Non Commutative Methods in Semiclassical Analysis

Jean Bellissard
Laboratoire de Physique Quantique Université Paul Sabatier
118, route de Narbonne F-31062 Toulouse Cedex, France

Contents

1 The kicked rotor problem

One considers a spinning particle submitted to rotate around a fixed axis. Let $\theta \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be its angle of rotation, $L \in \mathbf{R}$ its angular momentum, I its moment of inertia, μ its magnetic moment, and B a uniform magnetic field parallel to the axis of rotation. Its kinetic energy is given by :

$$\mathcal{H}_0 = \frac{L^2}{2I} + \mu B L \ , \tag{1}$$

We assume that this system is kicked periodically in time according to the following Hamiltonian :

$$\mathcal{H} = \frac{L^2}{2I} + \mu B L + k\cos(\theta) \sum_{n\in\mathbf{Z}} \delta(t - nT) \ . \tag{2}$$

where T is the period of the kicks, and k is a coupling constant representing the kicks strength. Here δ is the Dirac measure. Classically the motion is provided by the solution of the Hamilton-Jacobi equations :

$$\frac{d\theta}{dt} = \frac{\partial\mathcal{H}}{\partial L} \qquad \frac{dL}{dt} = -\frac{\partial\mathcal{H}}{\partial\theta} \ . \tag{3}$$

Between two kicks, $\partial\mathcal{H}/\partial\theta = 0$, so that L is constant whereas θ varies linearly in time. When the kick is applied, L changes suddenly according to $L(nT + 0) = L(nT - 0) + k\sin(\theta)$. If we set :

$$A_n = T\left(\frac{L(nT-0)}{I} + \mu B\right) \qquad \theta_n = \theta(nT - 0) \ , \tag{4}$$

the equation of motion can be expressed as :

$$A_{n+1} = A_n + K\sin(\theta_n) \qquad \theta_{n+1} = \theta_n + A_{n+1} \bmod 2\pi \ , \tag{5}$$

where K is the dimensionless coupling strength namely :

$$K = \frac{kT}{I} \ . \tag{6}$$

The phase space is the cylinder $\mathcal{C} = \mathbf{T}\times\mathbf{R}$, if A is considered as a real number. If we set

$$f(\theta, A) = (\theta', A') \qquad \theta' = \theta + A + K\sin(\theta) \qquad A' = A + K\sin(\theta) \ , \tag{7}$$

the solution of the equation of motion can be written as :

$$(\theta_{n+1}, A_{n+1}) = f(\theta_n, A_n) \ . \tag{8}$$

f is an analytic diffeomorphism of the cylinder $\mathcal{C}$, which is area preserving, namely $d\theta' \wedge dA' = d\theta \wedge dA$, and a twist map, namely $\partial\theta'/\partial A > 0$, which preserves the ends (see the course of John Mather in this issue). We remark that f also commutes with the translation $A \mapsto A + 2\pi$ of the action variable A in such a way that it also defines a map of the 2-torus $\mathbf{T}^2$.

The orthodox way of quantizing this model consists in choosing the Hilbert space $\mathcal{K} = L^2(\mathbf{T}, d\theta/2\pi)$ as the state space, and replacing L and θ by operators as follows :

$$\mathbf{L} = \frac{\hbar}{i}\frac{\partial}{\partial\theta} \qquad \mathcal{V} = \text{multiplication by } \mathcal{V}(\theta) \ , \tag{9}$$

whenever $\mathcal{V}$ is a continuous 2π -periodic function of the variable θ . Quantum Mechanics requires using a new parameter $\hbar$, the Planck constant which gives rise to a new dimensionless parameter :

$$\gamma = \frac{\hbar T}{I} = 4\pi \frac{\nu_{\mathrm{QM}}}{\nu_{\mathrm{CL}}} \ , \tag{10}$$

where $\nu_{\mathrm{CL}} = 1/T$ is the kicks frequency, whereas ν_{QM} is the eigenfrequency of the free quantum rotor in a zero magnetic field. To compute the motion, we need to solve Schrödinger's equation, namely, we look for a path $t \in \mathbf{R} \mapsto \psi_t \in \mathcal{K}$ such that :

$$i\hbar\dot{\psi}_t = H(t)\psi_t \qquad H(t) = \frac{\mathbf{L}^2}{2I} + \mu B\mathbf{L} + k\cos(\theta)\sum_{n\in\mathbf{Z}} \delta(t - nT) \ . \tag{11}$$

The δ-kicks may create a technical difficulty. To overcome it let us consider a smooth approximation δ_ϵ of δ given by a non negative L^1-function on $\mathbf{R}$ supported by $[0, \epsilon]$, with integral equal to 1. The solution can be given in term of a convergent Dyson expansion. Then letting ϵ converge to zero, we get the following result (see Appendix 1) :

Theorem 1 *The solution of (11) is given by the following evolution equation :*

$$\psi_{T-0} = F^{-1}\psi_{0-} \qquad F^{-1} = e^{-iA^2/2\gamma}e^{-iK\cos\theta/\gamma}e^{i\hat{y}} \tag{12}$$

where

$$A = T\left(\frac{L}{I} + \mu B\right) \ , \qquad \hat{y} = (\mu B)^2 \frac{TI}{\hbar} \ . \tag{13}$$

Let us also introduce the dimensionless magnetic field x :

$$x = -\mu BT \qquad \Rightarrow \qquad \hat{y} = \frac{x^2}{2\gamma} \ . \tag{14}$$

The operators of the form $\mathcal{V}$ whenever $\mathcal{V}(\theta)$ is a continuous 2π -periodic function of the variable θ, can be obtained as the norm limit of polynomials in the operator

$$U = e^{i\theta} \ . \tag{15}$$

In much the same way, one can quantize the action in the torus geometry by considering the operator :

$$V = e^{-iA} \ . \tag{16}$$

U and V are two unitary operators satisfying the following commutation rule :

$$UV = e^{i\gamma}VU \ . \tag{17}$$

The C^*-algebra generated by these two operators is the non commutative analog of the space of continuous functions on the 2-torus. By analogy with the commutative case, this algebra will be seen as the space of continuous functions on a virtual space, the "quantal phase space". Any such function will be the norm limit of polynomials of the form :

$$a = \sum_{\mathbf{m}\in\mathbf{Z}^2, |m|\leq N} a(\mathbf{m}) U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2/2} , \tag{18}$$

where the $a(m)$'s are complex numbers. We denote by $\mathcal{A}_\gamma$ the norm closure of this algebra. Whenever $\gamma = 0$, this algebra coincides with the space $\mathcal{C}(\mathbf{T}^2)$ of continuous functions on the 2-torus. One remarks that $\cos(\theta) \in \mathcal{A}_\gamma$, but there is no way of writing $F_0 = \exp(iA^2/\gamma)$ as an element of $\mathcal{A}_\gamma$ since it is not periodic with respect to A. Therefore $F_0 \notin \mathcal{A}_\gamma$ in general. However, the following properties hold :

$$\text{(i)} F_0 V F_0^{-1} = V \qquad \text{(ii)} F_0 U F_0^{-1} = U V^{-1} e^{i\gamma/2} , \tag{19}$$

so that, setting $\beta_0(a) = F_0 a F_0^{-1}$ for $a \in \mathcal{A}_\gamma$, β_0 defines an automorphism of $\mathcal{A}_\gamma$, which coincides for $\gamma = 0$ with the free rotation f_0 in $\mathbf{T}^2$, namely :

$$f_0(\theta, A) = (\theta + A, A) , \tag{20}$$

In particular if $\gamma \neq 0$, $a \in \mathcal{A}_\gamma$, we get :

$$\beta(a) = F a F^{-1} = e^{iK\cos(\theta)/\gamma} \beta_0(a) e^{-iK\cos(\theta)/\gamma} \in \mathcal{A}_\gamma , \tag{21}$$

which means that β is an automorphism of $\mathcal{A}_\gamma$.

At last, β admits a classical limit as $\gamma \mapsto 0$, namely the automorphism of $\mathcal{C}(\mathbf{T}^2)$ corresponding to the standard map (see section 3 below). For if $\mathcal{V} = K\cos(\theta)$, let us denote by $\mathcal{L}_v$ the "Liouville operator" defined by :

$$\mathcal{L}_v(a) = \frac{\mathcal{V}a - a\mathcal{V}}{i\gamma} , \tag{22}$$

the limit of $\mathcal{L}_v(a)$ as $\gamma \mapsto 0$ coincides with the Poisson bracket of $\mathcal{V}$ with a, and β can be written as :

$$\beta = e^{-\mathcal{L}_v} \circ \beta_0 . \tag{23}$$

To summarize, we have obtained an algebraic framework describing the quantal observables which is completely analogous to the classical description of the system, and which converges to the classical analog as $\gamma \mapsto 0$. In this framework,

(i) the observable algebra $\mathcal{A}_\gamma$ is the non commutative analog of the space $\mathcal{C}(\mathbf{T}^2)$ of continuous functions on the classical phase space $\mathbf{T}^2$.

(ii) the quantal evolution is described through the automorphism β of $\mathcal{A}_\gamma$ which admits the standard map as a classical limit.

Before leaving this section, let us describe the complementary point of view, given in wave Mechanics by the Feynman path integral, which happens to be exact and finite dimensional in this case.

Lemma 1 *If $\psi \in \mathcal{C}^\infty(\mathbf{T})$, then the following formula holds :*

$$\left(e^{-iA^2/2\gamma}\psi\right)(u) = e^{-i\pi/4} \int_{-\infty}^{+\infty} \frac{du'}{\sqrt{2\pi\gamma}} e^{i(u'-u-x)^2/2\gamma} e^{-ix^2/2\gamma} \psi(u') . \tag{24}$$

Proof : From (9)&(13), we get $A = -i\gamma\partial/\partial\theta - x$. If $\psi \in \mathcal{C}^\infty(\mathbf{T})$, let $(\psi_n)_{n\in\mathbf{Z}}$ its Fourier series, so that :

$$\left(e^{-iA^2/2\gamma}\psi\right)(\theta) = \sum_{n\in\mathbf{Z}} e^{-i(\gamma n-x)^2/2\gamma}\psi_n e^{in\theta} = \sum_{n\in\mathbf{Z}} \int_{-\pi}^{+\pi} \frac{d\theta'}{2\pi} e^{in(\theta-\theta')-i(\gamma n-x)^2/2\gamma}\psi(\theta') \ .$$

To compute the distribution kernel coming into this sum, we use the Poisson summation formula :

$$\sum_{n\in\mathbf{Z}} e^{in(\theta-\theta'+x)-i\gamma n^2/2} = \frac{e^{-i\pi/4}}{\sqrt{2\pi\gamma}} 2\pi \sum_{l\in\mathbf{Z}} e^{i(\theta-\theta'+x+2\pi l)^2/2\gamma} \ .$$

Now we perform the change of variables $u' = \theta' + 2\pi l$, $u = \theta$, and the sum over $l \in \mathbf{Z}$ will give rise to an integral over $\mathbf{R}$ with respect to u', leading to (24).

Using (12)&(24), we immediately get the following Feynman path integral representation :

Corollary 1 *For any $t \in \mathbf{N}$ and $\psi \in \mathcal{C}^\infty(\mathbf{T})$,we get :*

$$\left(F^{-t}\psi\right)(u) = \int_{\mathbf{R}^t} \frac{du_1\cdots du_t}{(2\pi\gamma)^{t/2}} e^{-it\pi/4} e^{i\left(\sum_{s=1}^{t}(u_s-u_{s-1}-x)^2/2-K\cos(u_s)\right)/\gamma}\psi(u_t) \ , \tag{25}$$

where $u_0 = u$, and the right-hand-side defines a convergent oscillatory integral which is periodic of period 2π with respect to u.

Remark : The expression contained in the phase factor

$$S(u_1,\cdots,u_t;u_0,x) = \sum_{1\le s\le t-2} \left(\frac{(u_s-u_{s-1}-x)^2}{2} - K\cos(u_s)\right) \ , \tag{26}$$

is nothing but the "Percival" Lagrangean or the "Frenkel-Kontorova" energy functional used by Aubry and Mather to describe the trajectories of the standard map. For indeed the stationnary points of such a Lagrangean are finite sequences $(u_s)_{1\le s\le t}$ satisfying the recursion relation :

$$2u_s - u_{s+1} - u_{s-1} + K\sin(u_s) = 0 \ , \ (1 \le s \le t-1) \ , \ u_t - u_{t-1} - x + K\sin(u_t) = 0 \ .$$

In particular if we set $p_s = u_s - u_{s-1}$ (for $1 \le s \le t$) we get $u_{s+1} = u_s + p_{s+1}$ for $0 \le s \le t-1$, and $p_{s+1} = p_s + K\sin(u_s)$ for $1 \le s \le t-1$, $x = p_t + K\sin(u_t)$, namely we recover the standard map (5) in $\mathbf{R}^2$ now instead of $\mathbf{T}^2$, for a trajectory $(\theta_0, A_0), \cdots, (\theta_t, A_t)$ such that $\theta_0 = u_0 \bmod 2\pi$, and $A_{t+1} = x \bmod 2\pi$.

2 The Rotation Algebra

2.1 The Polynomial Algebra $\mathcal{P}_I$

In this section we define properly the algebra $\mathcal{A}_\gamma$ and we will describe without proof its most important propeıties. We refer the reader to [BaBeFl] for more details. Actually given an interval I of $\mathbf{R}$, we will rather consider the algebra $\mathcal{A}_I$ which is

roughly speaking the set of continuous sections of the continuous field $\gamma \in I \mapsto \mathcal{A}_\gamma$. The semiclassical limit will be included whenever I contains $\gamma = 0$.
Let I be a compact subset of $\mathbf{R}$. The polynomial algebra $\mathcal{P}_I$ is defined as follows :
- the elements of $\mathcal{P}_I$ are the sequences $(a(\mathbf{m}))_{\mathbf{m}\in\mathbf{Z}^2}$ with finite support, where for each $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}^2$, $a(\mathbf{m}) : \gamma \in I \mapsto a(\mathbf{m}, \gamma) \in \mathbf{C}$ is a complex continuous function on I.
- $\mathcal{P}_I$ admits a natural structure of $\mathcal{C}(I)$-module by setting, for $a, b \in \mathcal{P}_I$, and $l \in \mathcal{C}(I)$:

$$(a+b)(\mathbf{m}) = a(\mathbf{m}) + b(\mathbf{m}) \qquad \lambda a(\mathbf{m};\gamma) = \lambda(\gamma) a(\mathbf{m};\gamma) \ . \tag{27}$$

- any element $a \in \mathcal{P}_I$ admits an adjoint a^* defined by :

$$a^*(\mathbf{m};\gamma) = \overline{a(-\mathbf{m};\gamma)} \ , \tag{28}$$

where $\overline{z}$ denotes the complex conjugate of z in $\mathbf{C}$.
-if $a, b \in \mathcal{P}_I$, their product is defined by :

$$(ab)(\mathbf{m};\gamma) = \sum_{\mathbf{m}'\in\mathbf{Z}^2} a(\mathbf{m}';\gamma) b(\mathbf{m}-\mathbf{m}';\gamma) e^{i\gamma \mathbf{m}'\wedge(\mathbf{m}-\mathbf{m}')} \ , \tag{29}$$

where we have set if $\mathbf{m}', \mathbf{m}'' \in \mathbf{Z}^2$:

$$\mathbf{m}' \wedge \mathbf{m}" = m'_1 m''_2 - m'_2 m''_1 \ . \tag{30}$$

- the topology on $\mathcal{P}_I$, is the direct sum topology obtained from the uniform norm on $\mathcal{C}(I)$.
Denoting by $\mathcal{P}_\gamma$ the algebra $\mathcal{P}_I$ whenever $I = \{\gamma\}$ it follows that $\mathcal{P}_\gamma = \mathcal{P}_{\gamma+4\pi}$. Moreover setting $\alpha(a) = ((-)^{m_1 m_2} a(m))_{\mathbf{m}\in\mathbf{Z}^2}$, α defines a $*$-isomorphism between $\mathcal{P}_\gamma$ and $\mathcal{P}_{\gamma+2\pi}$. Thus, as far as $\mathcal{P}_\gamma$ is concerned, one will consider that γ is defined mod. 2π. The same definition holds if we replace I by the torus $\mathbf{T}$ namely the continuous functions on I by the continuous 2π-periodic functions on $\mathbf{R}$. We will denote by $\mathcal{P}$ the corresponding algebra.

The following elements in $\mathcal{P}_I$ are remarkable :

$$\mathbf{I}(\mathbf{m};\gamma) = \delta_{\mathbf{m},\mathbf{0}} \quad U(\mathbf{m};\gamma) = \delta_{\mathbf{m},(1,0)} \quad V(\mathbf{m};\gamma) = \delta_{\mathbf{m},(0,1)} \ . \tag{31}$$

For indeed, $\mathbf{I}$ is the identity of $\mathcal{P}_I$ whereas U, V, are unitaries namely $UU^* = U^*U = VV^* = V^*V = \mathbf{I}$, and obey to the commutation rules (17). Moreover, $\mathcal{P}_I$ is algebraically generated by U, V as a $\mathcal{C}(I)$-algebra, namely if $a \in \mathcal{P}_I$, it can be written as :

$$a = \sum_{\mathbf{m}'\in\mathbf{Z}^2} a(\mathbf{m}) U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2/2} \ .$$

It will be convenient to introduce the "Weyl operators" as follows :

$$W(\mathbf{m}) = U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2/2} \ . \tag{32}$$

From the interpretation given in the previous section, it follows that $\mathcal{P}_I$ is the set of trigonometric polynomials over the "non-commutative" 2-torus. In particular if $I = \{0\}$, we recover the convolution algebra, which by Fourier transform is exactly the algebra of usual trigonometric polynomials.

The "evaluation" homomorphism η_γ is defined as the map from $\mathcal{P}_I$ into $\mathcal{P}_\gamma$ by :

$$\eta_\gamma(a) = (a(\mathbf{m};\gamma))_{\mathbf{m}\in\mathbf{Z}^2} \ . \tag{33}$$

It is immediate to check that η_γ is a $*$-homomorphism, namely it is linear, and preserves the product and the adjoint.

2.2 Canonical calculus

Using the analogy with the space of trigonometric polynomials on the 2-torus, we now define some rules for the differential calculus.
The integral is given by the trace defined by :

$$\tau(a) = a(0) \in \mathcal{C}(I) \ . \tag{34}$$

We will denote by $\tau_\gamma(a)$ the value of $\tau(a)$ at γ. The trace τ is a linear module map from $\mathcal{P}_I$ into $\mathcal{C}(I)$ satisfying :

(i) positivity : $\tau(a^*a) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} |a(\mathbf{m})|^2 \geq 0$, $a \in \mathcal{P}_I$,
(ii) normalization : $\tau(\mathbf{I}) = 1$,
(iii) trace property : $\tau(ab) = \tau(ba)$, $a, b \in \mathcal{P}_I$.

We remark that the value of $\tau(a)$ at $\gamma = 0$ is the 0^{th} Fourier coefficient of $\eta_0(a)$, namely the integral of its Fourier transform :

$$\tau(a)|_{\gamma=0} = \int_{\mathbf{T}^2} \frac{d\theta dA}{4\pi^2} a_{\text{cl}}(\theta, A) \ . \tag{35}$$

where a_{cl} is the Fourier transform of $\eta_0(a)$.
The angle average, is defined by the element $\langle a \rangle$ in $\mathcal{P}_I$ given by :

$$\langle a \rangle(\mathbf{m}) = \delta_{m_1,0} a(0, m_2) \ . \tag{36}$$

The map $a \mapsto \langle a \rangle$ is a module-map taking values in the commutative subalgebra $\mathcal{D}_I$ generated by V as a $\mathcal{C}(I)$-module. The usual Fourier transform permits to associate with any element b of $\mathcal{D}_I$ a continuous function of $(\gamma, A) \in I \times \mathbf{T}$ denoted by b_{av} as follows :

$$b_{av}(\gamma, A) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} b(0, m_2; \gamma) e^{-im_2 A} \ . \tag{37}$$

The mapping $b \in \mathcal{D}_I \mapsto b_{av} \in \mathcal{C}(I \times \mathbf{T})$, is a $*$-homomorphism, namely $(bc)_{av} = b_{av} c_{av}$ and $(b^*)_{av} = b^*_{av}$. We will say that $b \in \mathcal{D}_I$ is positive whenever b_{av} is positive. Using these definitions, the angle averaging satisfies :

$$\begin{array}{cll} \text{(i)} & \text{positivity property :} & \langle a^* a \rangle \geq 0 \ , \ a \in \mathcal{P}_I \\ \text{(ii)} & \text{projection property :} & \langle \langle a \rangle \rangle = \langle a \rangle \ , \\ \text{(iii)} & \text{normalization :} & \langle \mathbf{I} \rangle = 1 \ , \\ \text{(iv)} & \text{conditional expectation :} & \langle ab \rangle = \langle a \rangle b \ , \ \langle ba \rangle = b \langle a \rangle \ , \text{ if } b \in \mathcal{D}_I \ , \ a \in \mathcal{P}_I \ . \end{array} \tag{38}$$

A differential structure is defined on $\mathcal{P}_I$ through the data of two $*$-derivations ∂_θ and ∂_A given by :

$$(\partial_\theta a)(\mathbf{m}) = im_1 a(\mathbf{m}) \qquad (\partial_A a)(\mathbf{m}) = im_2 a(\mathbf{m}) \ . \tag{39}$$

These two derivations ∂_μ (if $\mu = \theta, A$) actually commute and satisfy :

$$\begin{array}{cll} \text{(i)} & \text{they are } \mathcal{C}(I)\text{-linear} & \\ \text{(ii)} & \partial_\mu(a^*) = (\partial_\mu a)^* & a \in \mathcal{P}_I \ , \\ \text{(iii)} & \partial_\mu(ab) = (\partial_\mu a)\, b + a\, (\partial_\mu b) & a, b \in \mathcal{P}_I \ , \\ \text{(iv)} & \partial_\theta U = iU \ , \partial_\theta V = 0 \ , \partial_A U = 0 \ , \partial_A V = -iV \ . & \end{array} \tag{40}$$

Moreover one can exponentiate them, namely defining by $\{\rho_{\theta,A}; (\theta, A) \in \mathbf{T}^2\}$ as the 2-parameter group of $*$-automorphisms given by :

$$\rho_{\theta,A}(a)(\mathbf{m}) = e^{i(m_1\theta - m_2 A)} a(\mathbf{m}) \ , \tag{41}$$

we get :

$$\partial_\mu a = \left(\frac{\partial \rho_{\theta,A}(a)}{\partial \mu}\right)_{\theta=A=0} \qquad \mu = \theta, A \ . \tag{42}$$

Actually $\rho_{\theta,A}$ is a module-$*$-homomorphism such that $(\theta, A) \in \mathbf{T}^2 \mapsto \rho_{\theta,A}(a) \in \mathcal{P}_I$ is continuous and :

$$\rho_{\theta,A} \circ \rho_{\theta',A'} = \rho_{\theta+\theta',A+A'} \ , \tag{43}$$

If $a, b \in \mathcal{P}_I$ their Poisson (or Moyal [Bou]) bracket $\{a, b\}$ is defined as follows :

$$\{a, b\}(\mathbf{m}; \gamma) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} a(\mathbf{m}'; \gamma) b(\mathbf{m} - \mathbf{m}'; \gamma) \frac{2}{\gamma} \sin\left(\frac{\gamma}{2} \mathbf{m}' \wedge (\mathbf{m} - \mathbf{m}')\right) \ , \tag{44}$$

where we set $(\sin x)/x = 1$ for $x = 0$. In particular that for $\gamma = 0$, it coincides with the usual Poisson bracket, namely :

$$\{a, b\}_{\text{cl}} = \{a_{\text{cl}}, b_{\text{cl}}\} = \partial_\theta a_{\text{cl}} \partial_A b_{\text{cl}} - \partial_A a_{\text{cl}} \partial_\theta b_{\text{cl}} \ , \tag{45}$$

