuous functions of α.*

A C^*-algebra can be represented by operators in a Hilbert space. Namely a representation π is the data of a Hilbert space $\mathcal{H}_\pi$, called the representation space, and of a $*$-homomorphism $\pi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H}_\pi)$ into the C^*-algebra of bounded operators on $\mathcal{H}_\pi$.

For the rotation algebra several representations have physical meaning. For instance, Harper in 1955 [49] used the following one : $\mathcal{H}_\pi = \ell^2(\mathbf{Z})$, namely the Hilbert space of quantum states on a discrete 1D chain, and the operator $\pi(U)$ is the translation by one, whereas $\pi(V)$ is the multiplication by $\mathrm{e}^{2i\pi(x-\alpha N)}$ if N is the position operator. It is not difficult to see then, that the Schrödinger equation $\pi(H_{\mathrm{Harper}})\psi = E\psi$, where H_{Harper} is given by (3) is nothing but the Harper equation :

$$\psi(n+1) + \psi(n-1) + 2\cos 2\pi(x - \alpha n)\psi(n) = E\psi(n) \ . \tag{7}$$

We can also represent U and V on $\ell^2(\mathbf{Z}^2)$ namely in a 2D lattice, by mean of the discrete magnetic translations. At last, one can also represent them on $L^2(\mathbf{R})$ by $\mathrm{e}^{i\sqrt{\gamma}K_1}$ and $\mathrm{e}^{i\sqrt{\gamma}K_2}$ where $\gamma = 2\pi\alpha$, K_2 is the position operator and $K_1 = -i\frac{\partial}{\partial x}$.
Many other representations do exist and are useful for practical purposes.

1.2 SEMICLASSICAL ANALYSIS. — To go forward on the knowledge of the Hofstadter spectrum, we will use the semiclassical analysis namely we will develop an asymptotic calculus as $\alpha \mapsto 0$. This can be done as follows : setting $\gamma = 2\pi\alpha$ we use the representation :

$$\pi(U) = \mathrm{e}^{i(k_1+\sqrt{\gamma}K_1)} \qquad \pi(V) = \mathrm{e}^{i(k_2+\sqrt{\gamma}K_2)} \ , \tag{8}$$

and expand the effective Hamiltonian (3) in powers of $\sqrt{\gamma}$.
If $\mathbf{k} = (k_1, k_2)$ is chosen as a maximum or a minimum $\mathbf{k} = \mathbf{k}_c$ of the band function $E(\mathbf{k})$, one gets for H an expression of the form :

$$H = E(\mathbf{k}_c) \pm \frac{\gamma}{2}(M^{-1})_{\mu\nu}K_\mu K_\nu + \mathrm{O}(\gamma^{3/2}) \ , \tag{9}$$

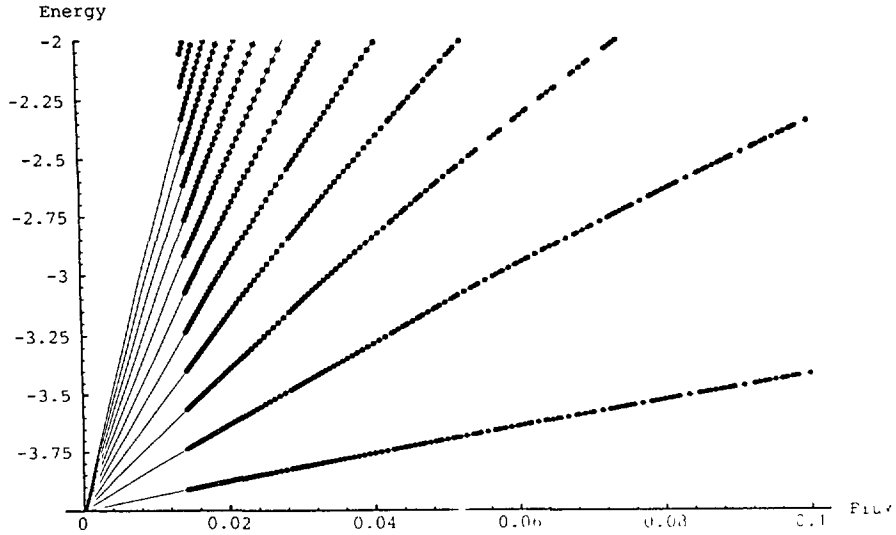

Fig.2. — Comparison between exact spectrum and semiclassical formulae for the Landau levels in the Harper model; points are extracted from the numerical exact spectrum, full curves represent semiclassical formulae given by Equation (11).

where $M_{\mu\nu}$ is the effective mass matrix near the extremum and + (resp. −) is chosen for a minimum (resp. maximum). The $O(\gamma)$ term is nothing but Landau's Hamiltonian in the effective mass approximation. The corresponding eigenvalues are easy to get, namely $(n \in \mathbf{N})$:

$$E_n = E(\mathbf{k}_c) \pm \frac{\gamma}{2}\frac{(2n+1)}{\sqrt{m_1 m_2}} + O(\gamma^2) \,, \tag{10}$$

where m_1, m_2 are the eigenvalues of M, and $n \in \mathbf{N}$ labels the corresponding Landau levels. The $O(\gamma^2)$ term can be obtained from (9) by perturbation theory. In particular if H is polynomial in U and V, one can get an expansion of E_n in (10) at every order in γ. In particular for Harper's equation, the Landau levels are given by (see Fig. 2) :

$$E_n^{\pm} = \pm\left[4 - \gamma(2n+1) + \frac{\gamma^2}{16}[1+(2n+1)^2] - \frac{\gamma^3}{64\times 3}[n^3+(n+1)^3] + O(\gamma^4)\right] \tag{11}$$

Such an expression can actually be generalized near each rational value $p/q \in \mathbf{Q}$ of α. For indeed, using the matrices u and v in eq.(5), and the generators $U_\beta = \rho_\beta(U)$ and $V_\beta = \rho_\beta(V)$, the sub-C^*-algebra of $M_q(\mathbf{C}) \otimes \mathcal{A}_\beta$ generated by $U' = u \otimes U_\beta$ and $V' = v \otimes V_\beta$ is isomorphic to $\mathcal{A}_\alpha$ if $\alpha = p/q + \beta$. Therefore in this representation, the effective Hamiltonian $\rho_\alpha(H)$ can be seen as a $q \times q$ matrix $\hat{H}_\beta$ with elements in $\mathcal{A}_\beta$. In much the same way we get an expansion near $\delta = 2\pi\beta \approx 0$, if we replace U_β and V_β by $e^{i(k_1+\sqrt{\delta}K_1)}$ and $e^{i(k_2+\sqrt{\delta}K_2)}$ and developing in powers of $\sqrt{\delta}$. The difference is that for $\delta = 0$ the "band function" $E(\mathbf{k})$ is replaced by a matrix valued functions $\mathcal{H}(\mathbf{k})$. Diagonalizing $\mathcal{H}(\mathbf{k})$ gives a sequence of q subbands $E_1(\mathbf{k}), \cdot \; , E_q(\mathbf{k})$, which gives the spectrum of $\rho_{p/q}(H)$, whenever we vary $\mathbf{k}$. Correspondingly we get $\mathbf{k}$-dependent eigenstates $|i\rangle_{\mathbf{k}}$, $i = 1, \cdot \; , q$.

Such a "classical" mechanics corresponds to motion in a fiber bundle. Here the $\{(\mathbf{k}, |i\rangle_{\mathbf{k}}) \in \mathbf{T}^2 \times \mathbf{C}^q; \mathbf{k} \in \mathbf{T}^2\}$ is a line bundle associated to the i^{th} subband.
To describe the spectrum near a band edge, we pick up a band (namely we choose $i \in [1, q]$), and let $\mathbf{k}_c$ be the value of $\mathbf{k}$ for which $E_i(\mathbf{k}_c)$ is the band edge we consider. Then $\mathbf{k}_c$ is a maximum or a minimum of E_i. In order to reduce the problem to the case $\alpha \mapsto 0$, we replace the matrix valued Hamiltonian $\hat{H}_\beta$ by an effective band Hamiltonian by mean of a standard Schur complement formula :

$$\tilde{H}_\beta(z) = \langle i|\hat{H}_\beta i\rangle + \langle i|\hat{H}_\beta Q_i \frac{1}{z\mathbf{I}_q - \hat{H}_\beta} Q_i \hat{H}_\beta |i\rangle \ , \tag{12}$$

where $|i\rangle = |i\rangle_{\mathbf{k}=\mathbf{k}_c}$, $Q_i = \mathbf{I}_q - |i\rangle\langle i|$ and $\mathbf{I}_q$ is the unit matrix in $M_q(\mathbf{C})$.
The eigenvalue E of $\hat{H}_\beta$ near $E_i(\mathbf{k}_c)$ is then given by the equation :

$$E = \text{eigenvalue of} \quad \tilde{H}_\beta(E) \ .$$

Expanding (12) in powers of $\sqrt{\delta}$ gives rise to an expansion of the same form as eq.(9) where now M is the effective mass for the subbands. So we get Landau sublevels at each band edge. However the $O(\delta)$ term splits into two pieces :

$$E_{i,n}^{(\delta)} = E_i(\mathbf{k}_c) \pm \frac{|\delta|}{2} \frac{(2n+1)}{\sqrt{m_1 m_2}} + \frac{\delta}{2} \langle i| \left[\frac{\partial \mathcal{H}}{\partial k_1} \frac{\partial \Pi_i}{\partial k_2} - \frac{\partial \mathcal{H}}{\partial k_2} \frac{\partial \Pi_i}{\partial k_1} \right]_{\mathbf{k}=\mathbf{k}_c} |i\rangle + O(\delta^2) \ , \tag{13}$$

where $\Pi_i(\mathbf{k}) = |i\rangle_{\mathbf{k}\mathbf{k}}\langle i|$. The first term is proportional to $|\delta|$ and gives rise to a discontinuity of the derivative of the band edges (namely for $n = 0$) at $\alpha = p/q$ (i.e. $\delta = 0$). Notice that $\sqrt{m_1 m_2}$ is the density of states at the band edge.
The other term is proportional to δ and accounts for the curvature of the i^{th} bundle introduced by the matrix Hamiltonian $\mathcal{H}(\mathbf{k})$. It produces an asymmetry of the derivative of the band edges near $\alpha = p/q$ (see Fig. 3).