From (44), the right-hand-side defines a continuous function of γ on I, so that the Poisson bracket $\{a, b\}$ still belongs to $\mathcal{P}_I$. The "Liouville operator" associated to $w \in \mathcal{P}_I$ is the module map defined by :

$$L_w(a) = \{w, a\} \ , \ a \in \mathcal{P}_I \ . \tag{46}$$

The properties of this operator are the following :

$$\begin{array}{llll} \text{(i)} & L_w \text{ is } \mathcal{C}(I)\text{-linear} & & \\ \text{(ii)} & L_w(a^*) = L_{w^*}(a)^* & w, a \in \mathcal{P}_I \ , & \\ \text{(iii)} & L_w(ab) = L_w(a)b + aL_w(b) & w, a, b \in \mathcal{P}_I \ , & \\ \text{(iv)} & [L_w, L_{w'}] = L_{\{w,w'\}} \text{ (Jacobi's identity)} & w, w' \in \mathcal{P}_I \ . & \end{array} \tag{47}$$

We also remark that

$$\tau\left(\rho_{\theta,A}(a)\right) = \tau(a) \quad \tau(\{a, b\}) = 0 \quad a, b \in \mathcal{P}_I \ , (\theta, A) \in \mathbf{T}^2 \ , \tag{48}$$

which is equivalent to the "integration by parts formula" :

$$\tau(\partial_\mu a \cdot b) = -\tau(a \cdot \partial_\mu b) \qquad \tau\left(L_w(a) \cdot b\right) = -\tau\left(a \cdot L_w(b)\right) \ , \tag{49}$$

2.3 The Rotation Algebra $\mathcal{A}_I$

In order to get all continuous functions on our non commutative torus, we ought to define the non commutative analog of the uniform topology on $\mathcal{P}_I$. This can be done by remarking that in the commutative case, the uniform topology is defined through a C^*-norm, namely a norm on the algebra which satisfies :

$$\|ab\| \leq \|a\| \|b\| \qquad \|a^* a\| = \|a\|^2 \ . \tag{50}$$

The importance of this relation comes from the fact that such a norm is actually entirely defined by the algebraic structure, namely it is given by the spectral radius of a^*a. Therefore, the algebraic structure is sufficient and the uniform topology becomes natural.
To construct such a norm, one uses the representations of $\mathcal{P}_I$. A "representation" of $\mathcal{P}_I$ is a pair $(\pi, \mathcal{H}_\pi)$, where $\mathcal{H}_\pi$ is a separable Hilbert space, and π is a $*$-homomorphism from $\mathcal{P}_I$ into the algebra $\mathcal{B}(\mathcal{H}_\pi)$ of bounded linear operators on $\mathcal{H}_\pi$. The formulæ (17)&(18) give an example of representation for which $\mathcal{H}_\pi = L^2(\mathbf{T}, d\theta/2\pi)$. In particular $\pi(U)$, $\pi(V)$ will be unitary operators on $\mathcal{H}_\pi$ so that if $a \in \mathcal{P}_I$, one gets (if $\|f\|_I$ denotes the sup norm in $\mathcal{C}(I)$) :

$$\|\pi(a)\| \leq \sum_{\mathbf{m}\in\mathbf{Z}^2} \|a(\mathbf{m})\|_I < \infty \ . \tag{51}$$

Two representations $(\pi, \mathcal{H}_\pi)$ and $(\pi', \mathcal{H}_{\pi'})$ are equivalent whenever there is a unitary operator S from $\mathcal{H}_\pi$ into $\mathcal{H}_{\pi'}$ such that for every $a \in \mathcal{P}_I$:

$$S\pi(a)S^{-1} = \pi'(a) \ . \tag{52}$$

Up to unitary equivalence, one can always assume that $\mathcal{H}_\pi = \ell^2(\mathbf{N})$, so that the family of all equivalence classes of representations of $\mathcal{P}_I$ is a set denoted by $\mathrm{Rep}(\mathcal{P}_I)$. We remark that the norm $\|\pi(a)\|$ depends only upon the equivalence class of π. We then define a seminorm on $\mathcal{P}_I$ by :

$$\|a\|_I = \sup\{\|\pi(a)\|; \pi \in \mathrm{Rep}(\mathcal{P}_I)\} \ . \tag{53}$$

This notation agrees with the sup-norm on $\mathcal{C}(I)$ if $a \in \mathcal{C}(I)$. Then one has [BaBeFl] :

Proposition 1 *The mapping $a \in \mathcal{P}_I \mapsto \|a\|_I \in \mathbf{R}_+$ is a C^*-norm.*

Remark : The only non trivial fact in this statement is that it is a norm, namely that $\|a\|_I = 0$ implies $a = 0$.

Definition 1 *The algebra $\mathcal{A}_I$ (resp. $\mathcal{A}$) is the completion of $\mathcal{P}_I$ (resp. $\mathcal{P}$) under the norm $\|\cdot\|_I$ (resp. $\|\cdot\|_{\mathbf{T}}$). $\mathcal{A}$ is called the "universal rotation algebra".*

Proposition 2 *1)-Any representation of $\mathcal{P}_I$ extends in a unique way to a representation of $\mathcal{A}_I$*
2)-If $\mathcal{B}$ is any C^*-algebra, and β is a $*$-homomorphism from $\mathcal{P}_I$ to $\mathcal{B}$, then β extends in a unique way as a $*$-homomorphism from $\mathcal{A}_I$ to $\mathcal{B}$.
3)-Any pointwise continuous group of $*$-automorphisms of $\mathcal{P}_I$ extends in a unique way as a norm pointwise continuous group of $*$-automorphisms of $\mathcal{A}_I$.
4)-The trace τ and the angle average $\langle\cdot\rangle$ satisfy :

$$\|\tau(a)\|_I \leq \|a\|_I \quad \|\langle a\rangle\|_I \leq \|a\|_I \quad a \in \mathcal{P}_I \ , \tag{54}$$

and therefore they extend uniquely to $\mathcal{A}_I$.
5)-The norm $\|\cdot\|_I$ satisfies :

$$||a||_I = \sup_{\gamma\in I} ||\eta_\gamma(a)|| \quad a \in \mathcal{P}_I \ . \tag{55}$$

In practice the explicit computation of the norm does not require the knowledge of every representation. It is enough to have a faithfull family, namely a family $\{\pi_j\}_{j\in J}$ where J is a set of indices, such that $\pi_j(a) = 0$ for all j's implies $a = 0$. In other words $\cap_{j\in J}\mathrm{Ker}(\pi_j) = \{0\}$. We recall that the spectrum $\mathrm{Sp}(a)$ of an element a of a C^*-algebra with unit $\mathcal{A}$, is the set of complex numbers z such that $z\mathbf{I} - a$ is non invertible in $\mathcal{A}$.

Proposition 3 *Let $(\pi_j)_{j\in J}$ be a faithfull family of representations of the C^*-algebra $\mathcal{A}$, then :*

$$\|a\|_I = \sup_{j\in J} \|\pi_j(a)\| \quad \mathrm{Sp}(a) = \mathrm{closure}\{\cup_{j\in J}\mathrm{Sp}(\pi_j(a))\}\ . \tag{56}$$

In particular if π is faithfull (namely if J contains only one point), $\|a\|_I = \|\pi(a)\|$ and $\mathrm{Sp}(a) = \mathrm{Sp}(\pi(a))$.

2.4 Smooth functions in $\mathcal{A}_I$

Beside $\mathcal{P}_I$, one can define many dense subalgebras of $\mathcal{A}_I$ playing the role of various subspaces of smooth functions.
(i) For $N \in \mathbf{N}$, the algebra $\mathcal{C}^N(\mathcal{A}_I)$ of N-times differentiable elements of $\mathcal{P}_I$ is the completion of $\mathcal{A}_I$ under the norm :

$$\|a\|_{\mathcal{C}^N,I} = \sum_{0\leq n,n';n+n'\leq N} \frac{1}{n!}\frac{1}{n'!}\|\partial_\theta^n\partial_A^{n'}(a)\|_I\ . \tag{57}$$

(ii) $\mathcal{C}^\infty(\mathcal{A}_I) = \cap_{N\geq 0}\mathcal{C}^N(\mathcal{A}_I)$. It coincides with the set of elements $a = (a(\mathbf{m}))_{\mathbf{m}\in\mathbf{Z}^2}$ with rapidly decreasing Fourier coefficients. It is a nuclear space, similar to the Schwartz space on the torus. Its dual space $S(\mathcal{A}_I)$ is a space of non commutative tempered distributions which can be very useful in investigating unbounded elements.
(iii) $\mathcal{H}^s(\mathcal{A}_I)$ is the Sobolev space, namely the completion of $\mathcal{P}_I$ under the Sobolev norm :

$$\|a\|_{\mathcal{H}^s,I} = \left(\tau(a^*a) + \tau(a^*(-\Delta)^{s/2}a)\right)^{1/2} \qquad \Delta = \partial_\theta^2 + \partial_A^2\ , \tag{58}$$

where $-\Delta$ is the Laplacean on the non commutative torus. The imbedding $\mathcal{H}^{s'}(\mathcal{A}_I) \mapsto \mathcal{H}^s(\mathcal{A}_I)$ is compact if $s' > s$ and $\mathcal{C}^\infty(\mathcal{A}_I) = \cap_{s\geq 0}\mathcal{H}^s(\mathcal{A}_I)$, showing that $\mathcal{C}^\infty(\mathcal{A}_I)$ is a nuclear space.
(iv) An element of $\mathcal{A}_I$ is holomorphic in some domain D of $(\mathbf{T}+i\mathbf{R})^2$ if the continuous mapping $(\theta, A) \in \mathbf{T}^2 \mapsto \rho_{\theta,A}(a) \in \mathcal{A}_I$, can be extended as a holomorphic function on D. A special interesting case consists in considering the algebra $\mathcal{A}_I(r)$ for $r > 0$, obtained by completing $\mathcal{P}_I$ with the norm :

$$\|a\|_{r,I} = \sup_{\gamma\in I} \sum_{\mathbf{m}\in\mathbf{Z}^2} |a(\mathbf{m};\gamma)|e^{r|m|_1}\ , \tag{59}$$

where $|m|_1 = |m_1| + |m_2|$. Then $\mathcal{A}_I(r)$ becomes a Banach $*$-algebra of holomorphic elements in the strip $D(r) = \{|\mathrm{Im}\theta| < r\ ,\ |\mathrm{Im}A| < r\}$.
(v) Let us consider now the case for which I is an open interval, and let $\mathcal{P}_I^\infty$ be the subalgebra of $\mathcal{P}_I$ the elements of which have Fourier coefficients given by $\mathcal{C}^\infty$-functions on I. Let us define the operator ∂_γ on $\mathcal{P}_I^\infty$ by :

$$\partial_\gamma a = \left(\frac{\partial a(\mathbf{m})}{\partial\gamma}\right)_{\mathbf{m}\in\mathbf{Z}^2}\ . \tag{60}$$

Then ∂_γ obeys the following rules (Ito's derivative) :

$$\begin{array}{lcl} \text{(i)} & \text{it is linear} & \\ \text{(ii)} & \partial_\gamma(a^*) = (\partial_\gamma a)^* & a \in \mathcal{P}_I^\infty \ , \\ \text{(iii)} & d\tau(a)/d\gamma = \tau(\partial_\gamma a) & a \in \mathcal{P}_I^\infty \ , \\ \text{(iv)} & \partial_\gamma(ab) = (\partial_\gamma a)b + a(\partial_\gamma b) + \frac{i}{2}\left(\partial_\theta a \partial_A b - \partial_A a \partial_\theta b\right) & a, b \in \mathcal{P}_I^\infty \ . \end{array} \tag{61}$$

One can extend ∂_γ to the dense subalgebra $\mathcal{C}^{N,L}(\mathcal{A}_I)$, obtained by completing $\mathcal{P}_I^\infty$ with respect to the norm :

$$\|a\|_{C^{N,L},I} = \mathrm{Max}_{l \leq L} \|\partial_\gamma^l a\|_{C^N,I} \ .$$

Let $\|\cdot\|$ be an algebraic $*$-norm. Then the following norm is also an algebraic $*$-norm.

$$\|a\|_{C^1} = \|a\|_I + \|\partial_\theta a\| + \|\partial_A a\| + \|\partial_\gamma a\| \ . \tag{62}$$

By recursion we will set $\|\cdot\|_{C^N} = (\|\cdot\|_{C^{N-1}})_{C^1}$ with $\|\cdot\|_{C^0} = \|\cdot\|$. It defines then an algebraic $*$-norm on $\mathcal{C}^{N,N}(\mathcal{A}_I)$ equivalent to $\|\cdot\|_{C^{N,N}}$

3 Continuity with respect to Planck's constant

Since the effective Planck constant γ is a tunable physical parameter in many examples, one can wonder whether the various quantities of interest such as mean values of observables, or the evolution, or the spectrum of observables, are continuous functions of γ. The main difficulty in dealing with this problem is that the family of algebras $\gamma \mapsto \mathcal{A}_\gamma$ even though continuous in the sense of Tomiyama [Tom], is not locally trivial. Indeed, $\mathcal{A}_\gamma$ is isomorphic to $\mathcal{A}_{\gamma'}$ if and only if $\gamma = \pm\gamma' \bmod 2\pi$ [Rie, PiVo]. Therefore such continuity properties must be carefully studied.

We will give in this section and again without proofs, three kinds of continuity properties. The first concerns the mean value of observables, namely the function $\tau(a)$ if $a \in \mathcal{A}_\gamma$. One important consequence is the Weyl formula for the semiclassical limit of the density of states. The second type of result concerns the continuity of the evolution. It requires the use of a non commutative analog of the Cauchy-Kovaleskaya theorem. In particular, the semiclassical limit of any time correlation function at fixed time, is equal to the corresponding classical expression. The last type of result is the continuity of the gap edges of the spectrum of any observable. This fact will permit to compute the spectrum numerically (see section 4).

It is to be noticed that the algebra $\mathcal{A}_\gamma$ can be constructed from the algebra of pseudodifferential operators of order zero acting on the unit circle. These results are well known in the context of pseudodifferential calculus. However, it turns out that all proofs given here are purely algebraic, and do not require any explicite reference to the form of the symbol. In particular they are valid for any element in the norm closure. But the closure contains much more than pseudodifferential operators, it also contains Fourier integral operators, and elements with no special behaviour.

3.1 Mean values of observables

Our first result is elementary in view of the definition of the algebra $\mathcal{A}_I$.

Proposition 4 *If $a \in \mathcal{A}_I$, the mapping $\gamma \in I \mapsto \tau_\gamma(a)$ is continuous.*

Proof : If $a \in \mathcal{P}_I$ the result comes from the definition of $\mathcal{P}_I$.If $a \in \mathcal{A}_I$, given $\epsilon > 0$, there is $a_\epsilon \in \mathcal{P}_I$ such that $\|a - a_\epsilon\|_I \leq \epsilon$, and therefore by (54), $\sup_{\gamma \in I} |\tau_\gamma(a) - \tau_\gamma(a_\epsilon)| \leq \epsilon$. Thus $\tau(a)$ is a uniform limit of continuous function on I namely it is continuous.

Let us now consider a self adjoint element $H = H^*$ in $\mathcal{A}_I$ and let Σ be its spectrum. Let f be a continuous function on Σ. Then the map $f \in \mathcal{C}(\Sigma) \mapsto \tau_\gamma(f(H)) \in \mathcal{C}$ is linear positive and bounded. Therefore there is a Radon measure $\mathcal{N}_\gamma$ on $\mathbf{R}$ supported by Σ such that :

$$\tau_\gamma(f(H)) = \int_{\mathbf{R}} d\mathcal{N}_\gamma(E) f(E) \ , \tag{63}$$

This measure is called the "density of states" of H. The "integrated density of states" (IDS) is :

$$N_\gamma(E) = \int_{E' \leq E} d\mathcal{N}_\gamma(E') \ . \tag{64}$$

It is a non decreasing function of $E \in \mathbf{R}$. From the proposition 4, we get [BaBeFl] :

Proposition 5 *If $H = H^* \in \mathcal{A}_I$ let $\mathcal{N}_\gamma$ be the integrated density of states of H. Then if E is a point of continuity of $\mathcal{N}_\gamma$, we get :*

$$\lim_{\gamma' \mapsto \gamma} \mathcal{N}_{\gamma'}(E) = \mathcal{N}_\gamma(E) \ , \tag{65}$$

If I contains $\gamma = 0$, since $\mathcal{A}_0 = \mathcal{C}(\mathbf{T}^2)$ it is easy to check that :

$$\mathcal{N}_0(E) = \int_{-\infty}^{E} d\mathcal{N}_0(E') = \int_{H_{cl}(\theta,A) \leq E} \frac{d\theta dA}{4\pi^2} \ . \tag{66}$$

where H_{cl} is the Fourier transform of $\eta_0(H)$. Thus $\mathcal{N}_0(E)$ is the area of the set $H_{cl}^{-1}(-\infty, E)$ in the 2-torus. A consequence of the Proposition 4 is the following :

Corollary 2 *If I contains $\gamma = 0$ and $H = H^* \in \mathcal{A}_I$ let $\mathcal{N}_\gamma$ be the integrated density of states of H. Then if E is a real number such that the level set $H_{cl}^{-1}(E)$ has zero Lebesgue measure in the 2-torus, we get :*

$$\lim_{\gamma \mapsto 0} \mathcal{N}_\gamma(E) = \mathcal{N}_0(E) \ , \text{ (Weyl's formula) } . \tag{67}$$

Let us also mention the following non trivial result [BaBeFl] :

Proposition 6 *If $H = H^* \in \mathcal{P}_I$, then its integrated density of states $\mathcal{N}_\gamma$ is continuous with respect to E for any $\gamma \in I$.*

3.2 The time evolution

Our next result concerns the continuity of the evolution with respect to γ. Let $w \in \mathcal{P}_I$, and let us consider the automorphism of $\mathcal{A}_\gamma$ (for $\gamma \neq 0$) given by :

$$\beta_\gamma(a) = e^{-i\eta_\gamma(w)/\gamma} a e^{i\eta_\gamma(w)/\gamma} , \tag{68}$$

Is it possible to prove that β can be continued at $\gamma = 0$ in such a way as to define an automorphism of $\mathcal{A}_I$? To show that it is actually possible, let us consider the algebra $\mathcal{A}_I(r)$ introduced in 2.4 with the norm defined by (58). Then we get [BeVi]:

Theorem 2 *Let $w = w^*$ be an element of $\mathcal{A}_I(r)$ where I is an interval containing $\gamma = 0$. Then for any ρ such that $0 < \rho < r$,*
(i) the Liouville operator L_w associated to w is well defined as a bounded linear operator from $\mathcal{A}_I(r)$ into $\mathcal{A}_I(r-\rho)$,
(ii) for t small enough, $\exp(tL_w)$ defines a linear bounded operator from $\mathcal{A}_I(r)$ into $\mathcal{A}_I(r-\rho)$,
(iii) $\exp(tL_w)$ can be extended as a $*$-automorphism of $\mathcal{A}_I$ for any $t \in \mathbf{R}$, in such a way as to satisfy :

$$\frac{de^{tL_w}(a)}{dt} = e^{tL_w}\left(L_w(a)\right) , \tag{69}$$

for any $a \in \mathcal{A}_I(r)$.

To prove this result, we will proceed in several steps. First of all :

Lemma 2 *If $w \in \mathcal{A}_I(r_0)$ and $a \in \mathcal{A}_I(r)$, $(r < r_0)$, then for any ρ such that $0 < \rho < r$ one has :*

$$\|\{w, a\}\|_{r-\rho} \leq \frac{2\|w\|_{r_0}\|a\|_r}{e^2\rho(r_0 - r - \rho)} . \tag{70}$$

Proof : From (44), using the inequalities $|\sin(x)| \leq |x|$, $|\mathbf{m}| \leq |\mathbf{m}'| + |\mathbf{m} - \mathbf{m}'|$ whenever $\mathbf{m}, \mathbf{m}' \in \mathbf{Z}^2$ and $|\mathbf{m}' \wedge \mathbf{m}"| \leq |m_1'||m"_2| + |m_2'||m"_1|$ we get :

$$\begin{aligned} \|\{w, a\}\|_{r-\rho}\| \leq \sup\nolimits_{\gamma\in I} \textstyle\sum_{m',m''\in\ Z^2} e^{r_0|m|}|w(m',\gamma)|e^{r\mathbf{m}''}|a(\mathbf{m}'',\gamma) \\ \cdots e^{-(r_0-r+\rho)|\mathbf{m}'|-\rho|\mathbf{m}''|}\left(|m_1'||m_2''| + |m_2'||m_1''|\right) . \end{aligned} \tag{71}$$

The inequality (70) will be obtained by using the estimate $\sup_{n\in\mathbf{Z}} |n|e^{-\rho|n|} = 1/e\rho$.

Lemma 3 *If $w \in \mathcal{A}_I(r)$ for any r such that $0 < \rho < r$ and any $n \in \mathbf{N}$ one has :*

$$\left\|\frac{L_w^n}{n!}\right\|_{r\mapsto r-\rho} \leq \left(\frac{2\|w\|_r}{\rho^2}\right)^n . \tag{72}$$

Proof : One can write

$$\left\|\frac{L_w^n}{n!}\right\|_{r\mapsto r-\rho} \leq \frac{1}{n!}\prod_{k=1}^{n} \|L_w\|_{r-\rho_{k-1}\mapsto r-\rho_k} ,$$

for any family $(\rho_k)_{1\leq k\leq n}$ such that $\rho_0 = 0 < \rho_1 < \cdots < \rho_{n-1} < \rho_n = \rho$. Using the inequality (70), we get :

$$\left\|\frac{L_w^n}{n!}\right\|_{r\mapsto r-\rho} \leq \frac{1}{n!}\left(\frac{2}{e^2}\right)^n \|w\|_r^n \prod_{k=1}^{n} \frac{1}{\rho_k(\rho_k-\rho_{k-1})} \ .$$

Let us choose $\rho_k = \rho k/n$; since $n^n e^{-n} \leq n!$, we immediately get the result.

Proof of theorem 2 : the point (i) is exactly the content of the Lemma 2. To prove (ii), it follows from the Lemma 3 that if $|t| < \rho^2/(2\|w\|_r) = T$, the expansion for $\exp(tL_w)$ in powers of t converges in norm as an operator from $\mathcal{A}_I(r)$ into $\mathcal{A}_I(r-\rho)$ and in addition :

$$\|e^{tL_w}\|_{r\mapsto r-\rho} \leq \frac{1}{1-2|t|\|w\|_r/\rho^2} \ . \tag{73}$$

Proving (iii) is more subtle : if we set $\beta_t = e^{tL_w}$, we observe that since L_w is a $*$-derivation, by (47), then, if $a, b \in \mathcal{A}_I(r)$:

$$\begin{array}{lrl} \text{(i)} & \beta_t(ab) = \beta_t(a)\beta_t(b) & \text{for } |t| < T \ , \\ \text{(ii)} & \beta_t(a^*) = \beta_t(a)^* & \text{for } |t| < T \ , \\ \text{(iii)} & \beta_{t+s}(a) = \beta_t\left(\beta_s(a)\right) & \text{for } |t| + |s| < T \ , \\ \text{(iv)} & d\beta_t(a)/dt = \beta_t(L_w(a)) & \text{for } |t| < T \ . \end{array} \tag{74}$$

Therefore given any representation π of $\mathcal{P}_I$, π can be extended as a representation of $\mathcal{A}_I$ thus of $\mathcal{A}_I(r)$. In particular, $\pi \circ \beta_t$ gives also a representation of $\mathcal{A}_I(r)$, so that by the same type of argument used in 2.3 (see (51)&(53)), one gets $\|\pi \circ \beta_t(a)\|_I \leq \|a\|_I$, and since π is arbitrary :

$$\|\beta_t(a)\|_I \leq \|a\|_I \ , \text{ for } |t| < T. \tag{75}$$

In particular, β_t can be extended by continuity to $\mathcal{A}_I$ and the extension still satisfies (74). Now if $t \in \mathbf{R}$, let n be a positive integer such that $|t/n| < T$. Then we set $\beta_t = \left(\beta_{t/n}\right)^n$. Thanks to (74) (iii), it is standard to check that this definition does not depend upon the choice of n. Moreover, (74) continues to hold at any value of t : this is obvious for (i), (ii), (iii). (iv) also holds once one notices that L_w commutes with β_t. Therefore $(\beta_t)_{t\in\mathbf{R}}$ defines a 1-parameter group of $*$-automorphisms of $\mathcal{A}_I$. At last it is norm-pointwise continuous, namely :

$$\lim_{t\mapsto 0} \|\beta_t(a) - a\|_I = 0 \ , \tag{76}$$

For indeed, by a 3ϵ-argument, it is enough to check it for $a \in \mathcal{A}_I(r)$, which is simply a consequence of the Lemma 3.

3.3 The spectrum of observables

Our last result concerns the continuity of the spectrum with respect to γ. Let $(\Sigma(t))_{t\in\mathbf{R}}$ be a family of compact subsets of a topological space X. This family is continuous at $t = t_0$ if the two following properties hold :

(i) it is continuous from the outside, namely given any closed set F in X, such that $\Sigma(t_0) \cap F = \emptyset$, there i $\delta > 0$, such that if $|t - t_0| \leq \delta$, then $\Sigma(t) \cap F = \emptyset$.

(ii) it is continuous from inside, namely given any open set O in X, such that $\Sigma(t_0) \cap O \neq \emptyset$, there is $\delta > 0$, such that if $|t - t_0| \leq \delta$, then $\Sigma(t) \cap O \neq \emptyset$.

If $X = \mathbf{R}$ a gap of $\Sigma(t)$ is a connected component of $\mathbf{R} - \Sigma(t)$. One can check that this definition is equivalent to the continuity of the gap edges of $\Sigma(t)$ at t_0.
For $a \in \mathcal{A}_I$ we set $\Sigma(\gamma) = \mathrm{Sp}(\eta_\gamma(a))$, whenever $\gamma \in I$. The main result of this subsection is [BaBeFl] :

Theorem 3 *For any normal element $a \in \mathcal{A}_I$, (namely such that $aa^* = a^*a$), the family $(\Sigma(\gamma))_{\gamma\in I}$ is continuous at every point of I.*

The proof of this theorem will not be given here. It can be found in [BaBeFl]. However, it is of very high importance in view of the numerical computation of the spectrum. For we will see in the next section that the spectrum can be easily computed on a computer for rational values of $\gamma/2\pi$. The continuity of the gap edges everywhere on I implies that this type of computation is sufficient to get an idea of the spectrum for irrational values of $\gamma/2\pi$.

Actually for smooth self adjoint elements of $\mathcal{A}_I$ one gets a better result [BaBeFl], namely :

Theorem 4 *For any self adjoint element $H \in \mathcal{C}^{3,1}\mathcal{A}_I$, the gap edges of any open gap of $(\Sigma(\gamma))_{\gamma\in I}$ are Lipshitz continuous at every point of I.*

Similar but weaker results have already been obtained previously by Choi et al. [ChElYu], and by Avron et al. [AvSi] on the almost Mathieu model. They found Hölder continuity only. Here we get a stronger result. However the Lipshitz constant depends explicitely of the width of the gap considered, and it diverges whenever the width tends to zero.

As we will see in section 4 below, there is no chance to get a better result because the gap edges have discontinuous derivative at each rational value of $\gamma/2\pi$. On the other hand, if a gap closes for some value of γ then generically with respect to $H \in \mathcal{C}^{3,1}\mathcal{A}_I$, we only get Hölder continuity with exponent $1/2$ near this point.

4 Structure of the Rotation Algebra $\mathcal{A}_I$

In this section we will give without proofs, a description of the structure of the rotation algebra. The reader interested in the proofs will be refered to [BaBeFl].

Let us consider first the case $\gamma = 0$. The algebra $\mathcal{P}_0$ is then the convolution algebra associated to the group $\mathbf{Z}^2$. Therefore by Fourier transform, one transforms it into the algebra of trigonometric polynomials with the pointwise multiplication. More precisely, if $a \in \mathcal{P}_0$, we set :

$$a_{\mathrm{cl}}(\theta, A) = \sum_{\mathbf{m}\in\mathbf{Z}^2} a(\mathbf{m}) e^{i(\theta m_1 - A m_2)} \ . \tag{77}$$

This a trigonometric polynomial. The main properties of the Fourier transform are :

$$(ab)_{\mathrm{cl}}(\theta, A) = a_{\mathrm{cl}}(\theta, A) b_{\mathrm{cl}}(\theta, A) \ , \ a, b \in \mathcal{P}_0 \ , \tag{78}$$

and

$$(a^*)_{\rm cl}(\theta, A) = a_{\rm cl}(\theta, A)^* \ , \ a \in \mathcal{P}_0 \ . \tag{79}$$

It follows that for every $(\theta, A) \in \mathbf{T}^2$, the map $a \in \mathcal{P}_0 \mapsto a_{\rm cl}(\theta, A) \in \mathbf{C}$ is a representation. Therefore :

$$\sup_{(\theta,A)\in\mathbf{T}^2} |a_{\rm cl}(\theta, A)| \leq \|a\|_0 \ , \ a \in \mathcal{P}_0 \ . \tag{80}$$

In particular, the Fourier transform $a \in \mathcal{P}_0 \mapsto a_{\rm cl} \in \mathcal{C}(\mathbf{T}^2)$, extends to $\mathcal{A}_0$ as a $*$-homomorphism. As a consequence of the Gelfand theorem we get the following result [BaBeFl] :

Theorem 5 *The Fourier transform $a \in \mathcal{P}_0 \mapsto a_{\rm cl} \in \mathcal{C}(\mathbf{T}^2)$, extends as a $*$-isomorphism from $\mathcal{A}_0$ to $\mathcal{C}(\mathbf{T}^2)$.*

Let us now consider the case $\gamma = 2\pi p/q$ where p, q are positive integers prime to each others. As we have seen in section 2, $\mathcal{A}_\gamma$ is isomorphic to $\mathcal{A}_{\gamma+2\pi}$, so that one can assume that $0 < p < q$ without loss of generality. One can extend the previous analysis to this case by introducing two $q \times q$ unitary matrices u and v satisfying :

$$u^q = \mathbf{I} = v^q \ , \qquad uv = e^{2i\pi p/q} vu \ . \tag{81}$$

The Fourier transform of $a \in \mathcal{P}_\gamma$ is then given by the following matrix valued function :

$$a_{\rm cl}(\theta, A) = \sum_{\mathbf{m}\in\mathbf{Z}^2} a(\mathbf{m}) e^{i(\theta m_1 - A m_2)} w(\mathbf{m}) \ , \tag{82}$$

where :

$$w(\mathbf{m}) = u^{m_1} v^{m_2} e^{-i\pi p m_1 m_2/q} \ . \tag{83}$$

We remark that in this last expression, $w(\mathbf{m} + q\mathbf{m}') = w(\mathbf{m})$, namely $\mathbf{m}$ is defined modulo q.
An example of such pair of matrices is given by :

$$u = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \ , \ v = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda^{q-2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda^{q-1} \end{bmatrix} \ , \tag{84}$$

where $\lambda = e^{2i\pi p/q}$. Actually, any such pair is unitarily equivalent to this latter one. Let us also set :

$$w'(m_1, m_2) = w(-m_2, m_1) \ , \tag{85}$$

to get the following characterization of $\mathcal{A}_{2\pi p/q}$ [BaBeFl] :

Theorem 6 *The Fourier transform $a \in \mathcal{P}_{2\pi p/q} \mapsto a_{\rm cl} \in \mathcal{C}(\mathbf{T}^2) \otimes M_q$, extends as a $*$-isomorphism from $\mathcal{A}_{2\pi p/q}$ to the subalgebra $\mathcal{C}_{\rm cov}(\mathbf{T}^2, q)$ of $\mathcal{C}(\mathbf{T}^2) \otimes M_q$, the element of which being continuous functions $a_{\rm cl}$ from $\mathbf{T}^2$ into M_q satisfying the covariance condition*

$$w'(\mathbf{m}) a_{\rm cl}(\theta, A) w'(\mathbf{m})^{-1} = a_{\rm cl}((\theta, A) + 2\pi \mathbf{m} p/q) \ . \tag{86}$$

The main interest of this result is that it makes it possible to compute the spectrum of an element $a \in \mathcal{A}_{2\pi p/q}$. For indeed for every $(\theta, A) \in \mathbf{T}^2$, the map $\pi(\theta, A) : a \in \mathcal{A}_{2\pi p/q} \mapsto a_{\rm cl}(\theta, A) \in M_q$ is a representation, and the family $\{\pi(\theta, A); (\theta, A) \in \mathbf{T}^2\}$ is faithfull. Therefore, thanks to prop.3, denoting by $e_k(\theta, A)$, $1 \leq k \leq q$, the eigenvalues of $a_{\rm cl}(\theta, A)$, the spectrum of a is :

$$\mathrm{Sp}(a) = \cup_{1 \leq k \leq q} \mathrm{Im}(e_k) \ , \ \mathrm{Im}(e_k) = \{e_k(\theta, A) \in \mathbf{C}; (\theta, A) \in \mathbf{T}^2\} \ . \tag{87}$$

Each set $\mathrm{Im}(e_k)$ is called a "band". The computation of the eigenvalues can be done numerically by matrix diagonalization. In many important examples, such as the "Harper" model (see Fig. 1) given by :

$$H_{\rm Harper} = U + U^* + V + V^* \ , \tag{88}$$

it is possible to compute analytically the points in the 2-torus for which the band edges are reached. In these cases, the numerical computation of the spectrum requires to diagonalize only few matrices for each value of p/q [BaBeFl].

The theorems 1 and 5 can be rephrased by characterizing the set of (closed two-sided) ideals of $\mathcal{A}_{2\pi p/q}$. If $p = 0$, any ideal J is given by the space of continuous functions on $\mathbf{T}^2$ vanishing on some closed subset Ω_J of $\mathbf{T}^2$. The map $J \mapsto \Omega_J$ is actually one-to-one. If $p \neq 0$, the same is true if we demand that Ω_J be invariant by the translations of period $2\pi/q$ in the 2-torus [BaBeFl]. However If $\gamma/2\pi$ is irrational, we get the following result [Sla, BaBeFl] :

Theorem 7 *If $\gamma/2\pi$ is irrational the algebra $\mathcal{A}_\gamma$ is simple, namely there is no other ideal than $\{0\}$ and $\mathcal{A}_\gamma$ itself.*

Corollary 3 *If $\gamma/2\pi$ is irrational every representation of the algebra $\mathcal{A}_\gamma$ is faithfull .*

A nice proof of theorem 6 was provided by Slawny [Sla], and the reader will find it in [BaBeFl]. The corollary is an immediate consequence of that theorem, for if π is a representation of the algebra $\mathcal{A}_\gamma$ its kernel is an ideal, namely it is either the algebra $\mathcal{A}_\gamma$, in which case $\pi = 0$, which is not possible since $\pi(\mathbf{I}) = 1$, or it vanishes, namely π is faithfull.

Thanks to this last result, we can choose any representation to produce explicit calculations. The three theorems of this section are sufficient to characterize the algebra $\mathcal{A}_I$ for any compact subset I of $\mathbf{R}$. For indeed thanks to (55) (prop. 2), J is an ideal of $\mathcal{A}_I$ if and only if for any $\gamma \in I$, $\eta_\gamma(J)$ is an ideal of $\mathcal{A}_\gamma$. The ideal structure is sufficient to characterize any C^*-algebra $\mathcal{B}$ which is a homomorphic image of $\mathcal{A}_I$.

5 Semiclassical asymptotics for the spectrum

In this section we will denote by $H = H^*$ a selfadjoint element of $\mathcal{A}_I$, and we intend to given a description of its spectrum when I is a small open interval around $\gamma = 2\pi p/q$. The same kind of results can be obtained for a unitary operator, and this will be left as an exercise to the reader.

5.1 2D lattice electrons in a magnetic field

In this subsection, we will describe a physical situation where the rotation algebra enters as an essential tool. It will give us a different intuitive description of the problem which may be useful.

Let us consider a two dimensional lattice, with lattice spacing δ, that we will identify with $\mathbf{Z}^2$, on which charged particles like electrons or holes, are supposed to move. Their quantum states will be wave functions $\psi = (\psi(\mathbf{m}))_{\mathbf{m}\in\mathbf{Z}^2} \in \ell^2(\mathbf{Z}^2)$. We suppose in addition that a uniform magnetic field B is applied on this lattice, perpendicularly to the plane of the lattice. Let $\mathbf{A} = (A_1, A_2)$ be the corresponding vector potential, namely a vector field on $\mathbf{R}^2$ solution of the equation $\partial_1 A_2 - \partial_2 A_1 = B$. In the Hilbert space $\ell^2(\mathbf{Z}^2)$, we consider the "magnetic translations" T_1, T_2 studied by Zak [Zak], associated to the two basis vectors $e_1 = (1,0)$ and $e_2 = (0,1)$ of $\mathbf{Z}^2$, namely the unitary operators defined by :

$$T_\mu \psi(\mathbf{m}) = e^{2i\pi \frac{e}{h} \int_{(\mathbf{m}-e_\mu)\delta}^{\mathbf{m}\delta} d\mathbf{l}\cdot\mathbf{A}} \psi(\mathbf{m} - e_\mu) \ , \qquad \mu = 1, 2 \ , \tag{89}$$

where e is the electric charge of the particle, h is the Planck constant and the integral in the phase factor is computed along the segment joining the sites $\mathbf{m} - e_\mu$ and $\mathbf{m}$. These two operators satisfy the following commutation rule :

$$T_1 T_2 = e^{2i\pi\phi/\phi_0} T_2 T_1 \ , \tag{90}$$

where ϕ is the magnetic flux through the unit cell, and $\phi_0 = h/e$ is the flux quantum. Therefore these two operators generate a representation of the C^*-algebra $\mathcal{A}_\gamma$ where now

$$\gamma = 2\pi\phi/\phi_0 = \text{const.} B \ , \tag{91}$$

is proportional to the magnetic field. In practice, if the lattice is given by the positions of the ions of a metal, δ is of the order of 1 Å so that even with the highest kind of magnetic field that can be produced in laboratories, namely $B \approx 18$ Teslas , we get $\gamma/2\pi \approx 0.5\ 10^{-4}$ which is fairly small, and shows that in this situation a semiclassical approximation will always be valid. However during the last ten years, networks with lattice spacings of the order of the micrometer have been built [PaChRa], leading to values of $\gamma/2\pi$ of the order of unity in magnetic fields not larger than 40 Gauss. This is why it has been necessary to go beyond the semiclassical regime.

¿From the band theory of metals [MeAs], the conduction properties are given only by those electrons sitting in the conduction bands, namely with energies within an interval of order $k\Theta$ from the Fermi level, if Θ denotes here the temperature, and k the Boltzmann constant. If we assume for simplicity that there is only one such band, thanks to the so-called "Peierls substitution" [Pei], one can prove rigorously that the restriction H of the Hamiltonian to that band is given by a selfadjoint element of the C^*-algebra generated by the two magnetic translations [Bel:Eva]. If several bands have to be considered, the Hamiltonian will be represented by a matrix with entries in this algebra. For $B = 0$, the band Hamiltonian H is represented through its Fourier transform (see section 4) by a continuous function H_{cl} on $\mathbf{T}^2$. In Solid State Physics, one usually uses the quasimomentum notation $\mathbf{k} = (k_1, k_2)$ instead of (θ, A) to represent a point in this 2-torus. Thus we get the following correspondence :

$$(T_1)_{\text{cl}} = e^{ik_1} \ , \ (T_2)_{\text{cl}} = e^{ik_2} \ , \text{if} B = 0 \ . \tag{92}$$

The advantage of this latter notation is that it restores the symmetry between the two directions in the lattice, a natural fact in the present context, even though it does not look so natural in the kicked rotor problem. In this subsection, we will prefer the use of the quasimomentum notations instead of the action-angle ones.

Let us now remark that the representation given by (89) is actually a very natural one from algebraic point of view. For indeed, thanks to subsection 2.2 (34&(35), the Hilbert space $\ell^2(\mathbf{Z}^2)$ can be seen as the completion $L^2(\mathcal{A}_\gamma, \tau)$ of the prehilbert space $\mathcal{P}_\gamma$ endowed with the scalar product $\langle a|b\rangle = \tau(a^*b)$. Let η be the natural imbedding of $\mathcal{P}_\gamma$ into $L^2(\mathcal{A}_\gamma, \tau)$. Since $\tau(a^*a) \leq \|a^*a\|_\gamma$, η can be extended to $\mathcal{A}_\gamma$ by continuity. Then let π_{GNS} be the representation of $\mathcal{A}_\gamma$ on this Hilbert space given by the left multiplication, namely

$$\pi_{\text{GNS}}(a)\eta(b) = \eta(ab) \ , \ a, b \in \mathcal{A}_\gamma \ . \tag{93}$$

The name "GNS" refers to Gelfand-Naimark-Segal [Dix], who defined and studied this representation in a C^*-algebra. Then we claim the following [BaBeFl]:

Theorem 8 *The representation of $\mathcal{A}_\gamma$ given by the magnetic translations in (89) is unitarily equivalent to the GNS representation relative to the trace of $\mathcal{A}_\gamma$. This representation is faithfull for any values of γ.*

Actually, the representation given by the magnetic translations depends upon the choice of a vector potential. The GNS representation corresponds to the so-called "symmetric gauge", namely $\mathbf{A} = B(-x_2, x_1)$. Every other gauge can be reached by a unitary transformation.

5.2 Low field expansion

We now consider an interval I of the form $I = [-\epsilon_0, \epsilon_0]$, for some $\epsilon_0 > 0$, and let $H = H^*$ belong to $\mathcal{A}_I$. We will describe a semiclassical expansion near a bottom well. In order to do so, let us assume that H_{cl} admits a local minimum or a local maximum at $\mathbf{k} = \mathbf{k}_0$. Moreover we will assume that this extremum is regular which is a generic property. More precisely, and without loss of generality we will assume :

(H0) $H = H^* \in C^N(\mathcal{A}_I)$ for $N > 2$, and all its Fourier coefficients are N-times differentiable with respect to γ.
(H1) H_{cl} admits a local minimum at $\mathbf{k}_0 = (0,0)$ and with $H_{\text{cl}}(0,0) = 0$.
(H2) The Hessian $D^2H_{\text{cl}}(0,0)$ of H_{cl} at $\mathbf{k} = (0,0)$ is a positive definite 2×2 matrix.

Our goal is to describe the spectrum of $\eta_\gamma(H)$ near the energies E close to $H_{\text{cl}}(0,0) = 0$ for $\gamma \in I$. To describe the result, let us assume that H can be written as :

$$\eta_\gamma(H) = \sum_{\mathbf{m}\in\mathbf{Z}^2} h(\mathbf{m};\gamma) W_\gamma(\mathbf{m}) \ . \tag{94}$$

with $h(\mathbf{m};\gamma)^* = h(-\mathbf{m};\gamma)$. Since H is smooth, one can check that this series converges absolutely in norm, so that this expansion is meaningfull. Let us introduce the following function :

$$H_{\text{scl}}(\mathbf{k};\gamma) = \sum_{\mathbf{m}\in\mathbf{Z}^2} h(\mathbf{m};\gamma) e^{i(m_1k_1+m_2k_2)} \ , \ \mathbf{k} \in \mathbf{T}^2 \ , \ \gamma \in I \ , \tag{95}$$

which coincides with $H_{\rm cl}$ for $\gamma = 0$. Then we get the following result [Bel:Eva, BaBeFl] :

Theorem 9 *Let H satisfy (H0),(H1)&(H2), and let $H_{\rm scl}$ be defined by (95). Then there are $\delta > 0$ and $0 < \epsilon \leq \epsilon_0$ such that if $|\gamma| \leq \epsilon$, the set $\mathrm{Sp}(\eta_\gamma(H)) \cap (-\delta, +\delta)$ is contained in the union over $n \in \mathbf{N}$ of the intervals $J_n(\gamma) = [E_n(\gamma) - \Delta(\gamma), E_n(\gamma) + \Delta(\gamma)]$ where $E_n(\gamma)$ admits a Taylor expansion in γ up to the $N^{\rm th}$ order of the form :*

$$E_n(\gamma) = |\gamma| \left(n + \frac{1}{2}\right) \left(\det D^2 H_{\rm cl}(0,0)\right)^{1/2} + \gamma \left(\frac{\partial H_{\rm scl}}{\partial \gamma}\right)_{\gamma=0,\mathbf{k}=\mathbf{0}} + \cdots + \mathrm{O}(\gamma^N) \ , \quad (96)$$

$$0 < \Delta(\gamma) \leq \mathrm{const.}|\gamma|^{N'} \ , \text{ for some } N' > N \ . \quad (97)$$

To understand more intuitively this result let us introduce a faithfull representation π of $\mathcal{A}_\gamma$ in a Hilbert space in which one can find two selfadjoint operators K_1, K_2 such that $[K_2, K_1] = i\mathrm{sgn}(\gamma)$, where $\mathrm{sgn}(x)$ denotes the sign of the real number x. That such a representation exists is a well known fact [BaBeFl], and is a consequence of the Weyl theorem on the canonical commutation relations. In this representation, one has :

$$\pi(U) = e^{i|\gamma|^{1/2}K_1} \ , \ \pi(V) = e^{i|\gamma|^{1/2}K_2} \ , \quad (98)$$

Therefore we get from (94) :

$$H_\gamma = \pi \circ \eta_\gamma(H) = \sum_{\mathbf{m} \in \mathbf{Z}^2} h(\mathbf{m}; \gamma) e^{i|\gamma|(m_1 K_1 + m_2 K_2)} \ , \ \gamma \in I \ . \quad (99)$$

Let us expand this expression formally in powers of $|\gamma|^{1/2}$ to obtain :

$$H_\gamma = \gamma \partial_\gamma H_{\rm scl}(\mathbf{0}; 0) + \frac{1}{2}|\gamma| \partial_\mu \partial_\nu H_{\rm scl}(\mathbf{0}, 0) K_\mu K_\nu + \mathrm{O}\left(|\gamma|^{3/2}\right) \ , \quad (100)$$

where we have used the Einstein convention on the repeated indices (here $\mu, \nu \in \{1, 2\}$). By a unitary transformation, the quadratic term can be transformed into $\omega(K_1^2 + K_2^2)/2$ where ω is the determinant of the Hessian matrix $\partial_\mu \partial_\nu H_{\rm scl}(\mathbf{0}, 0)$. We recognize here the Hamiltonian of a harmonic oscillator. Actually, if we choose the representation corresponding to a 2D free electron in a uniform magnetic field, namely the Hilbert space is $L^2(\mathbf{R}^2)$, and $K_\mu = \mathrm{const.}(P_\mu - eA_\mu)$ for some physical constant, then this Hamiltonian is the Landau one, namely the Hamiltonian describing a free electron in a uniform magnetic field. For this reason, the energy levels E_n are called the "Landau levels" and are equal to that order in γ to $\omega(n + 1/2)$, leading to the expression (96).