Formula (13) was written for the first time by Wilkinson [92] for the Harper model. Rammal [76] rederived it for Harper's model and related the derivative $\partial E_{i,n}(\delta)/\partial\delta$ to the magnetization of a superconducting network, using Abrikosov's theory. The formula (13) was derived in full generality (namely for any model given by Eq.(7)) in [13] (see also [51, 77]) where it was called the Wilkinson-Rammal formula.

1.3 BANDTOUCHING. — The Wilkinson-Rammal formula is valid only at a band edge separated from other bands, namely whenever $E_i(\mathbf{k}_c)$ is a simple eigenvalue of $\mathcal{H}(\mathbf{k}_c)$. On the contrary, if two bands touch at $\mathbf{k} = \mathbf{k}_c$, the asymptotic is different and depends upon the order of the contact between them. Whenever two bands touch each other, the contact is generically conical (see Fig. 4). Degenerate perturbation theory indicates that the effective Hamiltonian $\tilde{H}_\beta(z)$ in (12) must be a 2×2 matrix corresponding to the eigenprojection :

$$|i\rangle\langle i| + |i'\rangle\langle i'| = \Pi \ ,$$

if $E_i(\mathbf{k}_c) = E_{i'}(\mathbf{k}_c)$, i and i' being the labels of the touching bands, and $\mathbf{k}_c$ the conical point. One can show that the lowest order term in the $\tilde{H}_\beta(z)$ expansion is given (after a suitable choice of coordinates) by a Dirac operator, namely :

$$\tilde{H}_\beta(z) = E_i(\mathbf{k}_c) + \sqrt{\delta} \begin{pmatrix} 0 & K_1 + iK_2 \\ K_1 - iK_2 & 0 \end{pmatrix} + O(\delta) \ . \tag{14}$$

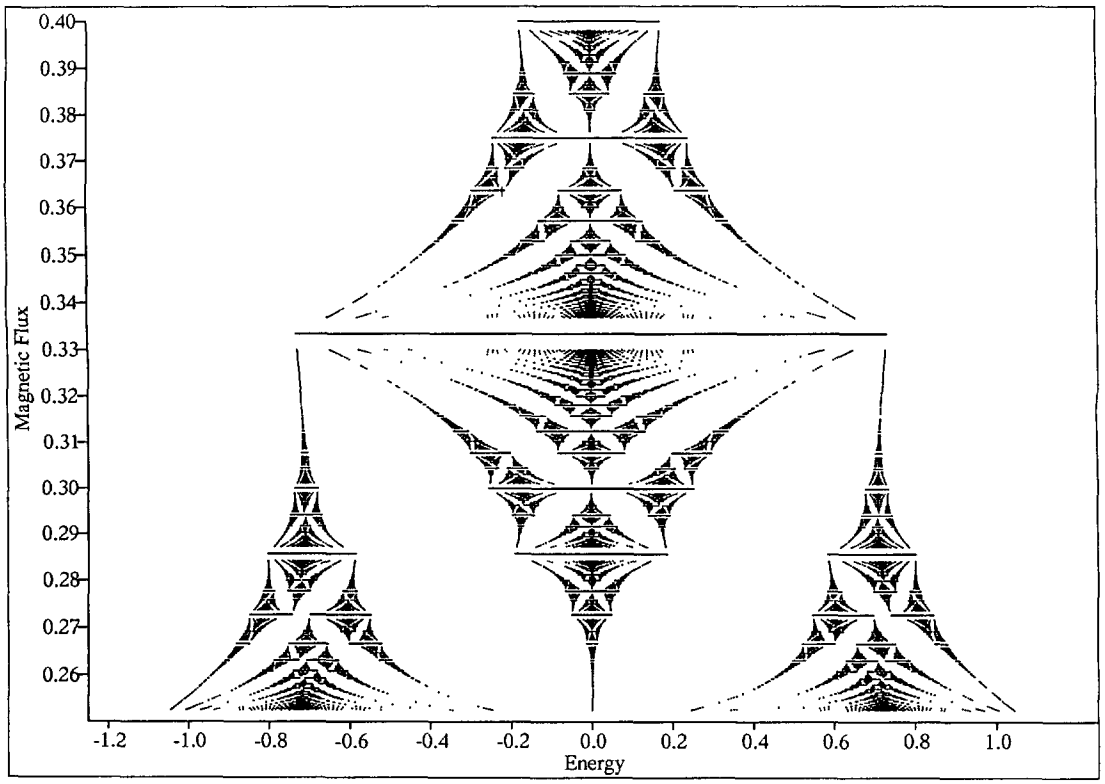

Fig.3. — Asymmetry of the central band edges for the Harper model near $\alpha = 1/3$.

Thus Landau sublevels are replaced by Dirac levels namely :

$$E_{i,n} = E_i(\mathbf{k}_c) + 2\sqrt{|\delta||n|}\mathrm{sgn}(n) + \mathrm{O}(\delta) \ , \tag{15}$$

where $n \in \mathbf{Z}$ is the Dirac label.

For Harper's model, it was shown in [17] that there is no gap at $E = 0$. In particular if $\alpha = p/q$, where q is even, the middle gap is closed producing a conical bandtouching. This is what we can see in figure 4 at $\alpha = 1/2$ where the Dirac levels are given by ($\delta = 2\pi(\alpha - 1/2)$) :

$$E^{\pm}_{i,n} = \pm 2(2n|\delta|)^{1/2}\left(1 - \frac{n|\delta|}{2}\right)^{1/2} + \mathrm{O}(|\delta|^{5/2}) \quad n \geq 0 \ . \tag{16}$$

Non generic touching had been studied in [9].

Playing with such a semiclassical analysis permits actually to reveal the topology of the classical bands underlying the system. A bunch of Landau sublevels will reveal the existence of a local extremum in a subband. The slope of these levels permits to measure the local curvature of the subband near this extremum. The expansion of this level in power of δ permits in principle to compute the higher order derivatives of the subband function. In much the same way, Dirac levels reveal the existence of a conical bandtouching. At last saddle points also produce a special packing of sublevels. This has been studied by Helffer and Sjöstrand [52]. It permits to understand in particular why the Hausdorff dimension of the Hofstadter spectrum may not vanish [88].

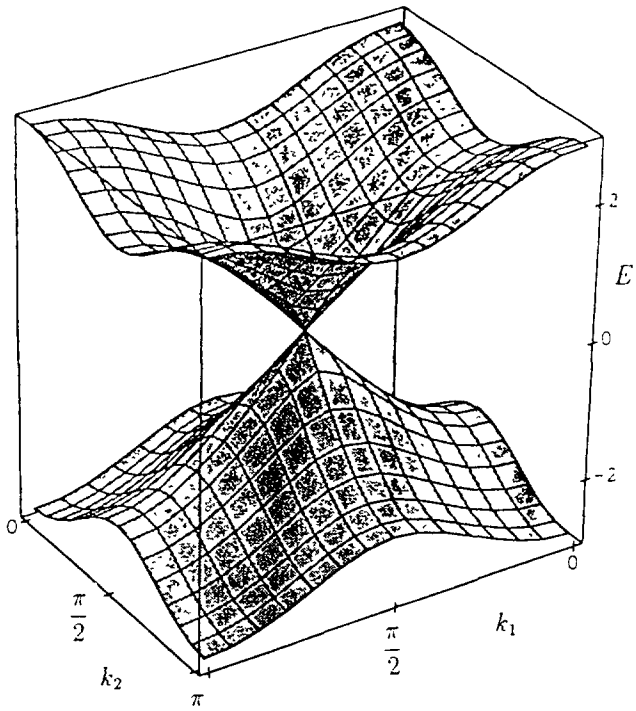

Fig.4. — Conical contact between bands at half flux in the Harper model.

1.4 TUNNELING EFFECT. — The previous analysis does not account for exponentially small terms in α (or $\alpha - p/q$) which may be produced by the tunneling effect. Actually considering $E(\mathbf{k})$ to be the classical Hamiltonian where k_1 (resp. k_2) represents classical momentum (resp. position) we may get tunneling between wells in the $\mathbf{k} = (k_1, k_2)$ phase space. Because $E(\mathbf{k})$ is periodic in $\mathbf{k}$, any well is actually infinitely degenerate by mean of the translation in reciprocal space. It will produce a broadening of the Landau levels and sublevels which gives an exponentially small correction as α or $\alpha - p/q \mapsto 0$. This broadening has been described by Wilkinson [92] and Helffer-Sjöstrand [52].

Before going into it, let us restrict ourselves to the simpler case for which we have classically degenerate wells in the cell of periods of the reciprocal lattice, close to each other. Such a phenomenon appears for the following model studied by Wilkinson [92] and Barelli-Kreft [10] :

$$H_{\mathrm{W}} = U + U^{-1} + V + V^{-1} + t_2(U^2 + U^{-2} + V^2 + V^{-2}) \,, \tag{17}$$

with $t_2 > 1/4$. Indeed, if $t_2 < 1/4$, H_{W} admits a unique classical minimum at $k_1 = k_2 = \pi \bmod 2\pi$ which bifurcates at $t_2 = 1/4$ into a local maximum and gives rise to four minima symmetric by a fourfold rotation around $k_1 = k_2 = \pi$. This symmetry produces an exact classical degeneracy. Thus the corresponding Landau sublevels are exactly degenerate modulo $\mathrm{O}(\alpha^{\infty})$.

To describe the tunneling effect between these four wells, we replace the Hamiltonian by a 4×4 effective Hamiltonian for each value of the Landau quantum number n. Using the rotation

symmetry, this matrix admits the following form :

$$H_{\text{tunnel}} = E_n(\gamma)\mathbf{I} + \begin{pmatrix} 0 & t & r & \bar{t} \\ \bar{t} & 0 & t & r \\ r & \bar{t} & 0 & t \\ t & r & \bar{t} & 0 \end{pmatrix} , \tag{18}$$

where $E_n(\gamma)$ is the n^{th} Landau sublevel computed as before, $r = \bar{r} \in \mathbf{R}$ and $t \in \mathbf{C}$. Using WKB techniques [50, 93], one can write t (resp. r) in terms of tunneling actions $S_{\text{n.n.}}$ (resp. S_{opp}) between neighbouring wells (resp. diagonally opposite wells) in the form :

$$t = \gamma \frac{\omega}{2\pi} e^{i(S_{\text{n.n.}}/\gamma + \nu)} \qquad r = \gamma \frac{\omega}{2\pi} e^{i(S_{\text{opp}}/\gamma + \nu')} , \tag{19}$$

where ω is the frequency of the harmonic oscillator associated to the bottom well and ν, ν' are Maslov's corrections.

It turns out that $|\text{Im}S_{\text{opp}}| > |\text{Im}S_{\text{n.n.}}|$ so that r becomes negligible compare to t. Moreover, contrary to the Schrödinger case, the real part of $S_{\text{n.n.}}$ does not vanish, producing oscillatory terms in the expression of the level splitting when γ varies.

Therefore the eigenvalues of $H_{\text{tunnel}} - E_n(\alpha)\mathbf{I}$ have the form :

$$\begin{aligned} E_{\pm} &= \pm\gamma \tfrac{\omega}{\pi} e^{-|\text{Im}(S_{\text{n.n.}})|/\gamma} \cos\left(\text{Re}(S_{\text{n.n.}})/\gamma + \tfrac{\pi}{4}\right) \\ E_{\pm} &= \pm\gamma \tfrac{\omega}{\pi} e^{-|\text{Im}(S_{\text{n.n.}})|/\gamma} \sin\left(\text{Re}(S_{\text{n n}})/\gamma + \tfrac{\pi}{4}\right) \end{aligned} \tag{20}$$

The corresponding Landau level splits into four levels like a braid as γ varies (see Fig. 5).

The same phenomenon has been also observed with Dirac sublevels [9]. In figure 6 is represented the braiding of Dirac sublevels produced by the touching of two bands along five conical points with an exact fourfold symmetry around the fifth one. One can see on figure 7 that the WKB formulae fit quite well with the diagonalization method.

The broadening of Landau sublevels can be described in much the same way by using the reciprocal group translation symmetry in the **k**-phase space instead of the fourfold rotation symmetry used above.

We replace then the Hamiltonian we started from, by an effective Hamiltonian H_{tunnel} acting in the Hilbert space generated by the Landau eigenstates of given quantum number n associated to each well. Since each such well is labelled by a point in a 2D lattice, H_{tunnel} can be seen as acting on $\ell^2(\mathbf{Z}^2)$ by mean of magnetic translations. However, when one computes the corresponding flux, one can see that now $H_{\text{tunnel}} \in \mathcal{A}_{\alpha'}$ with $\alpha' = 1/\alpha \bmod 1$ instead [92, 52]. This scale invariance of the spectrum was proposed for the first time by Azbel [6] who emphasized the role of the continuous fraction expansion of α. The Gauss map $\alpha' \mapsto 1/\alpha$ (see in [57]) precisely generates this expansion.

If H has the fourfold symmetry of the original lattice, so does H_{tunnel}. Moreover the matrix element of H_{tunnel} between two wells located at sites l and l' is proportional to $e^{iS_{ll'}/\gamma}$ where $S_{ll'}$ is the tunneling action between the wells l and l'. So that as $\gamma \mapsto 0$ only the nearest neighbouring wells should be accounted for. This leads to the following approximate Hamiltonian :

$$H_{\text{tunnel}} = \gamma \frac{\omega}{2\pi} e^{-|\text{Im}S|/\gamma} \left(U_{\alpha'} + U_{\alpha'}^{-1} + V_{\alpha'} + V_{\alpha'}^{-1}\right) + \text{O}\left(e^{-S'/\gamma}\right) , \tag{21}$$

with $S' > \text{Im}S$.

This formula shows that the broadening of each Landau sublevel is actually given by a Harper Hamiltonian, properly renormalized; thus the corresponding spectrum is again the Hofstadter

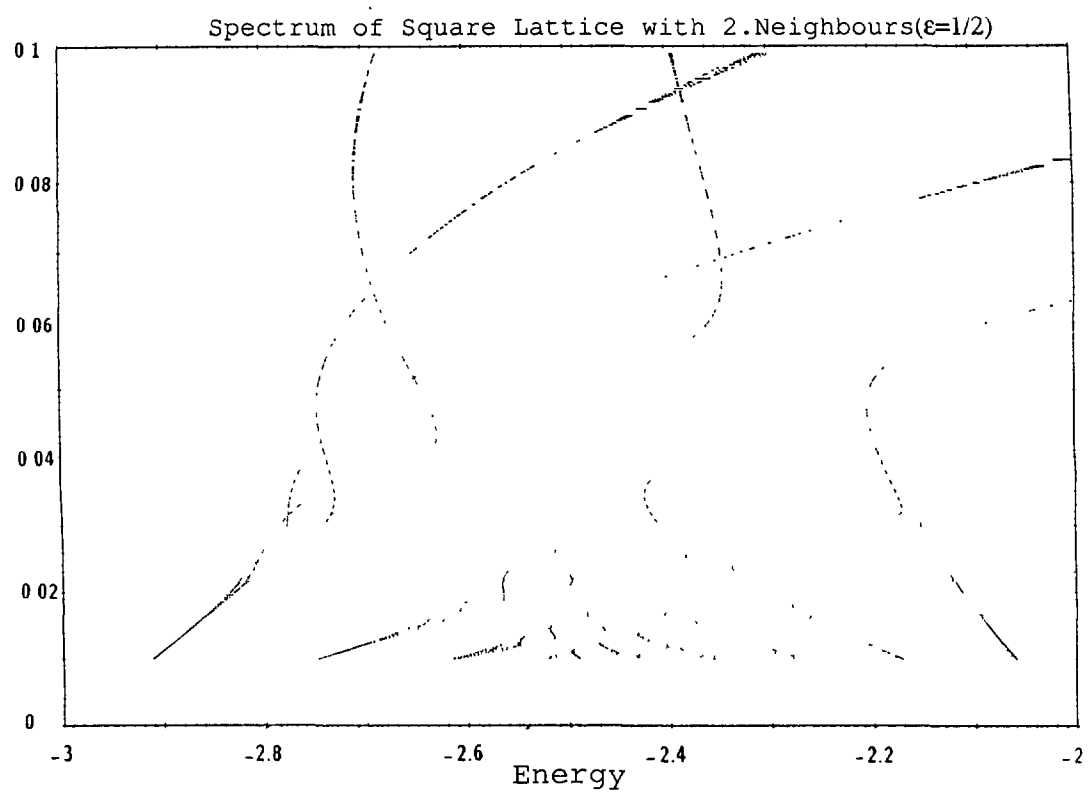

Fig.5. — Braid structure for Landau sublevels in a Harper-like model with second nearest neighbour interaction (from [10]).

one. Repeating this construction again and again gives rise in principle to the fractal structure of the spectrum.

However the previous analysis works only for well separated Landau sublevels. Near the band centers, Helffer and Sjöstrand have developed the same kind of ideas and they have shown that for Harper's model, each step of the renormalization analysis can be locally described by mean of four standard forms, one of them being the Harper model, the others taking the possible bandtouchings into account.

The same analysis has been performed by Kerdelhué [56] for the Harper model on a triangular or honeycomb lattice.

The main result of Helffer and Sjöstrand is contained in the following theorem [52]:

Theorem 2 *There is $C > 0$, such that if α admits a continuous fraction expansion of the form $[a_1, a_2, \cdot \ , a_n, \ ]$ with $a_n \geq C \ \forall n$, then the spectrum $\Sigma(\alpha)$ of Harper's equation has a zero Lebesgue measure.*

We remark however that if $C > 1$, the set of such α's has zero Lebesgue measure. A conjecture (The Ten Martini problem of Kac [17]) is that $\Sigma(\alpha)$ should have a zero Lebesgue measure for any irrational α.

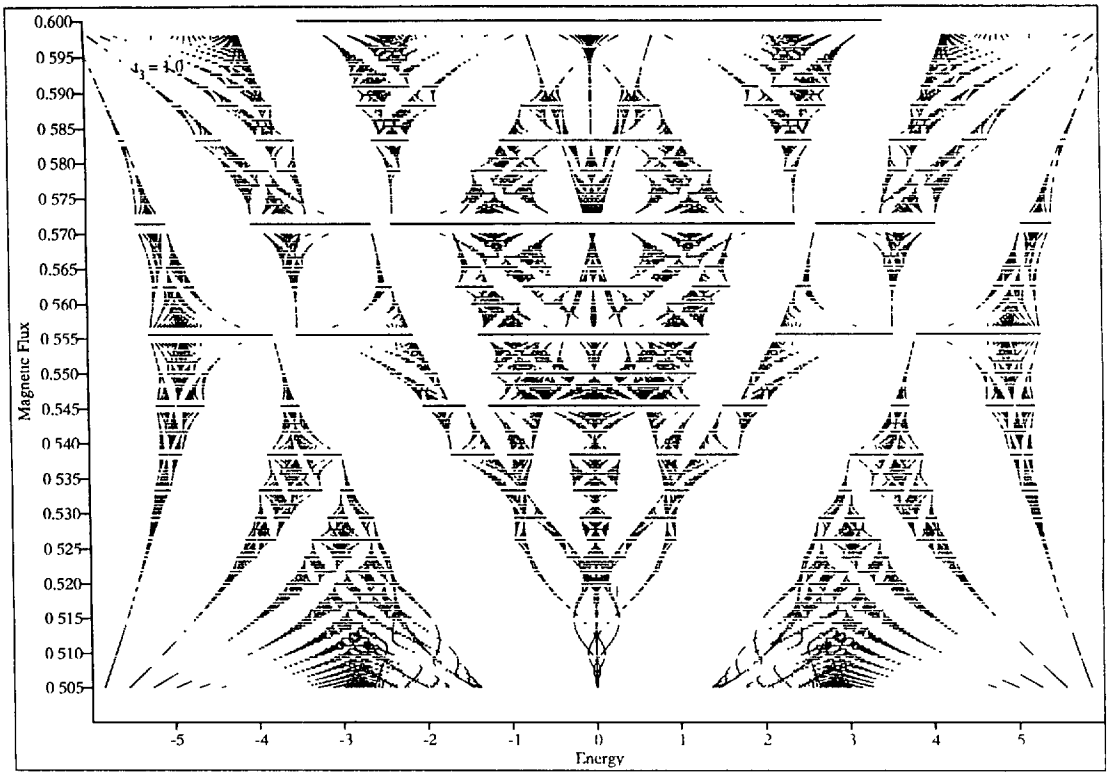

Fig.6. — Braid structure for Dirac sublevels in a Harper-like model with third nearest neighbour interaction (from [9]).