The proof of this theorem can be found in [BaBeFl, Bel:Eva]. The calculation of E_n to the next order has been done in [RaBe:Alg], in the case for which $\partial_\gamma H = 0$, and leads to (for $\gamma > 0$) :

$$E_n = \gamma\omega(2n+1)/2 + \gamma^2 \Delta^2 H_{\rm cl}(0) \left(1 + (2n+1)^2\right)/64$$
$$- \cdots \gamma^2 \left[9(3n^2 + 3n + 1)|\delta H_{\rm cl}(0)|^2 + (3n^2 + 3n + 2)|\partial^3 H_{\rm cl}(0)|^2\right]/288\omega + \mathrm{O}(\gamma^3) \ , \quad (101)$$

where :

$$\partial = \frac{\partial}{\partial k_1} - i\frac{\partial}{\partial k_2} \; , \; \overline{\partial} = \frac{\partial}{\partial k_1} + i\frac{\partial}{\partial k_2} \; , \; \Delta = \partial\overline{\partial} \; . \tag{102}$$

These formulæ have been checked numerically on several models. The calculation to the third order in powers of γ havsbeen computed for the Harper model (see (88)) [RaBe:Alg] and gives for the minimum (Fig. 1) :

$$E_n = -4 + \gamma(2n+1) - \frac{\gamma^2}{16}\left[1 + (2n+1)^2\right] + \frac{\gamma^3}{192}\left[n^3 + (n+1)^3\right] + \mathrm{O}(\gamma^4) \; . \tag{103}$$

Another example of interest has been investigated in [BeKrSe] and concerns the nearest neighbour model on a triangular lattice with two fluxes (Fig. 2). The corresponding Hamiltonian is given by :

$$H_\Delta = T_1 + T_2 + T_3 + T_1^* + T_2^* + T_3^* \; , \text{ with } \quad T_1T_2T_3 = e^{i2\pi\phi'/\phi_0} \; , \tag{104}$$

and (89). Here ϕ' represents the flux through the "up" triangles, while $\phi - \phi'$ represents the flux through the "down" triangles. The corresponding classical counterpart is given by :

$$H_{\Delta,\mathrm{cl}} = 2\cos k_1 + 2\cos k_2 + 2\cos(k_1 + k_2 + \frac{\gamma}{2} - \gamma') \; , \tag{105}$$

if we set $\gamma' = 2\pi\phi'/\phi_0$. The minima and maxima occur at the points $k_1 = k_2 = 2\pi\sigma/3 + \gamma'/3 = \theta_\sigma$, where $\sigma = -1, 0, +1$. If $\theta_\sigma \neq \pi/2$, one gets the following expression [BeKrSe] :

$$E_{\sigma,n} = 6\cos\theta_\sigma - \gamma\sqrt{3}(2n+1)\cos\theta_\sigma + \gamma\sin\theta_\sigma + \gamma^2\left[1 + (2n+1)^2\right]\cos\theta_\sigma/8 +$$

$$\cdots + \gamma^2\sin 2\theta_\sigma\cos\theta_\sigma[3(2n+1) + 5]/72 - \gamma^2/12\cos\theta_\sigma - \gamma^2\sqrt{3}\sin\theta_\sigma(2n+1)/6 \; , \tag{106}$$

giving rise to three bundles of Landau levels. For $\gamma' \approx 0$ two of these bundles are very close and actually intersect each other (Fig. 3) . The comparison between this formula and the exact spectrum obtained by matrix diagonalization is very good : they agree up to four digits for the coefficients of the power expansion in γ [BeKrSe].

The assumptions (H0, H1, H2) concern the generic case, for which the extremum is regular. However, some non generic case has been observed. For example, in [Wil:Cri, BaKr], the case of a square lattice with second nearest neighbour has been studied. The corresponding Hamiltonian is :

$$H_{\mathrm{WBK}} = T_1 + T_1^* + T_2 + T_2^* + t_2\left(T_1^2 + T_1^{2^*} + T_2^2 + T_2^{2^*}\right) \; . \tag{107}$$

For $t_2 < 1/4$, the classical Hamiltonian has only one absolute minimum, like in the Harper case. At the value $t_2 = 1/4$, this minimum bifurcates to give four degenerate minima for $\epsilon > 1/4$. At the bifurcation value, this minimum becomes flat namely the Hessian actually vanishes identically, giving rise to a normal form like :

$$H_{\mathrm{WBK},\gamma} = -3 + \frac{\gamma^2}{4}\left(K_1^4 + K_2^4\right) + \mathrm{O}(\gamma^4) \; . \tag{108}$$

A Bohr-Sommerfeld quantization condition gives at the lowest order in γ :

$$E_n = -3 + \frac{\gamma^2\pi}{4\Gamma(1/4)^4}(2n+1)^2 + \mathrm{O}(\gamma^4) \quad \text{for } n \text{ large} \; , \tag{109}$$

giving parabolic Landau levels, as can be observed in (Fig. 4).

5.3 Expansion near a rational field

The method outlined in the previous subsection for a low field expansion of the spectrum, can be extended to the expansion near a rational field, or also to the case of a matrix Hamiltonian, as can happen if several different bands contribute to the conduction. We will give here the method for the rational fields, leaving to the reader the case of a matrix Hamiltonian as an exercise.

We consider now an interval $I = [2\pi p/q - \epsilon_0, 2\pi p/q + \epsilon_0]$ and $H = H^* \in \mathcal{A}_I$. Using the matrices u, v given in section 4 (84), letting U_γ, V_γ be the generators of the algebra $\mathcal{A}_\gamma$ for $|\gamma| \leq \epsilon_0$, we consider the elements U', V' in $\mathcal{A}_\gamma \otimes M_q$ defined by :

$$U' = U_\gamma \otimes u \ , \ V' = V_\gamma \otimes v \ . \tag{110}$$

They are unitary and satisfy the commutation rule :

$$U'V' = e^{i(\gamma+2\pi p/q)} V'U' \ , \tag{111}$$

showing that they generate in $\mathcal{A}_\gamma \otimes M_q$ a subalgebra $*$-isomorphic to $\mathcal{A}_{\gamma+2\pi p/q}$. Letting γ vary in $I(0) = [-\epsilon_0, +\epsilon_0]$, we get a $*$-isomorphism between $\mathcal{A}_I$ and a closed subalgebra of $\mathcal{A}_I(0) \otimes M_q$.

Assuming that H is smooth enough, one can expand it as :

$$\eta_{\gamma+2\pi p/q}(H) = H_\gamma = \sum_{\mathbf{m}\in\mathbf{Z}^2} h(\mathbf{m};\gamma) W_\gamma(\mathbf{m}) \otimes w(\mathbf{m}) \ , \ \gamma \in I(0) \ , \tag{112}$$

and this series converges in norm. So we are left with the same problem as in 5.2, with now matrix Hamiltonians instead. Following the same scheme, the classical counterpart is the matrix valued function :

$$H_{\text{scl}}(\mathbf{k};\gamma) = \sum_{\mathbf{m}\in\mathbf{Z}^2} h(\mathbf{m};\gamma) e^{i(m_1k_1+m_2k_2)} w(\mathbf{m}) \ , \ \mathbf{k} \in \mathbf{T}^2 \ , \ \gamma \in I(0) \ . \tag{113}$$

As we already indicated, we must first diagonalize this matrix at $\gamma = 0$, giving q real eigenvalues (since H is selfadjoint), $e_1(\mathbf{k}), e_2(\mathbf{k}), \cdots, e_q(\mathbf{k})$, and therefore q bands $B_1, B_2, \cdots, B_q$, namely the set of values of the $e_j(\mathbf{k})$'s as $\mathbf{k}$ varies in the 2-torus. Since H is selfadjoint, it is always possible to choose the $e_j(\mathbf{k})$'s smooth. We will also denote by $P_1(\mathbf{k}), P_2(\mathbf{k}), \cdots, P_q(\mathbf{k})$ the corresponding eigenprojections; they are also smooth with respect to $\mathbf{k}$. We will now assume the following :

(H_q0) $H = H^* \in \mathcal{C}^N(\mathcal{A}_I)$ for some $N > 2$, and all coefficients in the expansion (113) are N-times continuously differentiable with respect to γ.
(H_q1) The eigenvalue e_j admits a minimum at $\mathbf{k} = \mathbf{0}$, and $e_j(\mathbf{0}) = 0$. Moreover, no other eigenvalue of $H_{\text{scl}}(\mathbf{0};0)$ coincides with $e_j(\mathbf{0}) = 0$.
(H_q2) The minimum of e_j is regular, namely the Hessian matrix $\partial_\mu\partial_\nu e_j(\mathbf{0})$ is positive definite.

Then we get the following result [Bel:Eva, BaBeFl] :

Theorem 10 *Let H satisfy (H_q0), (H_q1), (H_q2). Then there are $\delta > 0$ and $0 < \epsilon \leq \epsilon_0$ such that if $|\gamma| < \epsilon$, the set $\mathrm{Sp}(\eta_{\gamma+2\pi p/q}(H)) \cap (-\delta, +\delta)$ contains a subset Σ_j which is itself contained in the union over $n \in \mathbf{N}$ of the interval $J_{n,j}(\gamma) = [E_{n,j}(\gamma) -$*

$\Delta(\gamma), E_{n,j}(\gamma) + \Delta(\gamma)]$ *where* $E_{n,j}(\gamma)$ *admits a Taylor expansion in* γ *up to the* N^{th} *order of the form :*

$$E_{n,j}(\gamma) = |\gamma|(n+1/2)\left(\det D^2 e_j(\mathbf{0})\right)^{1/2} + \gamma\left(\frac{\partial e_j}{\partial\gamma}\right)_{\gamma=0,\mathbf{k}=\mathbf{0}} + \gamma E_{\text{RW}} + \mathrm{O}(\gamma^2) \ , \quad (114)$$

where $0 < \Delta(\gamma) \leq \text{const.}|\gamma|N'$, *for some* $N' > N$, *where* E_{RW} *is the "Rammal-Wilkinson" term given by the following expression :*

$$E_{\text{RW}} = \frac{i}{2}\mathrm{Tr}\left(P_j(0)[\partial_1 H_{\text{scl}}(0)\partial_2 P_j(0) - \partial_2 H_{\text{scl}}(0)\partial_1 P_j(0)]\right) \ , \quad (115)$$

The strategy used to prove this theorem is based upon the so-called "Schur complement formula". Let $H = H^*$ be a selfadjoint operator acting on a Hibert space of the form $\mathcal{H} = \mathcal{P} \oplus \mathcal{Q}$. Let P, Q be the orthogonal projections on each subspace of that decomposition and let D be a partial isometry from $\mathcal{H}$ to $\mathcal{P}$ such that $DD^* = \mathbf{I}_{\mathcal{P}}$ and $D^*D = P$. We define on $\mathcal{P}$ the family of operators :

$$H_{\text{eff}}(z) = DHD^* + DHQ(z\mathbf{I} - QHQ)^{-1}QHD^* \ , \quad (116)$$

whenever z is a complex number which does not belong to the spectrum of QHQ. Then it is possible to show that $z \in \mathrm{Sp}(H) - \mathrm{Sp}(QHQ)$ if and only if $z \in \mathrm{Sp}(H_{\text{eff}}(z))$. Moreover E is an eigenvalue of H not in $\mathrm{Sp}(QHQ)$ if and only if E is an eigenvalue of $H_{\text{eff}}(E)$.

We then denote by $P = \mathbf{I} - Q$ the projection $\mathbf{I} \otimes P_j(0)$ of $\mathcal{A}_I(0) \otimes M_q$. For $(\mathbf{k};\gamma) \approx (\mathbf{0}, 0)$, it follows that there is a small neighbourhood O of $e_j(0)$ such that if $z \in O$, $z \notin \mathrm{Sp}(QH_{\text{scl}}(\mathbf{k};\gamma)Q)$. Since the eigenvalue $e_j(\mathbf{k})$ is simple for $\mathbf{k} \approx 0$, the projector $P_j(\mathbf{k})$ is one dimensional for $\mathbf{k} \approx 0$, and therefore there exists a partial isometry $\hat{D} : \mathbf{C}^q \mapsto \mathbf{C}$ such that $\hat{D}\hat{D}^* = \mathbf{I}$, and $\hat{D}^*\hat{D} = P_j(0)$. If D is the partial isometry $\mathbf{I} \otimes \hat{D}$, let us introduce the effective Hamiltonian :

$$h_j(z) = DH_\gamma D^* + DH_\gamma Q\left(z\mathbf{I} - QH_\gamma Q\right)^{-1} QH_\gamma D^* \ . \quad (117)$$

By construction this is an analytic family of elements in $\mathcal{A}_I(0)$ now. We can therefore analyze it by the method developed in 5.2, and will give rise to a bundle of Landau sublevels $E_{n,j}(z)$ near the lower edge of the band B_j. The corresponding part of the spectrum of H_γ near $e_j(0) = 0$, will then be given by solving the implicit equation $E = E_{n,j}(E)$. The solution can be computed explicitely order by order in powers of γ, thanks to the hypothesis made on e_j. The Rammal-Wilkinson term comes from the first order contribution of the second term in (117). It reflects the fact that the matrices $H_{\text{scl}}(\mathbf{k};\gamma)$ do not mutually commute for various values of $\mathbf{k}$ in general, namely it reflects the existence of a curvature in the fiber bundle over the 2-torus defined by $P_j(\mathbf{k})$. The calculation of this term can be found in [RaBe:Alg, BeKrSe].

5.4 Qualitative analysis of the spectrum

Let us now comment on the formulæ (114)&(115). Due to the absolute value of γ appearing in the first term of (114), the right and left derivatives of the band edge with respect to γ are different, showing that the band edges eventhough continuous functions of γ by the theorem 3, have nevertheless a discontinuous first derivative at each rational point. On the other hand, even if $\partial_\gamma H = 0$ the Rammal-Wilkinson term may not vanish. This is the case for instance in the Harper model for $p/q = 1/3$ (see Fig. 1). We can see the effect of this term by the fact that the left and right derivative of the band edge are not symmetric around $\gamma = 2\pi p/q$. The difference between them reveals the occurrence of curvature effects.

On the other hand one can recognize whether the band edge is a maximum or a mini at the slope of the Landau sublevels emerging away from $\gamma = 2\pi p/q$.

For most values of p/q, all bands are separated by gaps. However, many non generic situation can be observed on examples.

(i) Two bands may overlap without touching each other. Then, each minimum or each maximum of the corresponding band will reveal itself by the occurence of a bundle of Landau levels emerging on both sides of $\gamma = 2\pi p/q$ (see Fig. 5), and given by the formula (114)&(115).

(ii) two bands B_j, $B_{j'}$, with or without overlap, may touch each other. In this case, generically they will touch on a conical point (see Fig. 6). This situation leads to a different canonical form. For indeed the previous analysis can be extended by replacing the projector $P_j(0)$ by $P_j(0) + P_{j'}(0)$. Then the effective Hamiltonian becomes a 2×2 matrix unitarily equivalent to the Dirac operator [HeSj:Har2, RaBe:Alg] :

$$H_{\text{Dirac}} = |\gamma|^{1/2} \begin{bmatrix} 0 & K_1 + iK_2 \\ K_1 - iK_2 & 0 \end{bmatrix} + \mathrm{O}(\gamma) \ . \tag{118}$$

This case will give "Dirac levels" which are parabolic namely :

$$E_{\pm n} = \pm \text{const.} |n\gamma|^{1/2} \ , \ n \in \mathbf{N} \ , \tag{119}$$

which is for instance what happens in the Harper model at $E = 0$ and $p/q = 1/2$ (see Fig. 1).

This formula must usually be corrected by a Rammal-Wilkinson term, giving a slope to the sublevel $n = 0$. This is what happens in the WBK-model (108), at $p/q = 1/2$ (see Fig. 7).

(iii) Two bands can also touch with a contact of order 2. There is another example proposed by M.Wilkinson [Wil:Cri] and studied in details by Barelli and Fleckinger [BaFl], which is the following :

$$H_{\mathrm{W}} = T_1 + T_2 + t_3 \left(T_1^2 T_2 e^{-i\gamma} + T_1^{-2} T_2 e^{i\gamma} + T_1 T_2^2 e^{-i\gamma} + T_1 T_2^{-2} e^{i\gamma} \right) + \text{h.c.} \ . \tag{120}$$

At $E = 0$,$p/q = 1/2$, we do get two families of Landau sublevels on either side of $p/q = 1/2$, corresponding to the bottom wells of the two bands. The generic parabolic touching can be seen on Fig. 8.

(iv) A maximum or a minimum can also be reached on a curve. This has been observed in the WBK model at $p/q = 1/2$. This case has been investigated in details by Helffer and Sjöstrand [HeSj:Har3], who remarked that the "subprincipal symbol" may break this degeneracy and create what they have called "miniwells", namely local extremas with deepness of order $\mathrm{O}(\gamma)$. Such an example has never been investigated numerically, but there are indications that such a phenomenon should occur on the WBK model.

At last we must point out the occurrence of tunneling effect. For indeed, the classical model gives a Hamiltonian on the phase space given by a 2-torus. This is equivalent to choosing $\mathbf{R}^2$ instead, but requiring that the Hamiltonian be periodic in both directions. This will be called the "extended picture". In this picture, each local extremum is repeated periodically, giving rise to an exact degeneracy. Therefore a tunneling effect should occur between the corresponding wells, ending into a broadening of the Landau levels or sublevels. The width of this broadening can be computed by the WKB method, and will give rise to terms of order $O(\exp(-S/\gamma))$ where S is some constant equal to the real part of the tunneling action between two neighbouring wells.

This effect has been studied in great details in the Harper model by Helffer & Sjöstrand [HeSj:Har1, HeSj:Har2, HeSj:Har3], and for the corresponding model on a honeycomb or triangular lattice by Kerdelhué [Ker]. By evaluating precisely the tunneling matrix representing the effective Hamiltonian restricted to each of the Landau sublevel, they could prove that it is again represented by a Hamiltonian with nearest neighbour interactions, having the symmetry of the original lattice (e.g. a Harper model for a square lattice), with a small correction. Therefore, each Landau sublevel is itself decomposed into subbands, and this explain the occurence of the fractal structure.

This tunneling effect has also been exhibited in a spectacular example by Barelli & Kreft [BaKr], in the WBK model for $t_2 > 1/4$ and $\gamma \approx 0$. As we already said, after the bifurcation the unique minimum splits into four degenerate minima surrounding one maximum. Since these four wells are very close to each other in each unit cell of the extended phase space, compare to the distance between cells, the tunneling effect between these four wells within the unit cell is likely to dominate over the other sources of tunneling. Each well gives rise to its own bunch of Landau levels, but the splitting due to the tunneling will separate them. It turns out that the tunneling action in this case is not purely imaginary, so that the Landau levels can be represented by if $n \in \mathbf{N}$, $t_2 = 1/2$ and $i = 1, 2, 3, 4$:

$$E_{n,i} = E_n(\gamma) + dE_{n,i}(\gamma) \ , \ E_n(\gamma) = -3 + \frac{3}{2}\gamma(2n+1) + O(\gamma^2) \ , \tag{121}$$

where the splitting is given by [BaKr] :

$$dE_{n,i} = \gamma\frac{3}{\pi}e^{-\mathrm{Im}(S_2)/\gamma}\cos(\mathrm{Re}(S_4)/4\gamma + \pi/4) + O(e^{-S'/\gamma}) \ , \tag{122}$$

where S_2 represents the action $\int_{AB} k_1 dk_2$ for a path AB in the complex energy surface $H_{cl}(\mathbf{k}) = E_n(\gamma)$ joining two neighbouring wells A and B, while S_4 is the tunneling action for a closed path in the same energy surface going through the four wells once. Moreover, S' is some action larger than S_2. Even though there are usually many non homotopic such paths in this complex energy surface, only the "shortest" ones (in terms of the corresponding action integral) do contribute to this order.

In this formula the width of the splitting is controlled by $\mathrm{Im}(S_2)$ which gives an exponentially small term. But the occurence of a non zero real part produces a nice braiding between these four sublevels as can be seen in (Fig. 9). In a recent work, Barelli and Fleckinger exhibited a braiding of Dirac sublevels near the half flux (see Fig. 10) [BaFl].

6 Elementary Properties of the Kicked Rotor

6.1 The Furstenberg Algebra

As we have seen the Floquet operator for the kicked rotor cannot be seen as an element of the rotation algebra. This is because the kinetic part is not a continuous function of U and V. However, we have seen that it defines a $*$-automorphism of the rotation algebra. To deal with that we have two choices. The first is to ignore the Floquet operator itself and to stick with its action on the non commutative torus. This is fine as long as we are interested only in the evolution of observables.

However, in many occasions do physicists need to know more on the spectrum of the Floquet operator itself, the so-called "quasi-energy" spectrum. One of its most important property is the "dynamical localization", a phenomenon similar to the Anderson localization in Solid State Physics of disordered metals [FiGrPr].

In order to deal with this latter problem, we can simply enlarge our algebra by brute force, adding the missing unitary F_0 equal to the kinetic energy defined in section 1 (12) by :

$$F_0 = e^{-iA^2/2\gamma} \ . \tag{123}$$

As we have seen in section 1 (19) this operator satisfies the following commutation rules

$$\text{(i)} \quad F_0 V F_0^{-1} = V \qquad \text{(ii)} \quad F_0 U F_0^{-1} = U V^{-1} e^{-i\gamma/2} \ . \tag{124}$$

As before we will denote by $\mathcal{B}_I$ the C^*-algebra generated by the polynomials in U, V, F_0 with coefficients in the set of continuous functions of γ in I. This algebra can be rigorously constructed along the line developed in section 2. However one can use the general method of C^*-algebras, namely the notion of crossed-product [Ped], to construct it.

One can indeed see $\mathcal{B}_I$ in two ways :

(i)-the first one comes from the previous definition, namely F_0 acts on the rotation algebra $\mathcal{A}_I$ by mean of the $*$-automorphism

$$\beta_0(a) = F_0 a F_0^{-1} \qquad\qquad a \in \mathcal{A} \ . \tag{125}$$

Therefore $\mathcal{B}_I$ can be seen as the crossed product $\mathcal{A}_I \times_{\beta_0} \mathbf{Z}$ of the rotation algebra $\mathcal{A}_I$ by the $\mathbf{Z}$-action defined by β_0. Using Weyl's operators defined in section 2 (32), we notice that

$$\beta_0\left(W(\mathbf{m})\right) = W(G\mathbf{m}) \ , \qquad \mathbf{m} \in \mathbf{Z}^2 \ , \tag{126}$$

provided G is the element of $SL(2, \mathbf{R})$ given by :

$$G = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \ . \tag{127}$$

(ii)-the second one consists in considering first the subalgebra generated by functions in $\mathcal{C}(I)$, together with the operators V and F_0. This is an abelian C^*-algebra isomorphic to $\mathcal{C}(I \times \mathbf{T}^2)$. This isomorphism associates to V and F_0 respectively the functions $f_V(\gamma, x, y) = e^{ix}$ and $f_{F_0}(\gamma, x, y) = e^{iy}$. Actually, the inner automorphism associated to U leaves this algebra invariant. This is because the commutation rules (19) can be written as

$$(i) \qquad UVU^{-1} = e^{i\gamma} V \ , \qquad\qquad (ii) \qquad UF_0U^{-1} = e^{i\gamma/2} V F_0 \ . \tag{128}$$

In other words, for $f \in \mathcal{C}(I \times \mathbf{T}^2)$, we get :

$$UfU^{-1} = f \circ \phi \ , \tag{129}$$

where ϕ is the "Furstenberg" map acting on $I \times \mathbf{T}^2$ as :

$$\phi(\gamma, x, y) = (\gamma, x + \gamma, y + x + \gamma/2) \ , \qquad (\gamma, x, y) \in I \times \mathbf{T}^2 \ . \tag{130}$$

This map was used by Furstenberg to study the ergodic properties of diophantine approximations in number theory. Thus $\mathcal{B}_I$ can be seen as the crossed product $\mathcal{C}(I \times \mathbf{T}^2) \times_\phi \mathbf{Z}$ by the Furstenberg map. This is why we propose to call this algebra the "Furstenberg algebra".