1.5 FLUX PHASES THEORY. — The discovery of copper oxide superconductors in 1986 has probably been one of the most important events in Solid State Physics in years. The main reason is that it provides probably a new mechanism for pairing of electrons. The corresponding forces are much stronger than the phonon coupling in BCS theory, producing higher critical temperature. Yet, this mechanism is still not understood.

Nevertheless the enormous effort since then by theoreticians has permitted to concentrate the studies on a small number of models. It is right now accepted by most of the experts that the key phenomena are to be found in a 2D model, representing the Fermi electrons in the Copper orbitals, the Oxygen being there only to create an effective doping by holes in these orbitals. One possible model is the so-called Hubbard model in two dimensions, with a strong electron-electron interaction, which can be approximated by the so-called $t - J$ model as an effective Hamiltonian.

A great deal of work has been done in finding a good approximate ground state for such a model. One of the approaches proposed was the so-called flux phase approximation. The basic idea in such an approach is that in the $t - J$ model, spin variables can be considered as slow whereas charge variables are rapid. Therefore a kind of Born-Oppenheimer approximation leads to a model in which the spins are frozen, creating an effective magnetic field acting on charges (here holes). In this adiabatic approximation, the charged particles become independent, and the Hamiltonian becomes similar to the Harper model. However the magnetic field is no longer

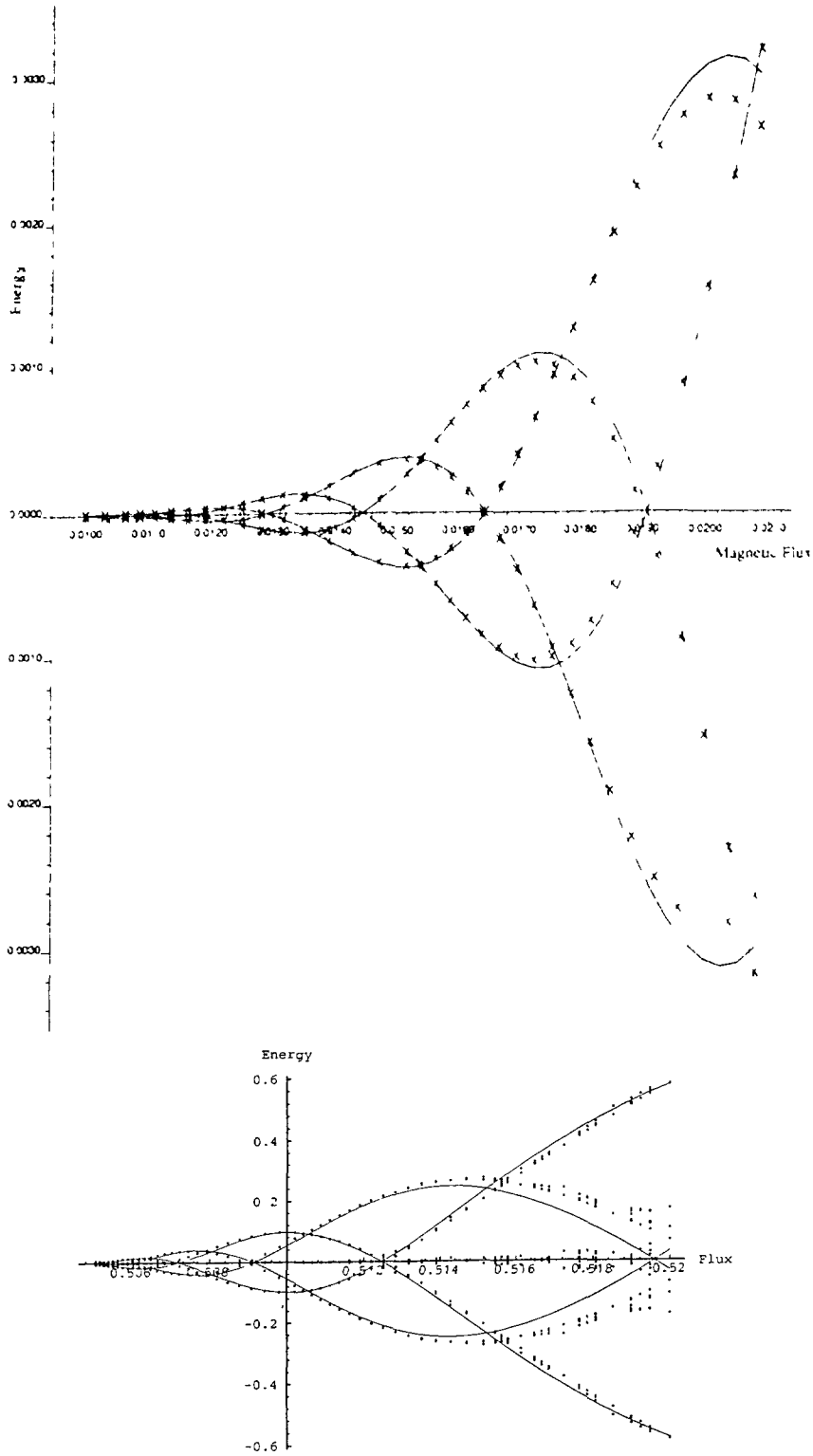

Fig.7. — Comparison between exact spectrum and semiclassical formulae for both braids; upper figure : crosses are from exact spectrum whereas full curves are coming from semiclassical formulae (from [10]); lower figure : points are extracted from numerical exact spectrum and curves correspond to semiclassical expansions (from [9]).

necessarily uniform, and it is still a dynamical variable in that it must be chosen to minimize the Fermi sea energy, given the holes concentration.

Because this problem is already difficult, many studies were dealing with the ansatz of a uniform minimizing magnetic field. So that we are back to Hofstadter's spectrum.

One of the main conjectures based upon this series of approximation, was that the minimizing dimensionless flux $\alpha = \phi/\phi_0$ should be equal to the concentration δ of charges (δ = number of charges per site). This conjecture was based upon numerical calculations on Hofstadter's spectrum (see in [59]) and was further proved to be correct in the semiclassical limit [77], namely $\delta \mapsto 0$ for the Hofstadter like models. This is what gave birth to anyons, a kind of quasiparticle with a magnetic flux attached to it. They can be reinterpreted in terms of parastatistics, thanks to the fact that the indiscernability of identical particles in 2D leads to a representation of the Braid Group [30].

Even though such an idea was very appealing because of its fundamental character, it has not been verified in experiments [39] and one can say that it has hardly any future by now [3] at least in the context of high T_c superconductors.

One of the main problems with such an approach is that the system should be driven spontaneously toward a state breaking the time reversal symmetry. Even though such a mechanism is a priori possible, there is some doubt that it can be broken in the $t-J$ model with spin 1/2 [37].

Another problem is that the minimizing magnetic flux is certainly not uniform. For indeed a slave boson approach shows that the uniform magnetic field ansatz gives rise to an unstable state [79].

However, at the half filled band and for a bipartite lattice, preliminary rigorous results by Lieb and Loss [62] indicate that the uniform magnetic field ansatz with flux 1/2 mod 1 should be correct. Nevertheless we may suspect that a small amount of holes should produce a slight disorder in the magnetic flux distribution.

One possible way to approach the non uniform magnetic flux problem was to consider the case of a periodic flux to use again the semiclassical approach for the stability of the Fermi sea. This was done in [8] to find out that the local disorder introduced by the periodicity, was producing only a global correction to the ground state energy by mean of a form factor, and the "anyon rule" $\overline{\alpha} = \delta$ was still valid as $\delta \mapsto 0$.

However as shown in [3] it has been recognized that the small field limit $\overline{\alpha} \mapsto 0$ and the long period one do not commute, probably because of the so-called "magnetic breakdown" caused by tunneling effect in phase space near separatrix [87]. Therefore, the understanding of the minimizing magnetic flux is still a pending question.

This long discussion was at the heart of the exchanges between Rammal and one of us (JB) during the years 1990-1991.

The very fact that the ground state of the Hubbard or the $t-J$ model was so hard to compute led them to think that the usual approaches were not suitable for a result.

2. Quantum chaos.

While one of us (JB) kept talking for months of quantum chaos with Rammal on the basis that such phenomena should be important and universal, one day in April 1990, Rammal called him on the phone to claim that in the Hubbard model on a finite lattice, the eigenvalues tend to exhibit many avoided crossings as the coupling constant was varied [38]. That was an indication that some chaotic behaviour, whatever that means, should occur in these models. Still, a clear test of such an idea had to be done, and even with it the question of the mechanism producing this effect remained unclear for the next year. Apparently Rammal took this idea

very seriously because he produced several works with his students on quantum chaos before May 1991 [80].

We find under the name of "Quantum Chaos" several topics mainly developed since the end of the seventies by a community of physicists who would probably disagree on a unified definition of the subject.

Most of them however would say that it concerns the quantum signature of classical chaos on the dynamical properties of a system. As its stands, such a definition reflects a pragmatic point of view taking into account the fact that most of the studies have concerned systems with two degrees of freedom so far. It is very significant that the title of the Les Houches 1989 summer school proceeding was "Chaos and Quantum Physics". As explained by the organizers in the foreword, "the expression "Quantum Chaos" at first makes us think of a quasi-exponential proliferation of publications from the last decade hiding more or less legitimately behind a scientific ill-defined label".

The subject developed around three kinds of technical approaches : Random Matrix Theory (RMT), Semiclassical Theory (ST) and Dynamical Localization (DL).

2.1 THE RANDOM MATRIX THEORY. — First of all, the oldest approach is Random Matrix Theory (RMT). It was proposed in the fifties by Wigner [91] as a convenient way of representing the Schrödinger operator for a nucleus. The idea is that such a Hamiltonian is very complicated and can be approximated in many respects by a finite (but large) dimensional matrix H_N, where N is the dimension, the elements of which are random variables. The simplest example consists in taking these variables to be gaussian, identically distributed and independent. This ensemble of matrices is called a GOE (gaussian orthogonal ensemble) or GUE (gaussian unitary ensemble) depending upon whether H_N is real symmetric or complex hermitian. The main features of these ensembles are :

(a) the density of states of such matrices is given by a semicircle law [91],

(b) the level spacing distribution exhibits a repulsion of neighbouring levels.

One can approximate the level spacing distribution by the one for 2×2 matrices called the Wigner surmise :

$$p(s) \approx Z^{-1} s^{\beta} \mathrm{e}^{-s^2/2} \qquad \begin{matrix} \beta = 1(\mathrm{GOE}) \\ \beta = 2(\mathrm{GUE}) \, . \end{matrix} \tag{22}$$

On the other hand if the random matrix is diagonal (or quasidiagonal) the level spacing distribution follows a Poisson law namely [21] :

$$p(s) = \mathrm{e}^{-s} \tag{23}$$

This is in particular the case in Anderson's random potential model [70] used in Solid State Physics to describe disordered systems giving rise to Anderson localization.

This RMT has been developed by Dyson and Mehta [40] to the point of becoming a very efficient tool of calculation. The relevance of such a theory for Nuclear Physics started to be checked in the sixties by Porter [74] and eventually by Bohigas *et al.* [48] in 1982.

In 1984, Bohigas et al. [26] proposed to use the RMT as a statistical tool to investigate spectra of quantum systems with small number of degrees of freedom. The first example was the Sinaï billiard. Following a general idea of Berry [21], they established a phenomenological rule, namely that a "generic classically integrable system" should have a level spacing distribution $p(s)$ given by a Poisson law implying some kind of level clustering, whereas a "generic" classically chaotic system should obey one of the Wigner distributions GOE or GUE depending upon the symmetry properties of the quantum Hamiltonian. In such a statement, the word

"generic" has not a precise meaning yet, even though recent rigorous results of Sinaï [83] permit to say more at least for integrable systems. But it indicates that special systems having extra symmetries (such as the harmonic oscillator [23] or geodesic motion on an arithmetic surface of constant negative curvature [24, 81]) may fail to follow the rule.

2.2 THE SEMICLASSICAL APPROACH : GUTZWILLER'S FORMULA. — The next technical tool used in "Quantum Chaos" is a formula based upon a WKB approximation and first established by Gutzwiller [46] in the context of impurity levels in Solid State Physics, and later by Balian and Bloch [7] in the case of eigenmodes of an optical cavity.
Let $\mathcal{H}(\mathbf{x})$ ($\mathbf{x} = (q,p) \in \mathbf{R}^N \times \mathbf{R}^N$, classical phase space) be a classical Hamiltonian corresponding to a quantum one $\hat{H}$. We consider the case of N degrees of freedom, and denote by h and $\hbar = h/2\pi$ the Planck constant. We assume that $\hat{H}$ has a discrete spectrum and denote by $E_0 \leq E_1 \leq \cdot \quad \leq E_n \leq \cdot$ the corresponding eigenvalues counted with their multiplicities. If

$$\delta_\epsilon(E) = \frac{\epsilon}{\pi} \frac{1}{(E^2 + \epsilon^2)} ,$$

(it is a smooth approximation of the Dirac measure) we define the smoothed density of states $d(E,\epsilon)$ by :

$$d(E,\epsilon) = \sum_n \delta_\epsilon(E - E_n) . \tag{24}$$

Using a Feynman path integral, and a stationary phase method [47] $d(E,\epsilon)$ can be approximately written as :

$$d(E,\epsilon) \approx \overline{d}(E) + \frac{1}{h} \sum_j A_j(E) \mathrm{e}^{-\epsilon T_j(E)/\hbar} \mathrm{e}^{i\left(\frac{S_j(E)}{\hbar} + \gamma_j\right)} \tag{25}$$

In this expression $\overline{d}(E)$ is the "Weyl asymptotics" [90] or classical term, namely :

$$\overline{d}(E) = \frac{1}{h^N} \int d^{2N}\mathbf{x} \delta(E - \mathcal{H}(\mathbf{x})) . \tag{26}$$

Moreover, in (25) we sum up over the family of periodic orbits indexed by j, corresponding to the energy E. Then, $T_j(E)$ is the period of this orbit, $S_j(E)$ is its action $\sum_{\mu=1}^N \oint_j p_\mu \mathrm{d}q_\mu$, and γ_j a phase shift (the Maslov index of j) which takes into account the existence of caustics or focal points in the phase space. At last $A_j(E)$ is a counting factor which can be written in terms of the Floquet matrix of orbit.

This formula shows that a large number (sometimes infinite) of oscillating terms (as $\hbar \mapsto 0$) must be taken into account to describe the fluctuations of the density of states away from its Weyl classical limit. This sum has very different properties depending upon whether the classical dynamics defined by $\mathcal{H}(\mathbf{x})$ on the energy shell $\mathcal{H}(\mathbf{x}) = E$ is quasi-integrable or chaotic. This is due to the exponential proliferation of classical periodic orbits in the latter case.

2.3 DYNAMICAL LOCALIZATION. — Several classical Hamiltonian systems in a strongly chaotic regime exhibit a diffusive behaviour in phase space. The prototype of such systems is provided by the standard map [32] given by :

$$\begin{aligned} p_{t+1} &= p_t + K \sin q_t \\ q_{t+1} &= q_t + p_{t+1} \bmod 2\pi , \end{aligned} \tag{27}$$

where t is the discrete time, and describing a rotator kicked periodically in time. Such a map is a generic model for describing the behaviour of a Hamiltonian system near a resonance which can be restricted to two degrees of freedom [63].

In his 1979 report, Chirikov [32] showed that for $K > 4$, the typical behaviour can be described via a diffusive process with diffusion constant (see Fig. 8) :

$$D = \lim_{t \mapsto \infty} \frac{\langle (p_t - p_0)^2 \rangle}{t} \approx_{K \mapsto \infty} \frac{K^2}{2} \tag{28}$$

Even though this result is not rigorous, recent mathematical progress has been made [18, 31, 89] on a slightly different model, the so-called "sawtooth map", which happens to be ergodic, and has a diffusion constant. For the standard map only rough estimates at finite time have been obtained [36].

This system can be quantized, and is known as the "kicked rotor problem" [28]. The quantum evolution is given by a unitary Floquet operator F given by :

$$F = \mathrm{e}^{-i\frac{\hat{P}^2}{2\hbar}} \mathrm{e}^{-i\frac{K}{\hbar}\cos\hat{Q}} , \tag{29}$$

where Q and P are the usual position and momentum operators on a circle, satisfying $[Q, P] = i\hbar$. Actually, here Q, P and $\hbar$ are dimensionless quantities. If one wants to introduce physical parameters, such as the period T between two kicks, the moment of inertia I of the classical rotor, one is led to the expression of the effective Planck constant

$$\hbar_{\mathrm{eff}} = \frac{\hbar T}{I} , \tag{30}$$

showing again that it can be tuned by varying the kicks frequency for instance.

It has been argued by Fishman, Grempel and Prange [42] that F should have a pure point spectrum on the basis of an argument similar to the one leading to Anderson localization. Then the momentum space is quantized according to $p_n = n\hbar$ ($n \in \mathbf{Z}$) and similar to 1D chain like in the Anderson model. As $\hbar \mapsto 0$ and fixed time the quantum evolution converges to the classical one, so that if we compare the average $\langle (p_t - p_0)^2 \rangle$ over an inital state in classical and quantum mechanics, the two functions agree for time $t \leq \tau$. Beyond this point the quantum average stay bounded while the classical one continues to increase (see Fig. 8).

An argument by Chirikov, Izrailev and Shepelyansky [33] supplemented by numerical calculations shows that the classical diffusion constant D is related to the breaking time τ and to the localization length ξ of the quantum system :

$$\tau \approx \xi \approx \mathrm{const} \frac{D}{\hbar^2} . \tag{31}$$

The discrepancy between classical and quantum behaviour is due to existence of quantum interference effects which trap the quantum state in a finite region of phase space.

The relationship between classical diffusion and quantum localization has been extended to systems with a higher number of degrees of freedom. A simple example is then given by a standard map in which K is actually a quasiperiodic function of time, namely

$$K(n) = K_0 f(\omega_1 n, \omega_2 n, \cdot \quad , \omega_{\nu-1} n) .$$

Such a system can be viewed as a ν-dimensional lattice in momentum space, for which the localization theory works : if $\nu = 2$, all states are localized but now the localization length (like the diffusion constant) increases exponentially fast with K. If $\nu \geq 3$ an Anderson metal-insulator transition is expected leading to absolutely continuous spectrum for large K [29, 82].