We see that ϕ leaves each fiber $\{\gamma\} \times \mathbf{T}^2$ invariant and we will denote by ϕ_γ the corresponding restriction. It is well-known that whenever $\gamma/2\pi$ is irrational, ϕ_γ is a minimal diffeomorphism [CoFoSi].

6.2 Calculus on $\mathcal{B}_I$

As for $\mathcal{A}_I$, a calculus can be defined on the Furstenberg algebra. Since the trace on $\mathcal{A}_I$ is β_0-invariant, it defines a trace on the crossed product in a natural way. It is actually defined by the formula :

$$\tau\left(W(\mathbf{m})F_0^l\right) = \delta_{\mathbf{m},0}.\delta_{l,0} \ , \qquad \mathbf{m} \in \mathbf{Z}^2 \ , \ l \in \mathbf{Z} \ . \tag{131}$$

Since we have defined originally (cf. section 1) U, V, F_0 in term of action-angle variables in the classical case, one also gets an angle average $\langle \cdot \rangle$ namely :

$$\langle W(\mathbf{m})F_0^l \rangle = \delta_{m_1,0} V^{m_2} F_0^l \ , \qquad if \quad \mathbf{m} = (m_1, m_2) \in \mathbf{Z}^2 \ , \ l \in \mathbf{Z} \ . \tag{132}$$

Thus, if $a \in \mathcal{B}_I$, $\langle a \rangle \in \mathcal{C}(I \times \mathbf{T}^2)$, and this average satisfies the properties described in (32).

In much the same way, a differential structure can be defined. The derivation ∂_θ can be extended immediately to $\mathcal{B}_I$ by :

$$\partial_\theta U = iU \ , \qquad \partial_\theta V = 0 \ , \qquad \partial_\theta F_0 = 0 \ . \tag{133}$$

We notice however that ∂_A cannot be extended as a derivation in $\mathcal{B}_I$ because $\partial_A F_0$ would be unbounded, namely outside $\mathcal{B}_I$. But a new derivation ∂_y appears defined by :

$$\partial_y U = 0 \ , \qquad \partial_y V = 0 \ , \qquad \partial_y F_0 = iF_0 \ . \tag{134}$$

Both ∂_θ and ∂_y are the infinitesimal generators of the following two-parameter group of $*$-automorphisms :

$$\hat{\rho}_{\theta,y}\left(W(\mathbf{m})F_0^l\right) = e^{i(m_1\theta + ly)} W(\mathbf{m}) F_0^l \ , \tag{135}$$

which leaves the trace invariant.

At last, the definition of a Poisson bracket is not obvious because for $\gamma = 0$ the algebra $\mathcal{B}_0$ is no longer commutative. Even though it is in principle possible to define such an object, we will not use it, and we skip this part of the calculus.

6.3 Representations and structure of $\mathcal{B}_I$

Among the representations of $\mathcal{B}_I$, we will select one family of special interest in view of the original definition of the kicked rotor in the physical Hilbert space $L^2(\mathbf{T})$ given in section 1. It is actually simpler to work in the momentum space, namely in $\ell^2(\mathbf{Z})$ where the integers of the chain $\mathbf{Z}$ are simply the quantum numbers for the angular momentum.

This family $\{\pi_{\gamma,x,y}; (\gamma,x,y) \in I \times \mathbf{T}^2\}$ is indexed by points in $I \times \mathbf{T}^2$ and acts on $\ell^2(\mathbf{Z})$ as follows :

$$\begin{array}{lll} \text{(i)} & (\pi_{\gamma,x,y}(f)\psi)(n) = f(\gamma)\psi(n) \;, & f \in \mathcal{C}(I) \\ \text{(ii)} & (\pi_{\gamma,x,y}(U)\psi)(n) = \psi(n-1) \;, & \\ \text{(iii)} & (\pi_{\gamma,x,y}(V)\psi)(n) = e^{i(x-n\gamma)}\psi(n) \;, & \\ \text{(iv)} & (\pi_{\gamma,x,y}(F_0)\psi)(n) = e^{i(y-nx+n^2\gamma/2)}\psi(n) \;, & if \quad \psi \in \ell^2(\mathbf{Z}) \;. \end{array} \tag{136}$$

Comparing with the equation (12) & (13), γ appears as an effective Planck constant, x as an effective magnetic field, and y as a phase factor entering in the definition of F_0.
With these definitions, the following result can be easily proved by standard technics :

Proposition 7 *1)-The family $\{\pi_{\gamma,x,y}; (\gamma,x,y) \in I \times \mathbf{T}^2\}$ is faithfull. In particular, the norm of $a \in \mathcal{B}_I$ is given by :*

$$\|a\| = \sup_{\gamma \in I} \sup_{(x,y)\in\mathbf{T}^2} \|\pi_{\gamma,x,y}(a)\| \;. \tag{137}$$

2)-The map $(\gamma,x,y) \in I \times \mathbf{T}^2 \mapsto \pi_{\gamma,x,y}(a)$ is strongly continuous for all $a \in \mathcal{B}_I$.
3)-For $\gamma \in I$, the trace is given by :

$$\tau_\gamma(a) = \int_{\mathbf{T}^2} \frac{dxdy}{4\pi^2} \; \langle 0|\pi_{\gamma,x,y}(a)|0\rangle \tag{138}$$

Moreover if $\gamma/2\pi$ is irrational, we get

$$\tau_\gamma(a) = \lim_{L\mapsto\infty} \frac{1}{2L+1} \mathrm{Tr}\left(\pi_{\gamma,x,y}(a)|_{[-L,L]}\right) \;,$$

uniformly in $(x,y) \in \mathbf{T}^2$. 4)-If T is the translation operator in $\ell^2(\mathbf{Z})$, namely if $(T\psi)(n) = \psi(n-1)$ for $\psi \in \ell^2(\mathbf{Z})$, then :

$$T\pi_{\gamma,x,y}(a)T^{-1} = \pi_{\phi(\gamma,x,y)}(a) \;, \qquad a \in \mathcal{B}_I \;, \; (\gamma,x,y) \in I \times \mathbf{T}^2 \;. \tag{139}$$

5)-If N is the position operator in $\ell^2(\mathbf{Z})$ defined by $(N\psi)(n) = n\psi(n)$, $\psi \in \ell^2(\mathbf{Z})$, we have :

$$\pi_{\gamma,x,y}(\partial_\theta a) = i[N, \pi_{\gamma,x,y}(a)] \;, \qquad \pi_{\gamma,x,y}(\partial_y a) = \frac{\partial}{\partial y}\pi_{\gamma,x,y}(a) \;. \tag{140}$$

Thanks to this result the elements of $\mathcal{B}_I$ can be described as follows. For $a \in \mathcal{B}_I$, we set :

$$a(\gamma,x,y;n) = \langle 0|\pi_{\gamma,x,y}(a)|n\rangle \;. \tag{141}$$

This is a continuous function on $I \times \mathbf{T}^2 \times \mathbf{Z}$ converging to zero at infinity. In terms of such functions the product and the $*$ in $\mathcal{B}_I$ can be expressed as follows :

$$ab(\gamma, x, y; n) = \sum_{l \in \mathbf{Z}} a(\gamma, x, y; l) b(\gamma, x - l\gamma, y - lx + l^2\gamma/2; n - l) \ , \tag{142}$$

$$a^*(\gamma, x, y; n) = a(\gamma, x - n\gamma, y - nx + n^2\gamma/2; -n)^* \ , \tag{143}$$

for $a, b \in \mathcal{B}_I$. Moreover, the representation $\pi_{\gamma,x,y}$ is given by :

$$(\pi_{\gamma,x,y}(a)\psi)(n) = \sum_{l \in \mathbf{Z}} a(\gamma, x - n\gamma, y - nx + n^2\gamma/2; l - n)\psi(l) \ , \qquad \psi \in \ell^2(\mathbf{Z}) \ . \tag{144}$$

In particular, due to the faithfullness of this family, $a = 0$ if and only if the function $a(\gamma, x, y; n)$ vanishes identically.

If we denote by $\mathcal{B}_\gamma$ the algebra $\mathcal{B}_I$ for $I = \{\gamma\}$, the following theorem characterizes its structure :

Theorem 11 *1)-If $\gamma/2\pi$ is irrational, $\mathcal{B}_\gamma$ is simple. In particular, every non zero representation is faithfull.*
2)-For $\gamma = 0$, the algebra $\mathcal{B}_0$ is isomorphic to the universal rotation algebra $\mathcal{A}$.
3)-If $\gamma = 2\pi p/q$ where p, q are positive integers prime to each others, $\mathcal{B}_{2\pi p/q}$ is isomorphic to the sub C^*-algebra of $M_q(\mathbf{C}) \otimes \mathcal{B}_0$ generated by :

$$\tilde{U} = u \otimes U_0 \ , \qquad \tilde{V} = v \otimes V_0 \ , \qquad \tilde{F}_0 = w \otimes F_{0,0} \ , \tag{145}$$

where $U_\gamma, V_\gamma, F_{0,\gamma}$ are the generators of $\mathcal{B}_\gamma$, and u, v, w are three unitary $q \times q$ matrices fulfilling the following conditions :

$$u^q = v^q = w^{2q} = \mathbf{I} \ , \tag{146}$$

$$uvu^{-1} = e^{2i\pi p/q} v \ , \qquad uwu^{-1} = e^{i\pi p/q} vw \ , \qquad vw = wv \ . \tag{147}$$

Proof : 1)-For $\gamma/2\pi$ irrational, the Furstenberg map $\phi_\gamma : (x, y) \in \mathbf{T}^2 \mapsto (x + \gamma, y + x + \gamma/2) \in T^2$ is a minimal diffeomorphism of the torus [CoFoSi]. Therefore, the crossed product $\mathcal{B}_\gamma = \mathcal{C}(\mathbf{T}^2) \times_{\phi_\gamma} \mathbf{Z}$ is simple [HiSk].
2)-For $\gamma = 0$ the commutation rules become :

$$UV = VU \ , \qquad VF_0 = F_0 V \ , \qquad UF_0U^{-1} = VF_0 \ . \tag{148}$$

These rules are precisely the ones defining the universal rotation algebra $\mathcal{A}$ if we identify V with the map $\gamma \in \mathbf{T} \mapsto e^{i\gamma} \in \mathbf{C}$ (cf. section 2).
3)-If one chooses the matrices u, v as in (84), the matrix w becomes :

$$w = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda' & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda'^{(q-2)^2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \lambda'^{(q-1)^2} \end{bmatrix} , \tag{149}$$

where $\lambda' = e^{i\pi p/q}$.

It is easy to check that $\tilde{U}, \tilde{V}, \tilde{F}_0$ satisfy the commutation rules for the algebra $\mathcal{B}_{2\pi p/q}$. Hence they define a $*$-homomorphism ρ from $\mathcal{B}_{2\pi p/q}$ into $M_q(\mathbf{C}) \otimes \mathcal{B}_0$.
4)-To achieve our result it is sufficient to prove that ρ is one-to-one. For $(x,y) \in \mathbf{T}^2$, let $\hat{\pi}_{x,y}$ be the representation of $M_q(\mathbf{C}) \otimes \mathcal{B}_0$ given by $\mathrm{id} \otimes \pi_{0,x,y}$ acting on $\mathbf{C}^q \otimes \ell^2(\mathbf{Z})$. Any $a \in \mathcal{B}_0$ can be seen as a function on $\mathbf{T}^2 \times \mathbf{Z}$ as (see (141)), and for $\phi \in \mathbf{C}^q \otimes \ell^2(\mathbf{Z})$ and $A \in M_q$ we get :

$$[\hat{\pi}_{x,y}(A \otimes a)\phi]_j(n) = \sum_{j=0}^{q-1} \sum_{l \in \mathbf{Z}} A_{j,j'} a(x, y - nx; l - n)\phi_{j'}(l) \ . \tag{150}$$

Let $\{e_j\}_{j=0}^{q-1}$ be the canonical basis of $\mathbf{C}^q$ with the convention that $e_{j+q} = e_j$, and let $\{\delta_n; n \in \mathbf{Z}\}$ be the canonical basis of $\ell^2(\mathbf{Z})$. We set :

$$|j, n\rangle = e_j \otimes \delta_n \ . \tag{151}$$

Then :

$$\langle j, 0|\hat{\pi}_{x,y}(A \otimes a)|j', l\rangle = A_{j,j'} a(x, y, ; l) \ . \tag{152}$$

It is not difficult to check that if now $b \in \mathcal{B}_{2\pi p/q}$ and $\phi \in \mathbf{C}^q \otimes \ell^2(\mathbf{Z})$ we get :

$$[\hat{\pi}_{x,y}(\rho(b))\phi]_j(n) = \sum_{l \in \mathbf{Z}} b(x - 2\pi jp/q, y - nx + j^2\pi p/q; l)\phi_{j+l}(n + l) \ , \tag{153}$$

where $j + l$ is defined modulo q. It is actually sufficient to check this formula on the generators $U_{2\pi p/q}, V_{2\pi p/q}, F_{0,2\pi p/q}$ since $\hat{\pi}_{x,y}$ and ρ are $*$-homomorphisms. In particular :

$$\langle 0, 0|\hat{\pi}_{x,y}(\rho(b))|l, l\rangle = b(x, y; l) \ . \tag{154}$$

Thus $\rho(b) = 0$ if and only if $b(x, y; l) = 0$ for any $(x, y; l)$, namely $b = 0$. Hence ρ is one-to-one.

Using the same strategy we can easily get :

Corollary 4 *1)-for $\gamma \in \mathbf{R}$ the algebra $\mathcal{B}_{2\pi p/q+\gamma}$ is isomorphic to the subalgebra of $M_q \otimes \mathcal{B}_\gamma$ generated by $u \otimes U_\gamma, v \otimes V_\gamma, w \otimes F_{0,\gamma}$.*
2)-for $\gamma, \gamma' \in \mathbf{R}$ the algebra $\mathcal{B}_{\gamma+\gamma'}$ is isomorphic to the subalgebra of $\mathcal{B}_\gamma \otimes \mathcal{B}_{\gamma'}$ generated by $U_\gamma \otimes U_{\gamma'}, V_\gamma \otimes V_{\gamma'}, F_{0,\gamma} \otimes F_{0,\gamma'}$.

6.4 Algebraic Properties of the Kicked Rotor

In section 1 we have expressed the Floquet operator of the kicked rotor as :

$$F_{K,\gamma,x}^{-1} = e^{-iA^2/2\gamma} e^{-iK\cos\theta/\gamma} e^{i\hat{y}} \ , \tag{155}$$

where $\gamma = \hbar T/I$ is the effective Planck constant and $x = -\mu BT$ is the effective magnetic field. Moreover in the momentum space representation, $A = \gamma N - x$ if N is the position operator (see prop. 7). Using the previous algebraic framework, it follows that :

$$F_{K,\gamma,x} = \pi_{\gamma,x,0}(F_K) \ , \tag{156}$$

with :

$$F_K = e^{iK(U+U^{-1})/2\hat{\gamma}} F_0 \ , \tag{157}$$

where $\hat{\gamma} : \gamma \in I \mapsto \gamma \in \mathbf{R}$. In this special case we notice the following property :

$$\pi_{\gamma,x,y}(F_K) = e^{iy} F_{K,\gamma,x} \ , \tag{158}$$

so that one can set $y = 0$ without loss of generality.

It follows that F_K belongs to $\mathcal{B}_I$ for any compact set I in the real line not containing the origin.

Our first set of results concerns the spectrum as a set of this Floquet operator. Since it is unitary its spectrum is necessarily contained in the unit circle S_1. Actually the following results are still valid if we replace $cos(\theta) = (U + U_{-1})/2$ by any real valued 2π-periodic continuous function $g(\theta)$ on the real line.

Theorem 12 *1)-For any $\gamma \neq 0$, the spectrum of $F_{K,\gamma} = \eta_\gamma(F_K)$ is the full circle.*
2)-If $\gamma/2\pi$ is irrational, the spectrum of $F_{K,\gamma,x}$ is the full circle for any $x \in \mathbf{T}$.
3)-If $\gamma/2\pi$ is rational, but $x/2\pi$ is irrational, the spectrum of $F_{K,\gamma,x}$ is the full circle.
4)-If $\gamma/2\pi$ and $x/2\pi$ are rational, $F_{K,\gamma,x}$ admits a band spectrum.

Proof : 1)-Since the family $\{\pi_{\gamma,x,y}; (x,y) \in \mathbf{T}^2\}$ of representations of $\mathcal{B}_\gamma$ is faithfull, we get :

$$\mathrm{Sp}_{\mathcal{B}_\gamma}(F_{K,\gamma}) = \cup_{(x,y)\in\mathbf{T}^2} e^{iy} \mathrm{Sp}(F_{K,\gamma,x}) \ . \tag{159}$$

Taking the union over y clearly gives the full circle.
2)-If $\gamma/2\pi$ is irrational, $\mathcal{B}_\gamma$ is simple. Thus each of the $\pi_{\gamma,x,y}$'s is faithfull, in particular :

$$S_1 = \mathrm{Sp}_{\mathcal{B}_\gamma}(F_{K,\gamma}) = \mathrm{Sp}(F_{K,\gamma,x}) \ , \qquad \forall x \in \mathbf{T}^2 \ . \tag{160}$$

3)-If $\gamma = 2\pi p/q$, the covariance condition (145) gives :

$$T^{nq} \pi_{\gamma,x,y}(F_K) T^{-nq} = \pi_{\gamma,x,y+nqx}(F_K) \ , \qquad n \in \mathbf{Z} \ . \tag{161}$$

In particular,

$$\mathrm{Sp}\left(\pi_{\gamma,x,y}(F_K)\right) = \mathrm{Sp}\left(\pi_{\gamma,x,y+nqx}(F_K)\right) \ , \qquad \forall n \in \mathbf{Z} \ . \tag{162}$$

If in addition $x/2\pi$ is irrational, given any $y' \in \mathbf{T}$ we can find a sequence (n_l) of integers such that $y' - y = \lim_{l\mapsto\infty} n_l q x \bmod 2\pi$. By the strong continuity of $\pi_{\gamma,x,y}$ with respect to y, it follows that :

$$\mathrm{Sp}\left(\pi_{\gamma,x,y'}(F_K)\right) \subset \mathrm{Sp}\left(\pi_{\gamma,x,y}(F_K)\right) \ . \tag{163}$$

Since y, y' are arbitrary, the same result holds after exchanging them. In particular for any y we have :

$$\mathrm{Sp}\left(\pi_{\gamma,x,y}(F_K)\right) = e^{iy} \mathrm{Sp}\left(\pi_{\gamma,x,0}(F_K)\right) = \mathrm{Sp}\left(\pi_{\gamma,x,0}(F_K)\right) \ , \tag{164}$$

showing the result.
4)-If $\gamma = 2\pi p/q$ and $x = 2\pi r/s$, the covariance property shows that $\pi_{\gamma,x,0}(F_K)$ is periodic. By the Bloch theorem we get a band spectrum. Actually one can easily see, using the corollary 4 that the algebra $\pi_{\gamma,x,0}(\mathcal{B}_I)$ is isomorphic to the subalgbera of $M_q \otimes M_s \otimes \mathcal{C}(\mathbf{T})$ generated by $u \otimes u' \otimes e^{ik}, v \otimes e^{ix} \otimes 1, w \otimes v' \otimes 1$ where u, v, w (resp. u', v') are the $q \times q$ matrices (resp. $s \times s$) defined in the theorem 12, and k is the

quasimomentum. Here we used the fact that $\pi_{\gamma,x,0}(F_K)^Q = \mathbf{I}$ if $Q = 2(q \vee s)$. This gives the band spectrum by diagonalizing the finite dimensional matrices and varying k.

The next set of results concerns the density of states. Let Δ be an interval in the unit circle, namely the image by $\omega \mapsto e^{i\omega}$ of an interval of the real line. Let us also call g_L the restriction to the finite set $[-L, L]$ of $g(\theta)$. This is a self adjoint matrix of dimension $2L + 1$. Let also $F^{(L)}_{0,\gamma,x}$ be the restriction of $F_{0,\gamma,x}$ to the same interval. Because it is diagonal it is a unitary $(2L+1) \times (2L+1)$ matrix. Then we set $F^{(L)}_{K,\gamma,x} = F^{(L)}_{0,\gamma,x} e^{iKg_L(\theta)/\gamma}$. Again, this is a unitary matrix of dimension $2L+1$. Let then $n_L(\Delta)$ be the number of eigenvalues of this matrix contained in Δ. As $L \mapsto \infty$, this number increases like $\mathrm{O}(L)$, so that we can define the Integrated Density of States (IDS) as the following limit, if it exists :

$$\mathcal{N}_{\gamma,x,y}(\Delta) = \lim_{L \mapsto \infty} \frac{n_L(\Delta)}{2L+1} \,. \tag{165}$$

The first important property is the "Shubin formula" [Bel:Gap]

Proposition 8 *If $\gamma/2\pi$ is irrational, the limit defining the IDS exists uniformly with respect to $(x, y) \in \mathbf{T}^2$ and is independent of (x, y). Moreover it is equal to :*

$$\mathcal{N}_\gamma(\Delta) = \tau_\gamma\left(\chi_\Delta(F_K)\right) \,, \qquad \text{(Shubin's Formula)} \,. \tag{166}$$

where χ_Δ is the characteristic function of the interval Δ.

The proof of this proposition can be found in [Bel:Kth, Bel:Gap, BeBoGh] for self adjoint operators. It can be easily adapt for the Floquet operator. We notice that the limit is reached uniformly with respect to (x, y). This is because the Furstenberg map is minimal and not only ergodic. Another remark is that the eigenprojection $\chi_\Delta(F_K)$ does not belong in general to the algebra $\mathcal{B}_\gamma$. However, it belongs to the von Neumann algebra $L^\infty(\mathcal{B}_\gamma, \tau_\gamma)$, namely the weak closure of $\mathcal{B}_\gamma$ in the GNS representation associated to the trace. Thus the Shubin formula is meaningfull.