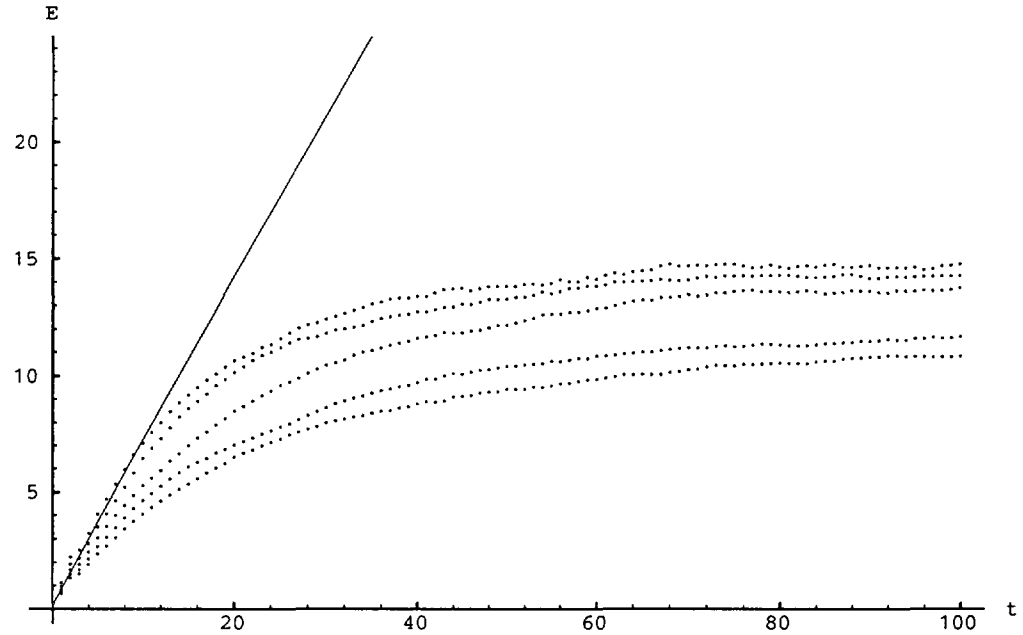

Fig.8. — Time evolution of the kinetic energy for the standard map in the chaotic regime $K = 4$; the straight line corresponds to the classical energy and points represent quantum curves for different values of the effective Planck constant (from [16]).

2.4 WHAT IS THEN QUANTUM CHAOS ? — Let us make more explicit the notion of "Quantum Chaos" lying behind the technicalities previously presented. First of all, as remarked by Berry [20], chaos is a property characterizing the long time behaviour of classical systems. While at short times the motion of a classical Hamiltonian system is deterministic, in the long run, a stochastic description becomes more suitable when chaos appears. This paradoxical situation has been explained in terms of a "sensitive dependence" on initial conditions [86].

In quantum mechanics there is always an extra parameter, namely Planck's constant $\hbar$ or some other parameter proportional to $\hbar$. We have seen in section 1 that a flux ratio plays the role of $\hbar$ and is no longer a universal constant. In the kicked rotor problem an effective Planck constant occurs which can be physically tuned as desired.

To simplify notation let us denote by $\hbar$ this effective Planck constant. The semiclassical limit corresponds to $\hbar \mapsto 0$. It is known that in most systems the quantum quantities at fixed time do converge to their classical counterpart as $\hbar \mapsto 0$.

However, there is a "clash" between the limits $\hbar \mapsto 0$ and $t \mapsto \infty$, even for systems as simple as a free rotor [20]! It means that these two limits definitely do not commute.

Where do long time quantities appear in quantum mechanics ? Precisely when we deal with stationary states. They are unchanged under time evolution and they give rise to the spectrum of the Hamiltonian. Therefore we would not be surprised to see effects of classical chaos in the behaviour of the spectrum as $\hbar \mapsto 0$. Notice however that the quantum states, for $\hbar > 0$, have no complexity in them at all if the system has a finite number of degrees of freedom [44].

Let us first consider the case of a classically integrable Hamiltonian $\mathcal{H}$. The Liouville-Arnold theorem [5] asserts that there is a canonical change of variables, the action and angle $(I_1, \cdot \quad, I_N, \theta_1, \cdot \quad, \theta_N) \in \mathbf{R}^N \times \mathbf{T}^N$, such that the Hamiltonian can be expressed as a function

of the actions $I = (I_1, \cdot \quad, I_N)$ only. The EBK [41] quantization procedure consists then in replacing each I_l by $n_l\hbar$ where n_l is an integer, to get the eigenvalue spectrum :

$$E^{\mathrm{EBK}}_{n_1,\cdots n_N} = \mathcal{H}(n_1\hbar, \cdot \quad, n_N\hbar) \ . \tag{32}$$

If $\mathcal{H}$ is smooth enough, this expression is the first term of an expansion in powers of $\hbar$. In some cases (as we have seen in section 1) exponentially small terms must be added to take into account the occurence of the tunneling effect due to exact classical degeneracies between classical regions. Still, as $\hbar \mapsto 0$, each eigenvalue is well labelled by the quantum numbers $(n_1, \cdot \quad, n_N) = \mathbf{n}$, and if two levels cross we can follow them independently.

If now $\mathcal{H}$ is not exactly integrable, say if $\mathcal{H}$ can be written as :

$$\mathcal{H}(\mathbf{I}, \theta) = \mathcal{H}_0(\mathbf{I}) + \epsilon f(\mathbf{I}, \theta) \ , \tag{33}$$

where f is a smooth enough function of $(\mathbf{I}, \theta)$, for ϵ small, the KAM theorem (see [43]) asserts that most of the unperturbed invariant tori, $\mathbf{I} = \text{const}$, survive but a theorem by Poincaré [73] asserts that only a discrete set of periodic orbits of $\mathcal{H}_0$ survives the perturbation.

Perturbation theory in quantum mechanics can be used uniformly with respect to Planck's constant [19]. The occurence of unstable periodic orbits of $\mathcal{H}_0$ corresponds in quantum mechanics to level crossings producing (as $\hbar$ varies), after perturbation, some kind of anticrossings (like the Jahn-Teller effect of molecular physics). On the other hand, classical chaos precisely develops near unstable periodic orbits. So we are led to admit that this local "hyperbolicity" is associated with avoided crossings of levels [45].

However in the fully chaotic regime, some new phenomena occur which cause the level splitting to increase. For indeed the tunneling effect between unperturbed resonant eigenstates is enhanced by the chaotic classical transport. This has been beautifully illustrated in [27]. In this latter case the usual WKB theory, requiring classical orbits in the complex phase space, does not provide the correct level splitting. Everything looks like if the transport of the wave function through the dynamical barrier was more efficient by using the classically chaotic motion than by using the usual tunneling effect. These results in level splitting are of the same order of magnitude as the mean spacing between levels.

A good example of such a phenomenon is given by the Zeeman effect on Rydberg Hydrogen atoms (see Fig. 9). If the magnetic field is small enough, each principal quantum number splits into a bunch of Zeeman sublevels roughly linear in B. However, one can see that rapidly these sublevels will start creating a lot of crossings. Due to higher order terms they tend to avoid each other. If the number of such avoided crossings becomes too high, each level will oscillate very rapidly as B increases. Moreover one can see that the splitting at the avoided crossings becomes larger in the classically chaotic regime (namely B large) than in the regular region [35].

We remark that if instead we consider $\hbar \mapsto 0$, and a given window of classical energies $[E_-, E_+]$, then $\hbar \mapsto 0$ will select quantum numbers "n" $\mapsto \infty$, and we will be in the same situation.

These arguments give an insight into what should be called Quantum Chaos. First of all, in the quasi-integrable regime, one can assign quantum numbers unambiguously to levels, even if they exhibit sometimes avoided crossings. For indeed, in this latter case, the level splitting is very small (typically of order $\mathrm{e}^{-S/\hbar}$ by WKB approximation) compared to the mean spacing separation of order $\hbar$.

In the chaotic regime however, the quantum numbers become meaningless. The spectrum is fuzzy due to level oscillations, and the level splitting is of the order of magnitude of the mean level spacing. For this reason (and this is somewhat paradoxical) the spectrum is more rigid in the chaotic case than in the regular one.

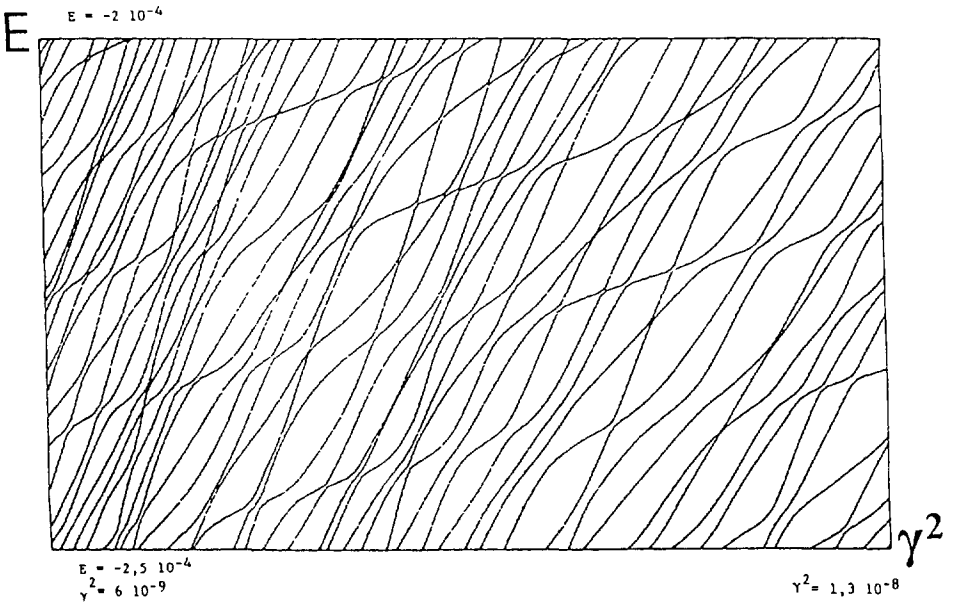

Fig.9. — Zeeman sublevels for Rydberg Hydrogen atoms in a strong magnetic field B. $\gamma = B/B_c$ is the dimensionless magnetic field expressed in atomic units ($\gamma = 1$ corresponds to $B = 2.3^5$ T). The natural parameter here is $\beta = \gamma^2/(-2E)^3$. $\beta \ll 1$ is the perturbative regime whereas here $\beta \geq 60$, namely the atom is in a strongly chaotic regime (from [35]).