Thanks to the Shubin formula, the IDS can be written as :

$$\mathcal{N}_\gamma(\Delta) = \int_\Delta d\mathcal{N}_\gamma(E) \,, \tag{167}$$

where $d\mathcal{N}_\gamma$ is a probability measure on the torus $\mathbf{T}$ (which we identify with the unit circle) called the Density of States (DOS). We can actually compute the DOS namely :

Proposition 9 *If $\gamma/2\pi$ is irrational, for any continuous real valued 2π-periodic function g on the real line, the DOS of the kicked rotor is equal to the normalized Lebesgue measure on the torus, namely :*

$$d\mathcal{N}_\gamma(E) = \frac{dE}{2\pi} \,. \tag{168}$$

Proof : The Shubin formula implies that the DOS is the unique probability measure on the torus such that :

$$\int_{\mathbf{T}} d\mathcal{N}_\gamma(E) e^{inE} = \tau_\gamma\left(F_K^n\right) , \qquad n \in \mathbf{Z} . \tag{169}$$

We claim that $\tau_\gamma\left(F_K^n\right) = 0$ unless $n = 0$ which will prove the result. For indeed, the trace is invariant by the automorphism group $\hat{\rho}_{0,k}$. On the other hand, we have :

$$\hat{\rho}_{0,k}\left(F_K\right) = e^{ik} F_K . \tag{170}$$

It follows that $\hat{\rho}_{0,k}\left(F_K^n\right) = e^{ink} F_K^n$ showing that

$$\tau_\gamma\left(F_K^n\right)\left(e^{ink} - 1\right) = 0 . \tag{171}$$

Our last result concerns the algebraic way of writing the kinetic energy. In order to study numerically the spectral properties of the kicked rotor, several physicists [CaChIzFo] have introduced the averaged kinetic energy. Giving an initial state $\phi \in \ell^2(\mathbf{Z})$, it is given by (see (9) & (11)) :

$$\mathcal{E}_c(t) = \langle\phi| F^t \frac{\mathbf{L}^2}{2I} F^{-t} |\phi\rangle , \qquad t \in \mathbf{Z} , \tag{172}$$

where $\mathbf{L}$ is the angular momentum, I is the moment of inertia and F the Floquet operator. Thanks to the definition of the position operator N (see Prop.7) and introducing the period T of the kicks, one can write it as :

$$\mathcal{E}_c(t) = \frac{I}{2T^2} \langle\phi| F^t \gamma^2 N^2 F^{-t} |\phi\rangle . \tag{173}$$

In order to keep only dimensionless quantities, we will redefine this kinetic energy by forgetting the prefactor $I/2T^2$. Moreover physicists usually choose an initial state localized on one value of the initial angular momentum. Using the covariance condition, it is always possible to choose $\phi = |0\rangle$ by changing the value of (x, y) if necessary. This why we will rather define the mean kinetic energy in the following way :

$$\mathcal{E}_{\gamma,x}(t) = \gamma^2 \langle 0| F_{K,\gamma,x}^t N^2 F_{K,\gamma,x}^{-t} |0\rangle . \tag{174}$$

We notice that varying y will not change this definition. Using now (146), it follows immediately that if $|A|^2 = AA^*$:

$$\mathcal{E}_{\gamma,x}(t) = \gamma^2 \langle 0| \pi_{\gamma,x,y}(|\partial_\theta F_{K,\gamma,x}^t|^2 |0\rangle . \tag{175}$$

The choice of the initial value of the angular momentum being arbitrary, we may average over the position of the initial state in momentum space, in order to get the generic properties of the system. This is equivalent to averaging over (x, y), namely to taking the trace. This why we will also consider the quantity :

$$\mathcal{E}_\gamma(t) = \gamma^2 \tau_\gamma\left(|\partial_\theta F_{K,\gamma,x}^t|^2\right) . \tag{176}$$

7 Localization and Dynamical Localization

7.1 Anderson's Localization

The localization phenomena was predicted in 1958 by Anderson [And] for conduction electrons in a disordered metal. The main idea underlying this effect is that the electronic wave in an infinite medium is reflected by the obstacles (ions, defects,etc,...). If the medium is a perfect crystal, the total reflection coefficient may not be equal to one due to constructive interference effects and allows the wave to travel freely towards the boundary. This happens whenever a Bragg condition is fulfilled, for special values of the total energy of the traveling particle, defining a band spectrum. This is the essence of Bloch theory for perfect metals. In such a case, the conductivity is infinite, if one neglects the influence of phonons and of the electron-electron interaction. If the medium is not periodic but quasiperiodic, such as quasicrystals, one may have also free Bloch waves if the quasiperiodic potential describing the influence of the ions on the travelling particle is not too strong [DiSi, BeLiTe, ChDe, BeIoScTe, BenSir].

However, in a disordered medium, the Bragg condition is unlikely, namely destructive interferences may force the electronic wave to vanish at infinity. Thus, the electonic wave is trapped in defects : in other words it is localized in a bounded region. Anderson proposed a tight binding model of such medium and could predict that 1-dimensional disordered chains always exhibit localization [Pas, Cyc]. Later on [AbAnLiRa] it was argued that in 2D the same effect occurs. But in higher dimension, localization holds only for strong disorder or at the band edges [FrSp, FrMaScSp]. Then if the disorder is not too strong, Ohm's law holds, leading to a finite conductivity, even if we ignore the phonons and the electron-electron interaction.

The Anderson model is extremely simple but contains most of the properties necessary to describe such a medium. In a tight binding representation, the electronic states can be represented as elements of the Hilbert space $\ell^2(\mathbf{Z}^D)$, if the crystal we start from is the D-dimensional lattice $\mathbf{Z}^D$. If there is no disorder, in the one electron approximation, the conduction electrons are approximately described by the free Laplacean Δ_D namely if $\psi \in \ell^2(\mathbf{Z}^D)$:

$$\Delta_D\psi(n) = t \sum_{|n-n'|=1} \psi(n') \ , \tag{177}$$

where t is the "hopping" parameter which measures the energy required for an electron to hop from one site to the next one. The energy spectrum is then given by the band $[-2Dt, 2Dt]$.

Adding one defect in the crystal can be described by adding to the previous Laplacean a local potential in the form of a sequence $V_{defect} = (V_{defect}(n); n \in \mathbf{Z}^D)$, as was shown in 1949 by Slater. To get a homogeneous distribution of defects it is therefore sufficient to replace V_{defect} by a homogeneous sequence V. To take into account the randomness of the defect distribution we will assume that the values $V(n)$ of this potential at each site are identically distributed random variables. Even though we expect some correlation between them in realistic systems, at least at short distances, Anderson proposed to consider the simplest case for which they are independent and uniformly distributed in an interval $[-W, W]$. Then W is a measure of the disorder strength. Let Ω be the corresponding probability space (in this example, $\Omega = [-W, W]^{\mathbf{Z}^D}$) and let $\mathbf{P}$ be the corresponding probability measure (in this example, $\mathbf{P} = \bigotimes_{n \in \mathbf{Z}^D} dV(n)/2W)$. The potential becomes a function of the random variable $\omega \in \Omega$ so that the Anderson Hamiltonian can be written as :

$$H_\omega = \Delta_D + V_\omega \ . \tag{178}$$

The probability space $(\Omega, \mathbf{P})$ can be seen as the configuration space for the disorder. The translation invariance of the original lattice is not completely lost. For indeed, translating this new system is equivalent to translate the distribution of defects back. More precisely, there is a measure preserving action of the translation group on Ω. For the Anderson model this action is given by $T^r\omega_n = \omega_{n-r}$. If we denote by $T(r)$ the translation by $r \in \mathbf{Z}^D$ in the Hilbert space, namely for $\psi \in \ell^2(\mathbf{Z}^D)$, $T(r)\psi(n) = \psi(n-r)$, we get the following "covariance condition":

$$T(r)H_\omega T(r)^{-1} = H_{T^r\omega} \ . \tag{179}$$

We will complete this framework by adding two conditions. The first one is the ergodicity of the probability measure $\mathbf{P}$. Thanks to Birkhoff's ergodic theorem, it expresses the fact that space averages coincide with $\mathbf{P}$-average. In this way, $\mathbf{P}$ can be constructed in practice simply by taking space averages, an unambiguous process. The second one concerns the existence of a topology on Ω which makes it a compact Hausdorff space, and such that the $\mathbf{P}$-measurable sets are generated as a σ-algebra by the Borel sets, namely $\mathbf{P}$ is a Radon measure. In the Anderson model the product topology will do it. Actually an intrinsic definition of homogeneous system has been proposed in [Bel:Kth, Bel:Gap] leading to the definition of a canonical topology on the disorder configuration space. For this topology, the mapping $\omega \in \Omega \mapsto H_\omega$ is strongly continuous (in the resolvent sense whenever H_ω is unbounded self adjoint).

To summarize, homogeneous media, such as crystals, quasicrystals, glasses, amorphous, aperiodic or disordered systems, may be mathematically described by the following axioms.

(D1)-The disorder configuration space is a compact Hausdorff topological space Ω endowed with a probability Radon measure $\mathbf{P}$

(D2)-The translation group is a locally compact abelian group G acting in Ω by mean of a continuous group of homeomorphisms $\omega \mapsto g\omega$. The probability $\mathbf{P}$ is G-invariant and ergodic.

(D3)-The quantum state space is a separable Hilbert space $\mathcal{H}$ in which G acts through a projective unitary representation $\{T(g); g \in G\}$.

(D4)-The Hamiltonian is a strong-resolvent continuous family $H = \{H_\omega; \omega \in \Omega\}$ of self adjoint operators acting on $\mathcal{H}$ with a common G-invariant domain $\mathcal{D}$.

(D5)-A covariance condition is satisfied, namely:

$$T(g)H_\omega T(g)^{-1} = H_{g\omega} \ . \tag{180}$$

In general we will prefer a projective unitary representation. For indeed there are concrete examples for which the translation group does not act as a true representation. This is the case for a crystal in a uniform magnetic field [Bel:Gap]. We have restricted ourself to abelian translation groups because no concrete useful example have been studied till now with non abelian groups. However, systems living on a Cayley tree admits a non abelian translation group which is usually a free group. We can also include in G other symmetries like rotations, reflections, if necessary. This has never been investigated in detail yet, even though we believe that it should be useful: classification of defects in crystal may be related to such groups.

The smallest observable algebra that can be of interest for physics, is the one constructed with the energy. In more concrete systems however, other observables like spins, may be relevant. For simplicity, we will consider the simplest case for which the only relevant observable is the energy. In a homogeneous medium, the choice of the origin is arbitrary, since the systems reproduces itself under translation. So that the physics of the system is described by any of the elements of the family $H = \{H_\omega; \omega \in \Omega\}$ representing the energy. In order to avoid choosing arbitrarily one of them, we will include all of them. We then define a non commutative C^*-algebra $C^*(H)$ as the smallest one in the space of bounded operators on$\mathcal{H}$ containing the resolvent of each of the elements of H. In general, we do not know the structure of such an algebra. However for most concrete examples construct till now, namely by using the Schrödinger operator for one electron systems [Bel:Kth, Bel:Gap], like the Anderson model, this algebra is nothing but the crossed product $\mathcal{C}(\Omega) \times G$ defined by the topological dynamical system (Ω, G) describing the disorder configurations in the original medium. This algebra must be slightly modified if a uniform magnetic field is turned on. We will ignore this latter case here.

Thanks to this framework, there is a very close analogy with aperiodic media in Solid State Physics and the dynamics of a kicked rotor. Even though the physical interpretation is very different, the C^*-algebra used to describe the observables is also a crossed product. However, in the kicked rotor model, the lattice G is the quantized momentum space instead, and the space Ω admits a fairly different interpretation since the variable γ plays the role of an effective Planck constant and is related to the period of the kicks, x plays the role of a magnetic field, whereas y represents a generic translation in momentum space. We also notice that the ergodicity of the measure holds only if $\gamma/2\pi$ is a fixed irrational number.

There is also a very close analogy with 2D-dimensional lattice electrons in a uniform magnetic field. We have already seen that the observable algebra is the rotation algebra $\mathcal{A}_I$ which can also be seen as the crossed product $\mathcal{C}(I \times \mathbf{T}) \times_\phi \mathbf{Z}$ if $\phi : (\gamma, x) \in I \times \mathbf{T} \mapsto (\gamma, x + \gamma) \in I \times \mathbf{T}$. Then γ plays the role of a dimensionless magnetic flux per plaquette, whereas x is a generic position of the origin in the x-direction of the lattice. Again, the ergodicity of the measure on $\Omega = I \times \mathbf{T}$ holds only if $I = \{\gamma\}$ where $\gamma/2\pi$ is a fixed irrational.

The main question now is whether this formal analogy between so different problems will produce phenomena similar to Anderson's localization. The common belief is that if H is a selfadjoint operator belonging to this algebra, with short range interactions, namely if it is smooth enough with respect to the differential structure that will be described in the next subsection, it will exhibit such phenomena at least if the dynamical system (Ω, G) is "sufficiently aperiodic". The precise meaning of "sufficiently aperiodic" is not completely understood yet. Several numerical studies have investigate this question, but they are far from having given a precise criterion yet [FiHuXX]. More precisely we define a 2-point function by $C(g) = \langle F F_g \rangle - \langle F \rangle^2$, where F is a continuous function on Ω and $F_g(\omega) = F(g^{-1}\omega)$ while $\langle \cdot \rangle$ is the ergodic average. If any 2-point function converges to zero fast enough as $g \mapsto \infty$, the localization is expected to occur. This is certainly not the case for a periodic or an almost periodic dynamics, describing for instance a perfect crystal with or whithout a uniform magnetic field. And indeed we do not expect in this case localization to occur. Still, a 1D model like the Almost Mathieu Hamiltonian [AuAn, ChDe, BeLiTe], has been proved to exhibit a metal-insulator transition at large coupling. But the Furstenberg map for instance, which satisfies this criterion, should give rise to localization. This is the basis of an argument by Fishman, Grempel and Prange [FiGrPr] predicting that localization occurs in the kicked rotor problem.

The next problem therefore is to describe mathematically what we expect to characterize the localization. One of the first criterion used by Anderson was connected to the time evolution of quantum states : if the time-average of the probability for the initial state to come back after time t is positive, then localization do occur. We will see later on, thanks to an early result of Pastur [Pas] that this criterion is related to the existence of a point spectrum for H_ω, $\mathbf{P}$-almost surely. This is essentially why mathematicians describe localization in term of the existence of a point spectrum. It is related to the finiteness of the so called "inverse participation ratio" (see below). Another way consists in defining the localization length: roughly speaking it gives a measure of the diameter of the region where a typical eigenstate is localized.

One of the main problems in dealing with the spectral property of the Hamiltonian, is that in many situations, this requires the choice of a fixed representation of the observable algebra. While in the Anderson model, this choice is quite natural, thanks to the description of the original disorders medium, in other models for which we would like to use the localization theory, it is not necessarily so. Two inequivalent representations of the same algebra may give different type of spectral measure for the same Hamiltonian. This happens for instance in the problem of Bloch electrons in a magnetic field. Therefore if this latter point of view were correct, localization would require to distinguish physically between different representations. However, the computation of the localization length requires a space average, in order to get a quantity insensitive to the specific configuration of the disorder, and therefore as we will see, it can be interpreted in a purely algebraic way. There is therefore an apparent contradiction between the two points of view. This is actually nothing but the usual opposition between the Schrödinger and Heisenberg point of view in Quantum Mechanics. Our main purpose in this section is to show how to reconcile them, and to show that in some sense they are equivalent.

Our last comment concerns the semiclassical limit. While this limit is meaningless in the Anderson problem, since the starting point is the band theory for perfect crystals, a fairly strong quantum theory, the kicked rotor problem gives a nice example where the semiclassical limit exists indeed together with a localization effect. It is therefore natural to consider what happens to the localization phenomena in this limit. The main discovery of Chirikov, Izrailev and Shepelyansky [ChIzSh] was to relate this limit to the diffusion constant in phase space of the classical kicked rotor. Even though this relation has not been proved to hold rigorously, many numerical studies show that it is probably correct at least under some unknown "generic condition". Therefore we have reached here one point of the so-called "quantum chaos". We will give only some pieces of this puzzle here.

7.2 The Observable Algebra

To avoid useless technical difficulties, we consider now the C^*-algebra $\mathcal{C}(\Omega) \times G$ where $G = \mathbf{Z}^D$.D will be called the dimension of the lattice. However, most of what will be described here can be extended to more general groups such as $\mathbf{R}^D$ for instance. As for the rotation or the Furstenberg algebra, we can develop a calculus as follows.

Elements of $\mathcal{C}(\Omega) \times \mathbf{Z}^D$ are continuous complex functions $a(\omega, n)$ on the space $\Omega \times \mathbf{Z}^D$ vanishing at infinity. To define this algebra properly, it is more convenient to start with the dense subalgebra $\mathcal{C}_c(\Omega \times \mathbf{Z}^D)$ of continuous functions on $\Omega \times \mathbf{Z}^D$ with compact support, endowed with the following operations:

$$ab(\omega; n) = \sum_{l \in \mathbf{Z}^D} a(\omega; l) b(T^{-l}\omega; n-l) \ , \tag{181}$$

$$a^*(\omega; n) = \overline{a(T^{-n}\omega; -n)} \ . \tag{182}$$

Since the functions a and b have compact support, the sum above is finite. Remarkable elements are given by :

$$\mathbf{I}(\omega; n) = \delta_{n,0} \ , \qquad U(r)(\omega; n) = \delta_{n,-r} \ , \qquad r \in \mathbf{Z}^D \ . \tag{183}$$

The first one $\mathbf{I}$ is a unit, whereas $U(r)$ is a group of unitaries namely $U(r)U(r') = U(r+r')$, $U(0) = \mathbf{I}$ and $U(r)^* = U(-r) = U(r)^{-1}$.

A family of representations in the Hilberts space $\ell^2(\mathbf{Z}^D)$ indexed by $\omega \in \Omega$ is given by:

$$\pi_\omega(a)\psi(n) = \sum_{n' \in \mathbf{Z}^D} a(T^{-n}\omega; n'-n)\psi(n') \ , \qquad a \in \mathcal{C}_c(\Omega \times \mathbf{Z}^D) \ , \qquad \psi \in \ell^2(\mathbf{Z}^D) \ . \tag{184}$$

In particular we get $a(\omega; n) = \langle 0 | \pi_\omega(a) | n \rangle$. Then a C^*-norm is defined by:

$$\|a\| = \sup_{\omega \in \Omega} \|\pi_\omega(a)\| \ , \qquad a \in \mathcal{C}_c(\Omega \times \mathbf{Z}^D) \ . \tag{185}$$

Then $\mathcal{C}(\Omega) \times_T \mathbf{Z}^D$ is the completion of $\mathcal{C}_c(\Omega \times \mathbf{Z}^D)$ under this norm. To shorten the notations we will denote it by $\mathcal{A}$.

Given an invariant probability measure $\mathbf{P}$ on Ω, a normalized trace $\tau_{\mathbf{P}}$ (or τ for short whenever no confusion arises) is defined by:

$$\tau(a) = \int_\Omega d\mathbf{P} a(\omega; 0) \ , \qquad a \in \mathcal{A} \ . \tag{186}$$

It is easy to see, by using the Birkhoff ergodic theorem, that if $\mathbf{P}$ is ergodic,

$$\tau(a) = \lim_{\Lambda \uparrow \mathbf{Z}^D} \frac{1}{|\Lambda|} \mathrm{Tr}_\Lambda \left(\pi_\omega(a)\right) \ , \qquad \text{for } \mathbf{P} - \text{almost all } \omega \ . \tag{187}$$

At last, the differential structure is related to the group action and defined as follows. If $n = (n_1, \ldots, n_D) \in \mathbf{Z}^D$, we define the $*$-derivation ∂_μ by:

$$\partial_\mu a(\omega; n) = i n_\mu a(\omega; n) \ , \qquad \mu = 1, \ldots, D \ . \tag{188}$$

These derivations commute together and are the infinitesimal generators of the D-parameter group of automorphism $\{\rho_\theta; \theta \mathbf{T}^D\}$ (the so-called dual action of Takasaki [Ped]) defined by:

$$\rho_\theta(a)(\omega; n) = e^{i\theta n} a(\omega; n) \ , \qquad \theta n = \theta_1 n_1 + \ldots + \theta_D n_D \ . \tag{189}$$

Moreover, denoting by N_μ the position operators defined by $N_\mu \psi(n) = n_\mu \psi(n)$ in $\ell^2(\mathbf{Z}^D)$, we get:

$$\pi_\omega(\partial_\mu a) = i[N_\mu, \pi_\omega(a)] \ . \tag{190}$$

7.3 Localization Criteria

In this subsection we give several criteria for the localization and discuss the relation between its finiteness and the nature of the spectrum. We will consider a self adjoint element $H = H^*$ in the algebra $\mathcal{A} = \mathcal{C}(\Omega) \times_T \mathbf{Z}^D$ previously described. In view of the study of a Floquet operator we may consider a unitary element $F = (F^*)^{-1}$ of this algebra instead. This latter case reduces to the former provided we identify F with e^{iTH} for some $T > 0$ and the Borel sets Δ are subset of the unit circle. In the physical representation π_ω we consider the operator $\pi_\omega(H) = H_\omega$ instead.

If Δ is some Borel subset of $\mathbf{R}$ we denote by P_Δ t eigenprojection of H corresponding to energies in Δ namely :

$$P_\Delta = \chi_\Delta(H) \ , \tag{191}$$

where χ_Δ is the characteristic function of the interval Δ. Again, we notice that in general P_Δ may not belong to $\mathcal{A}$. However it always belongs to the so-called Borel algebra $\mathcal{B}(\mathcal{A})$ [Ped], formally generated by Borel functions of elements of $\mathcal{A}$. The Borel functional calculus permits to extend any representation of $\mathcal{A}$ to its Borel algebra. Hence the previous definition makes sense. The price we pay for it is that the mapping $\omega \in \Omega \mapsto \pi_\omega(a)$ may not necessarily be strongly continuous any more, but it is always strongly borelian if $a \in \mathcal{B}(\mathcal{A})$.

If H_ω has a pure point spectrum in Δ we get the following decomposition:

$$\pi_\omega(P_\Delta) = \sum_{E \in \Delta} \Pi_E(\omega) \ , \tag{192}$$

where $\Pi_E(\omega)$ is the eigenprojection of H_ω corresponding to the eigenvalue E. If E is a simple eigenvalue, one gets $\Pi_E(\omega) = |\psi_{E,\omega}\rangle\langle\psi_{E,\omega}|$, where $\psi_{E,\omega}$ is a normalized eigenstate namely:

$$\|\psi_{E,\omega}\|^2 = \sum_{n \in \mathbf{Z}^D} |\psi_{E,\omega}(n)|^2 = 1 < +\infty \ . \tag{193}$$

The first quantity measuring the localization is the probability of staying at the origin. It was introduced by [And] and studied by Pastur [Pas]. To define it let us first consider the time-average $A_{n,n'}(\Delta, \omega)$ of the probability for an initial state at n to be localized at n' after time t:

$$A_{n,n'}(\Delta, \omega) = \lim_{T \mapsto \infty} \int_0^T \frac{dt}{T} \, |\langle n|\pi_\omega(e^{itH} P_\Delta)|n'\rangle|^2 \ . \tag{194}$$

If H_ω has a pure point spectrum, the decomposition (192) leads to :

$$A_{n,n'}(\Delta,\omega) = \sum_{E\in\Delta} |\langle n|\Pi_E(\omega)|n'\rangle|^2 \ . \tag{195}$$

The covariance condition implies $A_{n,n'}(\Delta,\omega) = A_{0,n'-n}(\Delta,T^{-n}\omega)$, so that the staying probability is entirely given by the function $A_{0,0}(\Delta,\omega)$, provided we consider it as a function of the disorder. We remark that if the eigenvalues are simple, since the eigenstates are normalized we get:

$$A_{0,0}(\Delta,\omega) = \frac{\sum_{E\in\Delta} |\psi_{E,\omega}(0)|^4}{\left(\sum_{E\in\Delta} |\psi_{E,\omega}(0)|^2\right)^2} \ , \tag{196}$$

namely $A_{0,0}(\Delta,\omega)$ is the mean inverse participation ratio for energies in Δ. To get a quantity insensitive to the disorder, let us average it with respect to $\mathbf{P}$ defining the averaged inverse participation ratio :

$$\xi_\Delta = \int_\Omega d\mathbf{P}\, A_{0,0}(\Delta,\omega) \ . \tag{197}$$

Using now the automorphism group defined in eq.(189) and the eq.(184,186), an elementary calculation leads to the following expression for ξ_Δ:

$$\xi_\Delta = \lim_{T\mapsto\infty} \int_0^T \frac{dt}{T} \int_{\mathbf{T}^D} \frac{d^D\theta}{(2\pi)^D}\, \tau(e^{itH} P_\Delta \rho_\theta(e^{-itH} P_\Delta)) \ . \tag{198}$$

So we see that the staying probability or the inverse participation ratio, admits a purely algebraic expression. The Pastur theorem [Pas] can then be established as follows:

Theorem 13 *For almost all $\omega \in \Omega$, the number of eigenvalues of H_ω in Δ is either zero or infinity. The latter is realized, namely H_ω has some point spectrum in Δ, if and only if the averaged inverse participation ratio ξ_Δ is positive.*

Comment: this criterion is not sufficient to eliminate continuous spectrum.

We now introduce a stronger notion of localization giving a measurement of the localization length. Whenever $\pi_\omega(H)$ has pure point spectrum, the eigenstate may decay faster at infinity on the lattice. We are led to introduce quantities like:

$$\ell^{(p)}(E,\omega) = \left[\sum_{n\in\mathbf{Z}^D} |\psi_{E,\omega}(n)|^2 |n|^p\right]^{1/p} \ , \tag{199}$$

for $p \geq 1$. If the eigenstates decrease exponentially fast one can also consider the quantity

$$\ell(E,\omega) = \limsup_{n\mapsto\infty} \frac{-ln|\psi_{E,\omega}(n)|}{|n|} \ . \tag{200}$$

However such expressions are very badly behaving with ω in general and they are not suited for comparison with experiments or numerical calculations. The following definition will be more convenient and will give rise to a quantity independent of ω. We consider the averaged fluctuation of the position in the form:

$$\Delta X_{\omega,n}(T)^2 = \int_0^T \frac{dt}{T} \langle n|(N_\omega(t) - N)^2|n\rangle \ , \tag{201}$$

where $N_\omega(t) = e^{iH_\omega t} N e^{-iH_\omega t}$, and $N = (N_1, \ldots, N_D)$ is the position operator. The covariance property gives $\Delta X_{\omega,n}(T) = \Delta X_{T^{-n}\omega,0}(T)$, so that after averaging over the disorder, we get a quantity indcpendent of n, namely $\Delta X(T)^2 = \int_\Omega d\mathbf{P}(\omega) \Delta X_{\omega,0}(T)^2$. An elementary calculation shows that:

$$\Delta X(T)^2 = \int_0^T \frac{dt}{T} \tau(|\nabla e^{iHt}|^2) \ . \tag{202}$$

We will generalize this expression by considering, for every Borel subset Δ of the real line, the corresponding quantity $\Delta X_\Delta(T)^2$ obtained in the same way if we replace e^{iHt} by $e^{iHt} P_\Delta$. The main result in this respect is the following:

Theorem 14 *If*

$$\ell^2(\Delta) = \limsup_{T \mapsto \infty} \Delta X_\Delta(T)^2 < \infty \ , \tag{203}$$

then H_ω has a pure point spectrum in Δ for almost all $\omega \in \Omega$. Moreover, if $\mathcal{N}$ denotes the density of states of H, there is an $\mathcal{N}$-measurable non negative function ℓ on $\mathbf{R}$ *such that for every Borel subset Δ' of Δ,*

$$\ell^2(\Delta') = \int_{\Delta'} d\mathcal{N}(E) \ell(E)^2 \ , \tag{204}$$

Comment: we will see in the proof that if $\sigma_{pp}(\omega)$ denotes the set of eigenvalues of H_ω :

$$\ell^2(\Delta') = \int_\Omega d\mathbf{P}(\omega) \sum_{n \in \mathbf{Z}^D} n^2 \sum_{E \in \sigma_{pp}(\omega) \cap \Delta} |\langle 0|\Pi_E(\omega)|n\rangle|^2 \ . \tag{205}$$

In particular, letting Δ shrink to the point E, the function $\ell(E)^2$ represents a kind of average (over the disorder and over a small spectral set around E), of the quantity $\sum_{n \in \mathbf{Z}^D} n^2 |\psi_{E,\omega}(n)|^2$. Namely it is a measure of the extension of the eigenstate corresponding to E. This is the reason for the definition below.

Definition 2 *The function ℓ will be called the localization length for H.*

Proof of the theorem: (i)-The basic argument we will use here is due to Guarneri [Gua, Bel:Tre]. We will denote by $\sigma_{pp}(\omega)$ the set of eigenvalues of H_ω (the point spectrum), whereas $\Pi_{pp}(\omega)$ will denote its spectral projection on the point spectrum and $\Pi_c(\omega) = \mathbf{I} - \Pi_{pp}(\omega)$ will be its spectral projection on the continuous part of its spectrum. Using the definition of the trace in $\mathcal{A}$, we get :

$$\Delta X_\Delta(T)^2 = \int_\Omega d\mathbf{P}(\omega) \sum_{n \in \mathbf{Z}^D} n^2 p_T(\omega, n) \ , \tag{206}$$

where,

$$p_T(\omega, n) = \int_0^T \frac{dt}{T} |\langle 0|\pi_\omega(e^{iHt} P_\Delta)|n\rangle|^2 \ . \tag{207}$$

We will set $p_T(n) = \int_\Omega d\mathbf{P}(\omega) p_T(\omega, n)$. By definition, we have :

$$0 \leq p_T(\omega, n) \leq 1 \ , \tag{208}$$

$$\sum_{n \in \mathbf{Z}^D} p_T(\omega, n) = 1 \ , \tag{209}$$

whereas the Wiener criterion gives

$$\lim_{T \mapsto \infty} p_T(\omega, n) = \sum_{E \in \sigma_{pp}(\omega) \cap \Delta} |\langle 0|\Pi_E(\omega)|n\rangle|^2 \ . \tag{210}$$

In particular, if L is a positive integer, and $|n|_\infty = \max_{1 \leq j \leq D} |n_j|$,

$$\lim_{T \mapsto \infty} \sum_{|n|_\infty < L} p_T(\omega, n) \leq \langle 0|\Pi_{pp}(\omega)|0\rangle = 1 - \langle 0|\Pi_c(\omega)|0\rangle \ . \tag{211}$$

If we set $r = \int_\Omega d\mathbf{P}(\omega) \langle 0|\Pi_c(\omega)|0\rangle$ we obtain after averaging over the disorder

$$\lim_{T \mapsto \infty} \sum_{|n|_\infty < L} p_T(n) \leq 1 - r \ . \tag{212}$$

Since $r \geq 0$, one can find $T_L > 0$ such that if $T \geq T_L$, $\sum_{|n|_\infty < L} p_T(n) \leq 1 - r/2$. Thus :

$$\Delta X_\Delta(T)^2 \geq L^2 \int_\Omega d\mathbf{P}(\omega) \sum_{|n|_\infty \geq L} p_T(\omega, n) \geq L^2 (1 - \sum_{|n|_\infty < L} p_T(n)) \geq \frac{L^2 r}{2} \ . \tag{213}$$

Taking the limit $T \mapsto \infty$ leads to $L^2 r \leq 2\ell^2(\Delta) < \infty$ for any $L \in \mathbf{N}$. Thus $r = 0$ showing that for almost all ω's, $\langle 0|\Pi_c(\omega)|0\rangle = 0$. Using the covariance condition we also get for all n's $\langle n|\Pi_c(\omega)|n\rangle = 0$ almost surely, and since $\mathbf{Z}^D$ is countable, there is $\Omega' \subset \Omega$ of probability one such that for any $\omega \in \Omega'$, $\Pi_c(\omega)|n\rangle = 0$ for all $n \in \mathbf{Z}^D$, namely the continuous spectrum is empty.

(ii)-Given two Borel subsets $\Delta_1, \Delta_2 \subset \Delta$, we define the following expression:

$$\mathcal{E}^{(L)}_{T,\omega}(\Delta_1, \Delta_2) = \int_0^T \frac{dt}{T} \sum_{|n|_\infty < L} n^2 \langle 0|\pi_\omega(e^{iHt} P_{\Delta_1})|n\rangle \overline{\langle 0|\pi_\omega(e^{iHt} P_{\Delta_2})|n\rangle} \ . \tag{214}$$

In particular $\mathcal{E}^{(L)}_{T,\omega}(\Delta, \Delta) = \sum_{|n|_\infty < L} n^2 p_T(\omega, n)$. This expression gives a Borel function of ω. In addition, using the Wiener criterion, we have:

$$\lim_{T \mapsto \infty} \mathcal{E}^{(L)}_{T,\omega}(\Delta_1, \Delta_2) = \sum_{|n|_\infty < L} n^2 \sum_{E \in \sigma_{pp}(\omega) \cap \Delta_1 \cap \Delta_2} |\langle 0|\Pi_E(\omega)|n\rangle|^2 = \mathcal{E}^{(L)}_\omega(\Delta_1 \cap \Delta_2) \ . \tag{215}$$

From this definition of $\mathcal{E}^{(L)}_\omega(\Delta')$ whenever $\Delta' \subset \Delta$ is Borel, it follows that

(a) $0 \leq \mathcal{E}^{(L)}_\omega(\Delta') \leq L^2$,
(b) If $\Delta_1 \cap \Delta_2 = \emptyset$, then $\mathcal{E}^{(L)}_\omega(\Delta_1 \cup \Delta_2) = \mathcal{E}^{(L)}_\omega(\Delta_1) + \mathcal{E}^{(L)}_\omega(\Delta_2)$,
(c) If $(\Delta_i)_{i \in \mathbf{N}}$ is a decreasing sequence of Borel susbets of Δ converging to the empty set, namely $\bigcap_{i \in \mathbf{N}} \Delta_i = \emptyset$, then $\mathcal{E}^{(L)}_\omega(\Delta_i)$ decreases to zero,
(d) $\mathcal{E}^{(L)}_\omega(\Delta') \leq \mathcal{E}^{(L+1)}_\omega(\Delta')$,
(e) $\mathcal{E}^{(L)}_\omega(\Delta')$ is a Borel function of ω as a pointwise limit of Borel functions.

After averaging over the disorder we obtain $\mathcal{E}^{(L)}(\Delta') = \int_\Omega d\mathbf{P}(\omega)\mathcal{E}^{(L)}_\omega(\Delta')$ which fulfill (a), (b), (c) using the dominated convergence theorem, and (d). From (202,206), we also get:

$$\int_\Omega d\mathbf{P}(\omega)\mathcal{E}^{(L)}_{T,\omega}(\Delta',\Delta') \leq \int_0^T \frac{dt}{T}\tau(|\nabla e^{iHt}P_{\Delta'}|^2) \ . \tag{216}$$

Using the dominated convergence theorem we conclude that $\mathcal{E}^{(L)}(\Delta') \leq \ell^2(\Delta')$ and also thanks to the property (b), $\mathcal{E}^{(L)}(\Delta') \leq \ell^2(\Delta)$ for $\Delta' \subset \Delta$. It follows that $\lim_{L\mapsto\infty}\mathcal{E}^{(L)}(\Delta') = \mathcal{E}(\Delta')$ exists and defines a non negative σ-additive set function over the set of Borel subsets of Δ, namely a Radon measure. Moreover it satisfies $\mathcal{E}(\Delta') \leq \ell^2(\Delta')$, and by the monotone convergence theorem, eq.(215) above implies:

$$\mathcal{E}(\Delta') = \int_\Omega d\mathbf{P}(\omega) \sum_{n\in\mathbf{Z}^D} n^2 \sum_{E\in\sigma_{pp}(\omega)\cap\Delta} |\langle 0|\Pi_E(\omega)|n\rangle|^2 \ . \tag{217}$$

(iii)-On the other hand the definition of $\ell^2(\Delta')$ and Fatou's lemma imply:

$$\ell^2(\Delta') \leq \int_\Omega d\mathbf{P}(\omega) \sum_{n\in\mathbf{Z}^D} n^2 \limsup_{T\mapsto\infty} \int_0^T \frac{dt}{T}|\langle 0|\pi_\omega(e^{iHt}P_{\Delta'})|n\rangle|^2 \ . \tag{218}$$

By the Wiener criterion the right hand side is nothing but $\mathcal{E}(\Delta')$ showing that $\ell^2(\Delta') = \mathcal{E}(\Delta') \leq \ell^2(\Delta)$ for $\Delta' \subset \Delta$. Hence it is a nonnegative Radon measure on Δ.
(iv)-To finish the proof it is sufficient to show that this measure is absolutely continuous with respect to the DOS $\mathcal{N}$ of H. Let then $\Delta' \subset \Delta$ be such that $\mathcal{N}(\Delta') = \tau(\Delta') = 0$. From the definition of the trace it follows that $\langle 0|\pi_\omega(P_{\Delta'})|0\rangle = 0$ almost surely. By covariance and because $\mathbf{Z}^D$ is countable this gives $\pi_\omega(P_{\Delta'})|n\rangle = 0$ for all n's almost surely, namely $\pi_\omega(P_{\Delta'}) = 0$ almost surely. Then (214) above implies $\mathcal{E}^{(L)}_{T,\omega}(\Delta',\Delta') = 0$ for any L, T, and almost every ω. Consequently $\mathcal{E}(\Delta') = 0$, and the representation (204) holds.

7.4 Localization in the Kicked Rotor

As claimed previously, one can use the same formalism for investigating the localization properties of the kicked rotor. It is then sufficient to work with the Floquet operator instead of a Hamiltonian, and with Borel subsets of the circle. However the C^*-algebra we are using, $\mathcal{B}_\gamma$, is parametrized by the effective Planck constant γ, an additional parameter here. Apart from this remark, we get the previous structure if we set $\Omega = \mathbf{T}^2$, $D = 1$, and the action is provided by the Furstenberg map. The Lebesgue measure $dxdy/4\pi^2$ gives the probability measure, which is ergodic whenever $\gamma/2\pi$ is irrational. In view of the theorem (14) above, we cannot expect any finite localization length otherwise, because the action is no longer ergodic and from a result of Izrailev and Shepelyanski [IzSh] it follows that we get an absolutely continuous spectrum for $\pi_{\gamma,x,y}(F)$ whenever $\gamma/2\pi$ and $x/2\pi$ are rational. Now, we remark that the definition of the localization length coincides with the definition of the mean kinetic energy given by (176) up to the constant γ^2. Hence $F_{K,\gamma,x}$ will have a pure point spectrum in Δ whenever the mean kinetic energy

$$\overline{\mathcal{E}}_\Delta(\gamma) = \gamma^2 \limsup_{T\mapsto\infty} \frac{1}{T}\sum_{t=0}^{T-1} \tau_\gamma\left(|\partial_\theta(F^t_{K,\gamma}P_\Delta)|^2\right) \ , \tag{219}$$

is finite. Moreover we get the elementary formula $\gamma^2\ell_\gamma^2(\Delta) = \overline{\mathcal{E}}_\Delta(\gamma)$ whenever ℓ_γ denotes the localization length in term of γ.

The case of the kicked rotor permits to go a little bit further. First of all, the definition of the Floquet operator permits to show that it is $\mathcal{C}^\infty$ with respect to ∂_θ. Moreover we get the folowing result:

Proposition 10 *If the localization length ℓ_γ exists for the Floquet operator $F_{K,\gamma}$ of the kicked rotor model, it is constant over the circle.*

Proof: Clearly $\eta_y = \hat{\rho}_{\theta=0,y}$ (see (135)) commutes with the derivation ∂_θ. Moreover η_y translates the spectrum of $F_{K,\gamma}$ by y along the circle because for any Borel subset Δ of $\mathbf{T}$:

$$\eta_y(F_{K,\gamma}) = e^{iy}F_{K,\gamma} \ , \qquad \eta_y(P_\Delta) = P_{\Delta+y} \ . \tag{220}$$

Thus

$$\tau_\gamma\left(|\partial_\theta(F_{K,\gamma}^t P_\Delta)|^2\right) = \tau_\gamma\left(\eta_y|\partial_\theta(F_{K,\gamma}^t P_\Delta)|^2\right) = \tau_\gamma\left(|\partial_\theta(F_{K,\gamma}^t P_{\Delta+y})|^2\right) \ , \tag{221}$$

because η_y is an automorphism. It implies $\ell_\gamma^2(\Delta) = \ell_\gamma^2(\Delta + y)$ for any Borel set Δ, and therefore $\ell(E) = \text{const}$. Thus Δ is not needed anymore so that :

Corollary 5 *For the kicked rotor model the following formula holds*

$$\ell_\gamma^2 = \frac{\overline{\mathcal{E}_\gamma}}{\gamma^2} = \lim_{T\mapsto\infty}\frac{1}{T}\sum_{t=0}^{T-1}\tau_\gamma\left(|\partial_\theta(F_K^t)|^2\right) \tag{222}$$

A result by Casati & Guarneri [CaGu] shows that, the spectral measure of $F_{K,\gamma,x}$ is purely continuous generically in γ. Thus :

Proposition 11 *For the kicked rotor model there is a dense G_δ-set Γ of zero Lebesgue measure in $[0,1]$ such that for any $\gamma \in \Gamma$ the localization length diverges.*

However many numerical calculations [CaChIzFo, BeBa] have shown that the mean kinetic energy for the quantum kicked rotor model is bounded in time. So we expect the localization length to be finite on a "large set" of γ's, presumably for almost all γ's in $[0,1]$. Before discussing this question let us mention without proof another result which supplement the previous one namely

Proposition 12 *For the kicked rotor model the localization length is a lower semi-continuous function of γ.*

We may also expect $\gamma^2\ell^2(\gamma)$ to converges to some finite quantity as $\gamma \mapsto 0$. This is the content of the Chirikov-Izrailev-Shepelyansky formula [ChIzSh] found on the basis of a numerical work. The well-known observation is that despite the diffusive behavior of the classical model (namely for strong coupling) the quantized version exhibits, up to a certain breaking time τ^*, a diffusion-like motion in phase space and then for $t > \tau^*$ its kinetic energy saturates as a function of time. This numerical result allows us to write

$$\overline{\mathcal{E}_\gamma} = \mathcal{E}_\gamma(\tau^*) \simeq D\tau^* \ , \tag{223}$$

where D is the classical diffusion coefficient.

There is here a mathematical difficulty. First of all, never was a diffusion coefficient shown to exist rigorously for the standard map. Moreover, averaging it over all possible initial conditions will not give a finite quantity due to the "Pustilnikov acceleration modes" (or "islands of stability"). This means that we should not average over the full torus. It raises the question of which quantum average should be considered. However, recent works [BeVa, Vai, Cher] have shown that for the sawtooth map, a diffusion coefficient does exist. Moreover, a conjecture states that for the standard map there is a "large" set of values of K for which no island of stability occurs, and a diffusion constant does exist.

To get the Chirikov-Izrailev-Shepelyansky formula, we argue as follows. Since the eigenstates of the Floquet operator are localized, only a finite number $\ell \simeq \ell_\gamma$ of eigenvalues contribute effectively to the evolution of the initial state $|0\rangle$. Therefore we can approximate this Floquet operator by a $\ell \times \ell$ matrix $F^{(\ell)}$. The existence of classical chaos will lead to a strong level repulsion. Hence one can consider that the mean distance between the quasienergies is $\Delta E \approx 2\pi/\ell = \mathrm{O}(\gamma)$ on the torus.

For times short enough, the discrete spectral sum arising from the previous approximation can be approximated by an integral, which will be precisely the classical approximation. Hence for t small, $\mathcal{E}_\gamma(t) \approx \mathcal{E}_{\mathrm{cl}}(t) \approx Dt$. This is fine as long as $t\Delta E \ll 2\pi$. But after a breaking time $\tau^* \approx 2\pi/\Delta E \approx \ell$, the quantization dominates and gives an almost periodic function of time for $\mathcal{E}_\gamma(t)$. Thus, $\ell_\gamma^2\gamma^2 = \overline{\mathcal{E}_\gamma} \simeq \mathcal{E}_\gamma(\tau^*) \simeq D\tau^*$. Since $\ell \simeq \ell_\gamma$, we get :

$$\ell_\gamma \simeq \frac{D}{\gamma^2} \,. \tag{224}$$

Numerical calculations are in a fairly good agreement with this prediction, but no rigorous mathematical work has been produced to justify this formula yet. We may expect that :

$$\lim_{\gamma \mapsto 0} \ell_\gamma \gamma^2 = D \quad \text{at } K \text{ large} \,, \tag{225}$$

under certain conditions. For indeed, we have seen that ℓ_γ diverges on a generic set of γ's. Moreover, D does not exist for all K's.
For the moment we do not know how to define mathematically the breaking time τ^*.

We would like now to study the behavior of the kinetic energy for the quantized version of the kicked rotor model as the effective Planck constant γ tends to zero. For that, we perform a numerical calculation giving the classical and quantum energies of the KR for $K = 4$ corresponding to the diffusive regime (Fig. 11). We computed the quantal energy for different values of Planck's constant γ in both cases; it is easy to see that as γ is decreased the quantal curves tend to the classical one.

One could think that this energy converges to its classical limit as $\gamma \mapsto 0$ but a problem arises because of the uniformity of the semiclassical limit with respect to time.

That the breaking time be $\mathrm{O}(\gamma^2)$ can be shown by the following heuristic argument [HeTo]. The semiclassical approximation [Gut:Hou] for the evolution is correct modulo error terms of $\mathrm{O}(\hbar(\gamma^2)$. Therefore, the quantum and classical evolutions for observables should agree up to time $\mathrm{O}(\hbar^{-2})$. Whenever the semiclassical approximation is exact, however, such as in the hydrogen atom, the harmonic oscillator, the Arnold Cat map, we should not see any breaking time.

Appendix

Our aim in this appendix is to prove the theorem 1. The following theorem 15 actually implies the theorem 1.

Let $\mathcal{H}$ be a separable Hilbert space and H be a self adjoint operator on $\mathcal{H}$ with domain $\mathcal{D}$. This domain becomes a Hilbert space when endowed with the norm $\|\psi\|_H^2 = \|\psi\|^2 + \|H\psi\|^2$. Let also V be a bounded self adjoint operator on $\mathcal{H}$ leaving the domain $\mathcal{D}$ invariant and bounded on it for the domain norm $\| \cdot \|_H$. Let also f be a periodic continuous function on $\mathbf{R}$ with period T. Then the solution of the Schrödinger equation :

$$i\hbar\psi_t = (H + f(t)V)\psi_t \ , \tag{226}$$

with $\psi(s) = \psi$, is given by

$$\psi_t = U(t,s)\psi \ , \tag{227}$$

where $U(t,s)$ is a unitary operator such that :

(i) it is strongly continuous with respect to s, t ,
(ii) $U(s,s) = \mathbf{I}$ for all $s \in \mathbf{R}$,
(iii) $U(t,s) = U(t,t')Ut', s)$ for all $t' \in \mathbf{R}$,
(iv) $U(t+T, s+T) = U(t,s)$ for all $s, t \in \mathbf{R}$,
(v) for any $\psi \in \mathcal{D}$, the vector $U(t,s)\psi$ belongs to $\mathcal{D}$, is strongly differentiable with respect to s and t and is a solution of the Schrödinger equation.