2.5 LEVEL REPULSION. — To illustrate this point, let us compute approximately the statistics of level repulsion. Let us consider a Hamiltonian H depending upon some parameter $\hbar$, the spectrum of which being given by a discrete set $E_0(\hbar) \leq \cdot \quad E_l(\hbar) \leq E_{l+1}(\hbar) \leq \cdot$ of eigenvalues only sensitive to the variation of $\hbar$.

We will define three scales of variation for $\hbar$:

(i) a microscopic scale δ_μ, such that if $\hbar' - \hbar = O(\delta_\mu)$ perturbation theory for individual levels $E_l(\hbar)$ is convergent;

(ii) a short scale δ_s, such that in the range $\hbar' - \hbar = O(\delta_s)$ the macroscopic shape of the eigenvalue distribution does not change, but each pair (E_l, E_{l+1}) of individual levels oscillates a large number of times;

(iii) a macroscopic scale δ_M along which the macroscopic shape of the spectrum is modified.

We assume that :

$$\delta_\mu \ll \delta_s \ll \delta_M \ .$$

Now let us consider the level separation between $E_l(\hbar)$ and $E_{l+1}(\hbar)$. Consider an interval $[\hbar_0, \hbar_1]$ of size $O(\delta_\mu)$ along which $E_l(\hbar)$, $E_{l+1}(\hbar)$ exhibit one avoided crossing. As $\hbar$ varies from $\hbar_0$ to $\hbar_1$, the main variation of $H(\hbar)$ can be described by a 2×2 matrix corresponding to the subspace generated by states l and $l+1$. Thus one can write :

$$H(\hbar_1) - H(\hbar_0) = \delta H(\hbar) \approx \bigoplus_{\text{pairs}(l,l+1)} (e_l + \vec{\sigma} \cdot \vec{v}_l) \ , \tag{34}$$

where the sum concerns all pairs of eigenvalues $(l, l+1)$ which anticross, and we have used the representation of 2×2 matrices given by Pauli's one $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$. The constant e_l

represents the mean variation of $(E_l + E_{l+1})/2$ which we can neglect here because e_l has an appreciable value only on the scale δ_M by hypothesis. So $e_l \approx 0$. Here $\mathbf{v}_l$ is some vector in $\mathbf{R}^3$. Remark that if $H(\hbar)$ is real symmetric $\vec{v}_l \in \mathbf{R}^2$ (namely the second component of $\mathbf{v}_l$ vanishes).

Let us now consider an interval $[\hbar_-, \hbar_+]$ with size $O(\delta_s)$ and let us decompose it into $N = O(\delta_s/\delta_\mu)$ subintervals $\cup_{k=0}^{N-1}[\hbar_k, \hbar_{k+1}]$ of size $O(\delta_\mu)$.
Hence the total variation of $H(\hbar)$ along this interval can then be written as :

$$H(\hbar_+) - H(\hbar_-) = \bigoplus_l \left(\sum_{k=0}^{N-1} \sigma \cdot \mathbf{v}_l^{(k)} \right) \tag{35}$$

By hypothesis on such an interval N is large, but the macroscopic shape of the spectrum does not change, so that one can consider the $\mathbf{v}_l^{(k)}$'s (as k varies) as identically distributed random variables, whereas the rapid oscillations make them independent.
Therefore the sum :

$$\sum_{k=0}^{N-1} \sigma \cdot \mathbf{v}_l^{(k)} = \sigma \cdot \mathbf{V}_l$$

becomes a Gaussian random 2×2 matrix with mean zero as N gets large.
The level spacing $E_{l+1} - E_l$ is then given by :

$$s = E_{l+1} - E_l = 2\,|\mathbf{V}_l| \ ,$$

namely its distribution is given by the Wigner surmise :

$$p(s) = Z^{-1} s e^{-s^2/2a} \quad \text{if } \mathbf{V} \in \mathbf{R}^2 \quad \text{real symmetric case}$$
$$p(s) = Z^{-1} s^2 e^{-s^2/2a} \quad \text{if } \mathbf{V} \in \mathbf{R}^3 \quad \text{complex hermitian case .}$$

We have here a beginning of an argument explaining why the Wigner law should be universal.

2.6 CHAOS IN STRONGLY CORRELATED FERMIONS SYSTEMS. — Let us now consider the $t - J$ model describing a 2D strongly correlated fermion lattice gas :

$$H_\Lambda = t \sum_{|x-y|=1, x, y \in \Lambda} (1 - n_{x,-\sigma})\, c^+_{x,\sigma} c_{y,\sigma}\, (1 - n_{y,-\sigma}) + J \sum_{|x-y|=1, x, y \in \Lambda} \mathbf{S}_x \cdot \mathbf{S}_y \ , \tag{36}$$

Λ is a finite lattice in $\mathbf{Z}^2$, taken with periodic boundary conditions, $c_{x,\sigma}$ is the fermion creator operator at the site $x \in \Lambda$ of an electron of spin σ, $n_{x,\sigma} = c^+_{x,\sigma} c_{x,\sigma}$ is the number of electrons of spin σ at x, and $\mathbf{S}_x$ is the spin operator for electrons at site x. H_Λ acts on the Hilbert subspace of states with occupation number $n_x = n_{x,\uparrow} + n_{x,\downarrow} \leq 1$ at each site (the so-called "Gutzwiller projection").
The total number of particles $\sum_x (n_{x,\uparrow} + n_{x,\downarrow}) = N_\uparrow + N_\downarrow$ is fixed and equal to $N - N_0$ where N_0 is the number of holes. To avoid frustration we will assume that Λ is bipartite and in particular that N is even. Since the total spin is conserved we will fix $2S^{(3)} = N_\uparrow - N_\downarrow$ to its value which maximizes the dimension of the corresponding sector. Since N is even, $2S^{(3)} = 0$ or 1 depending upon whether N_0 is even or odd. The physics will correspond to the thermodynamic limit namely $N \mapsto \infty$, $N_0/N \mapsto \delta$ (δ is the doping concentration), $\mathbf{S}/N \mapsto 0$ (no magnetization here).

This model describes an electron gas with a large number of quantum particles. It is quite unclear *a priori* whether it has any classical limit, the particles being fermions. So how can we speak of any chaos in such a model?

Even though we do not know yet the answer to that question, we can check whether the eigenvalue spectrum is very sensitive to physical parameters. So one can measure its level spacing distribution. To do so, we must however be careful enough, not to introduce extra phenomena liable to hide a possible level repulsion. For instance if our Hamiltonian admits a symmetry, the different sectors will not be coupled together and the eigenvalues of H in two sectors will be independent. One can then check that the level spacing distribution will satisfy a Poisson law.

In our $t-J$ model it is therefore important to restrict H to a sector of each symmetry. This is why we must choose N, N_0, the total spin $S^{(3)}$ Moreover, due to translation invariance, we must choose a given quasi-momentum sector indexed by k. Lastly the J term admits a spin rotation invariance which forces us to choose a spin sector.

From the numerical point of view, the choice of the finite lattice Λ is done by defining a sublattice $\mathcal{L}$ of $\mathbf{Z}^2$ generated by two vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbf{Z}^2$. In order to preserve the square symmetry, we will choose :

$$\mathbf{x} = (q,p) \qquad \mathbf{y} = (-q,p) \ ,$$

where q and p are two integers. Thus $\mathcal{L} = \mathbf{x}\mathbf{Z} + \mathbf{y}\mathbf{Z}$ and $\Lambda = \mathbf{Z}^2/\mathcal{L}$.
Λ can be identified with the square whose sides are $\mathbf{x}$ and $\mathbf{y}$ and the number of sites becomes :

$$N = |\Lambda| = p^2 + q^2$$

In this way we can have 16 or 18 sites (we choose N even to avoid spin frustrations) as shown in figure 10. To avoid matrix size too big for the computer memory, we choose $N_0 = 1$ (one hole).

Now the main technical difficulty lies in the spin symmetry. It is actually too hard to compute matrix elements in each spin sector. To avoid such a long calculation it is better to break this symmetry by modifying the J-term as :

$$\sum_{|x-y|=1, x,y\in\Lambda} \left(J_\perp (S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)}) + J_\parallel S_x^{(3)} S_y^{(3)} \right) \ , \tag{37}$$

with $J_\perp \neq J_\parallel$. The limit $J_\parallel \mapsto \infty$ is actually the Ising model which is obviously integrable.

At last, the level spacing must be normalized in order to allow a comparison between levels far apart in the spectrum. The normalization is usually chosen in such a way as the local mean level separation be one. This can be done by calculating the density of states $\rho(E)$ (see Fig. 11) and replacing the levels E_l by the number $\xi_l = \int_{-\infty}^{E_l} \overline{\rho}(E) \mathrm{d}E$ where $\overline{\rho}$ is some smooth approximation of $\rho(E)$. This operation is called "unfolding the spectrum" [25].

The calculation has been performed in [65] (see Fig. 12) and exhibits a GOE distribution. In [65] it is also shown that as $J_\parallel \mapsto \infty$ the level spacing distribution becomes Poisson.

2.7 DISCUSSION AND SPECULATIONS. — The previous analysis is certainly very preliminary. But it shows that RMT can be used to investigate whether a system is integrable. Actually recent and still unpublished investigations of various models indicate that such a method is justified. For instance, a Heisenberg spin chain in 1D, known to be integrable by mean of a Bethe Ansatz [22] exhibits a Poisson distribution [4].

In much the same way a 1D $t-J$ model with one hole should also be integrable [60]. A numerical calculation shows that it satisfies a Poisson distribution [84]. Presumably with two holes the situation should be different [61].

One could use such a method to test the integrability of a 2D Ising model with magnetic field, by computing the level spacing distribution on the transfer matrix. While for a zero

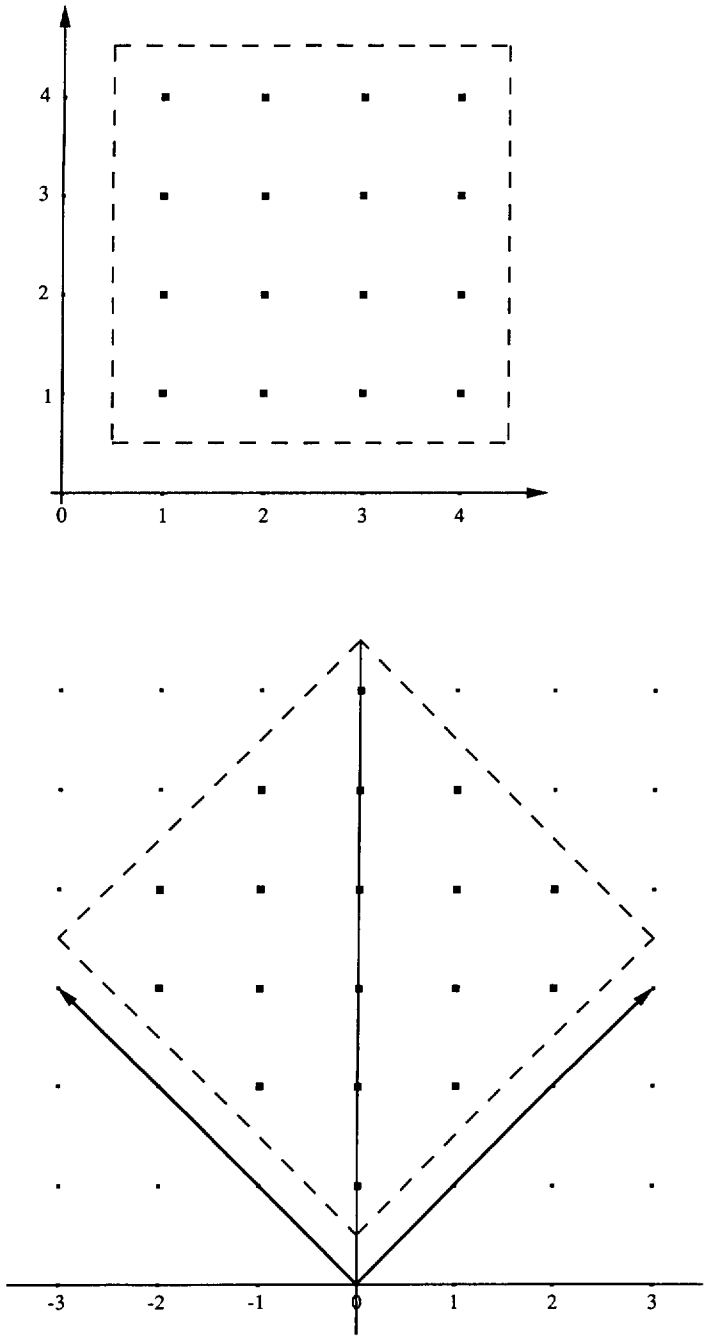

Fig.10. — Lattice sites for the $t - J$ model; upper figure : 16 sites; lower figure : 18 sites.

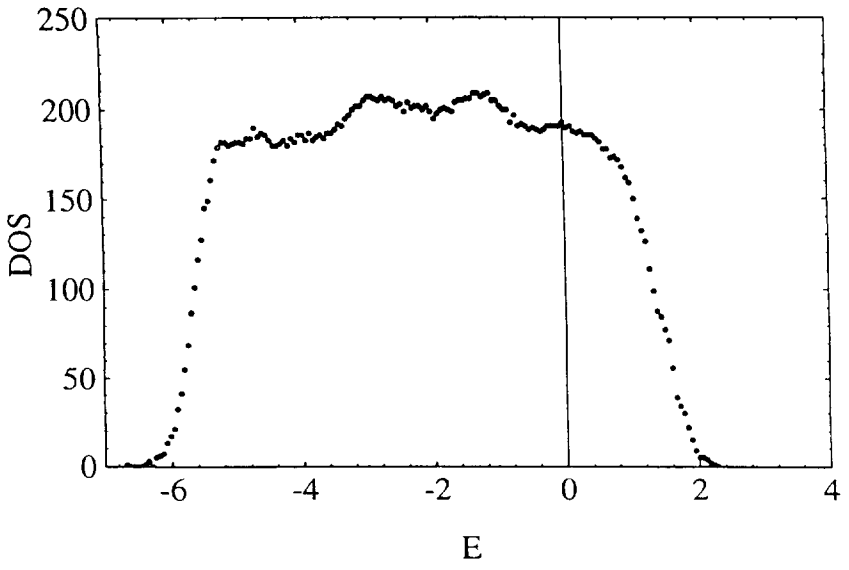

Fig.11. — Density of states of the $t - J_z$ model for $N = 18$ and $J_z = 0.25$ (from [65]).

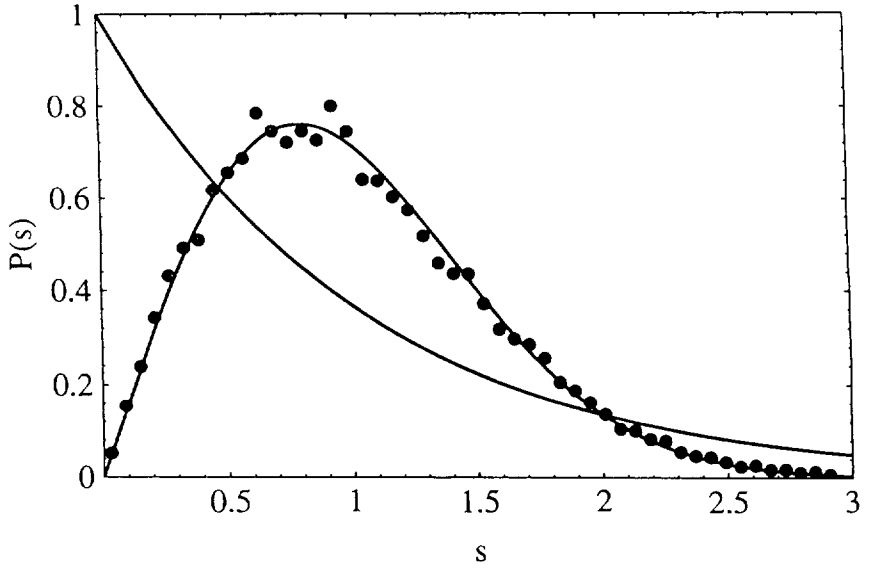

Fig.12. — Level spacing distribution $P(s)$ in the $t - J_z$ model for $N = 18$ and $J_z = 0.25$. Ideal GOE and Poisson distributions are shown as full lines (from [65]).

magnetic field [67] it is known to be integrable, it is unclear whether integrability holds for every magnetic field. Such a test could give an indication.

The next question concerns the GOE character of the result for the $t-J$ model. All matrix elements of the Hamiltonian are obviously real, so that this result should not be surprising. However the approach in terms of slave bosons, flux phases or anyons superconductivity suggests that, at least for low energy excitations, the time reversal symmetry should be broken. We do not see anything like that here. It is however not a sufficient argument, because this breaking of the time reversal symmetry affects only the ground state. For one hole, the Nagaoka theorem [66] requires that the ground state be unique so that time reversal symmetry should not be broken. Moreover, the path integral approach used in quantum Monte Carlo calculations [54] shows that for spin 1/2, no time reversal symmetry breaking is required, namely there is no need to introduce a complex gauge field. Therefore, one does not expect any of the approximation like flux phases or anyons to work unless in trivial case, namely whenever the dimensionless flux is 0 or 1/2 mod 1.

Another important question raised by this result concerns the validity of the quasiparticle approximation in terms of a Fermi liquid theory. Even though quasiparticle peaks have been observed in a $t-J$ model or in the 2D Heisenberg antiferromagnet, if the theory were made of well defined independent quasiparticles we would expect a Poisson distribution to occur. How to reconcile the two points of view ? Presumably, while level repulsion do occur at high enough energy, in the region where the density of states is large, whereas at low energy, elementary excitations may produce level clustering. However this has not been tested yet. If this is so a liquid theory should be only a low energy approximation, and RMT should give the explanation for the incoherent background in Green's functions.

If such an explanation fails, namely if GOE still holds near the groundstate energy, then we ought to find another explanation for the appearance of quasiparticles peaks. We suggest that "scars" [53] may be relevant in such case. We would then have a new kind of excitations, neither bosons nor fermions, but scars conspiring to produce a stable excitation.

Lastly, an important question concerns the conductivity. We conjecture that the linear behaviour of the resistivity with respect to temperature [11] should be a consequence of the random matrix approach. However, again, only low energy excitations are relevant for such calculations. Therefore if RMT holds at small energy and explains the linear law of resistivity, the quasiparticles approach becomes questionnable.

The computation of conductivity requires the knowledge of the matrix elements for the current operator. This involves the locality properties of the Hamiltonian. It means that some basic structure is required to describe the Hamiltonian in terms of random matrices. Like in Quantum Field Theory, translation invariance and locality are essential ingredients there. Randomness will occur only for the part of the Hamiltonian which is not constrained by these two requirements. We are currently investigating this question [14].

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