The operator $F_s = U(s+T, s)$ is called the Floquet operator for the family $H(t) = H + f(t)V$. Notice that if $t = s$ the corresponding Floquet operators are unitarily equivalent thanks to (iii) and (iv).

Now for ϵ a positive real number, let ρ_ϵ be a non negative function on $\mathbf{R}$ with support in the interval $[-\epsilon, \epsilon]$ and of integral equal to one. We will set :

$$f_\epsilon(t) = \sum_{n \in \mathbf{Z}} \rho_\epsilon(t - nT). \tag{228}$$

Let F_ϵ be the corresponding Floquet operator with $t = -\epsilon$. Then the following result holds :

Theorem 15 *As ϵ tends to zero, the Floquet operator F_ϵ converges strongly to the unitary operator F given by :*

$$F = e^{-i\left(\frac{TH}{\hbar}\right)} . e^{-i\left(\frac{V}{\hbar}\right)} \ . \tag{229}$$

Proof : Denoting by $U_\epsilon(t,s)$ the evolution operator, it is a classical result that it admits the following Dyson expansion, which converges in norm :

$$\begin{aligned} U_\epsilon(t,s) \quad &= \quad \textstyle\sum_{n \geq 0} \left(-\frac{i}{\hbar}\right)^n \int_{s \leq s_n \leq \ldots \leq s_1 \leq t} ds_1 \ldots ds_n f_\epsilon(s_1) f_\epsilon(s_2) \ldots f_\epsilon(s_n) \\ & \qquad e^{-(t-s_1)\frac{iH}{\hbar}} V e^{-(s_1-s_2)\frac{iH}{\hbar}} V \ldots V e^{-(s_n-s)\frac{iH}{\hbar}} \ . \end{aligned} \tag{230}$$

Each term is a well defined strong integral. Taking $t = T - \epsilon$ and $s = -\epsilon$ we get an expansion for the Floquet operator.

As ϵ tends to zero, the restriction of the measure $f_\epsilon(s)ds$ to the interval $[-\epsilon, T-\epsilon]$ converges weakly to the Dirac measure supported by $\{0\}$. Since the integrand is strongly continuous, the term of order n in the Dyson expansion of F_ϵ converges strongly to $\left(-\frac{i}{\hbar}\right)^n e^{-i\left(\frac{HT}{\hbar}\right)} V^n/n!$. Summing up all these terms gives the result.

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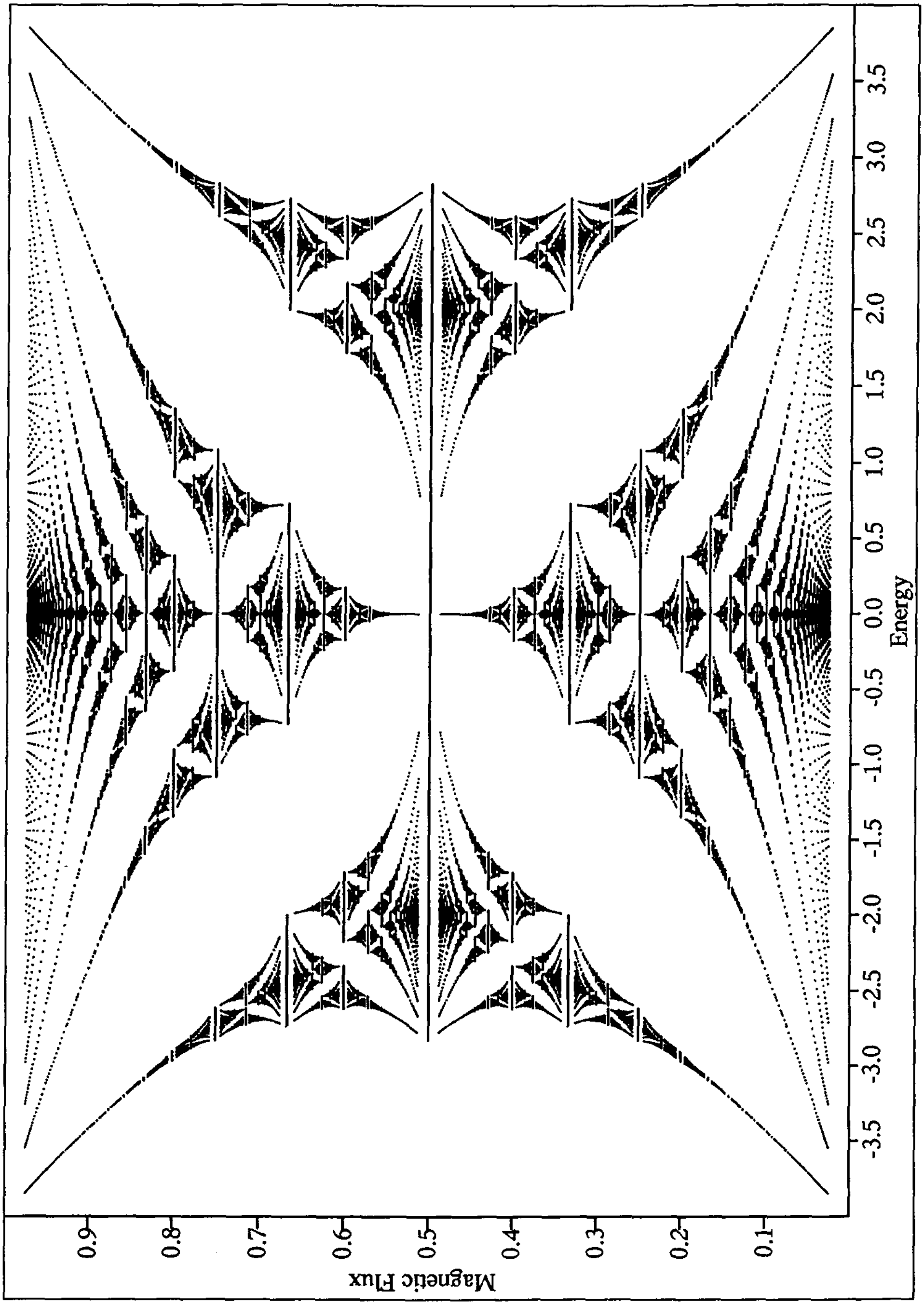
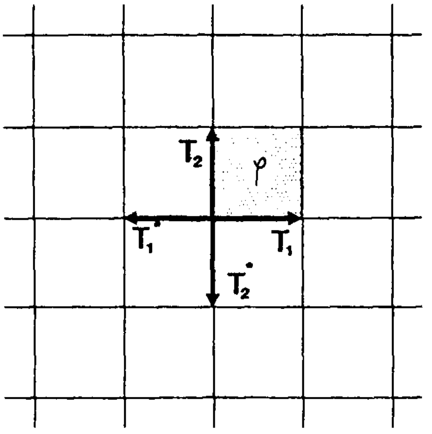
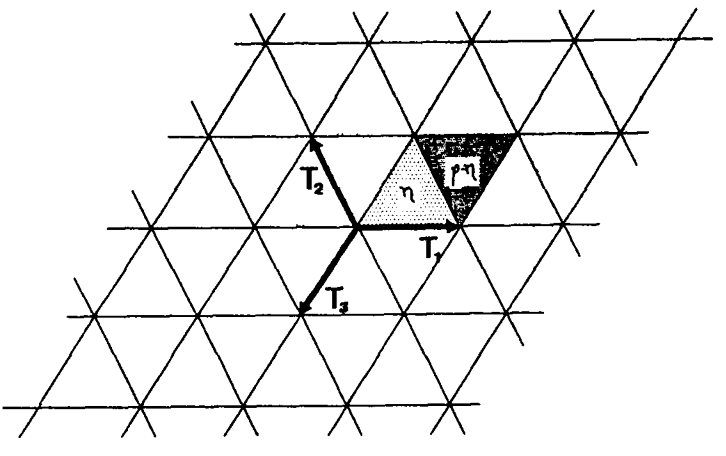
Figure captions

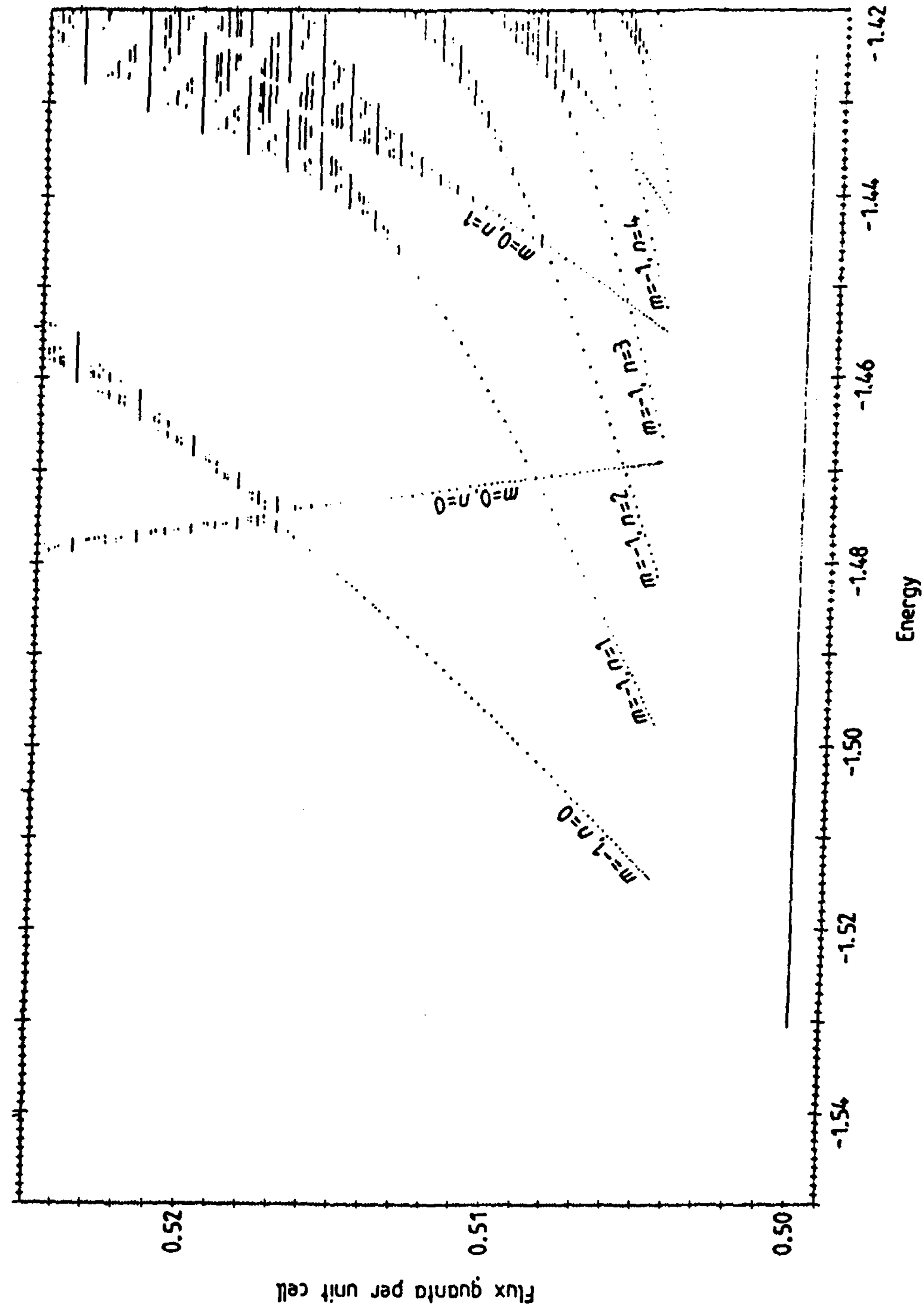
Fig.1 Spectrum of Harper's model (Hofstadter's butterfly).
Fig.2 Magnetic translations and fluxes through elementary cells ; upper figure : square lattice ; lower figure : triangular lattice, from [BeKrSe].
Fig.3 Spectrum of triangular lattice with $\eta = 2\pi 0.0175$ around half flux, from [BeKrSe].
Fig.4 Spectrum of square lattice with second nearest neighbour interaction, from [BaKr].
Fig.5 Asymmetry of the central band edges for the Harper model near $\alpha = 1/3$.
Fig.6 Conical contact between bands at half flux in the Harper model.
Fig.7 Spectrum of the Hamiltonian with second nearest neighbour interaction near half flux, from [BaKr].
Fig.8 Parabolic contacts between bands at half flux in a Harper-like model, with third nearest neighbour interaction, from [BaFl].
Fig.9 Braiding of Landau sublevels in a model with second nearest neighbour interaction, from [BaKr].
Fig.10 Braiding of Dirac sublevels near half flux in a model with third nearest neighbour interaction, from [BaFl].
Fig.11 Time evolution of the kinetic energy for the standard map in the chaotic regime $K = 4$; the staight line corresponds to the classical energy and points represent quantum curves for different values of the effective Planck constant.

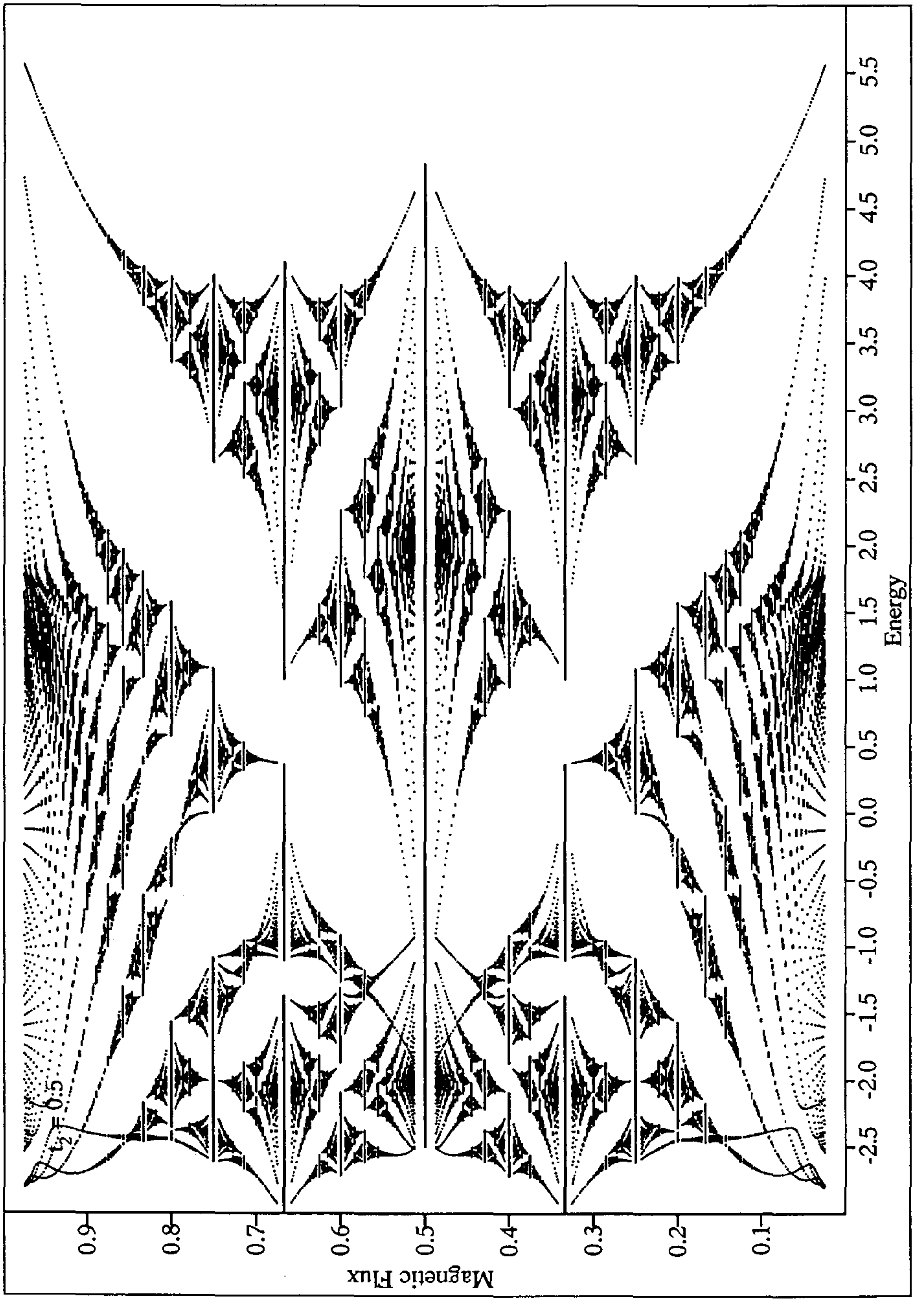


Magnetic Flux
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
Energy
-3.5
-3.0
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5

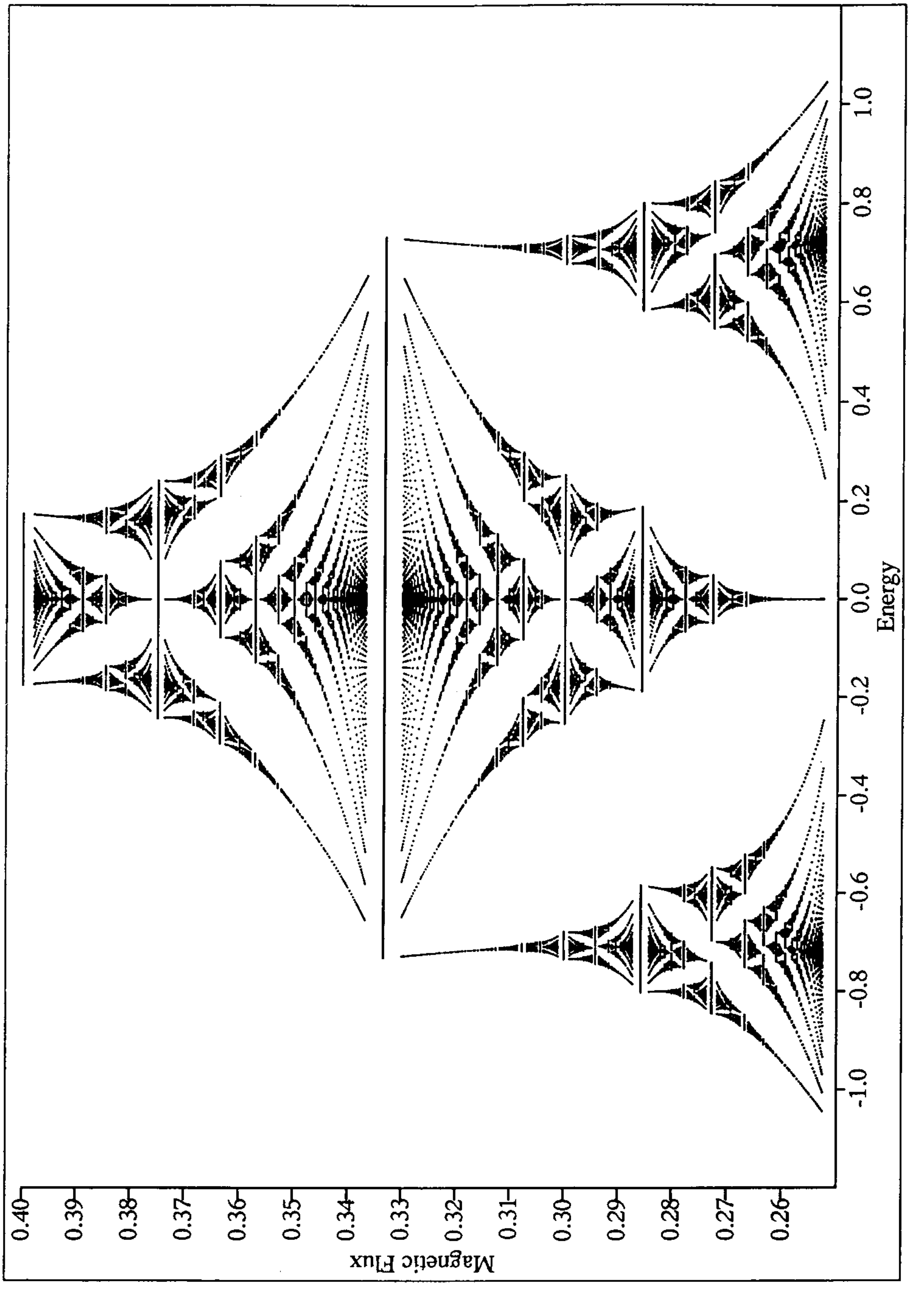

T_2
φ
T_1^*
T_1
T_2^*


T_2
η
$\varphi\text{-}\eta$
T_1
T_3

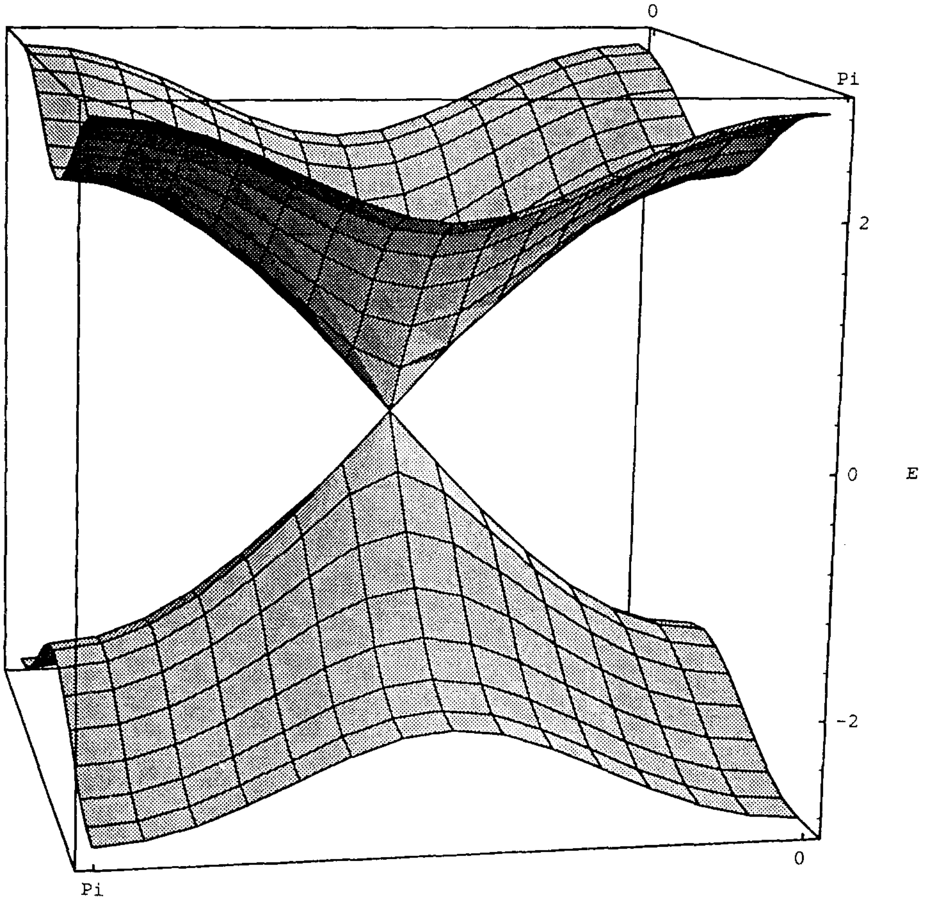

m=0, n=1
m=-1, n=4
m=-1, n=3
m=0, n=0
m=-1, n=2
m=-1, n=1
m=-1, n=0
-1.42
-1.44
-1.46
-1.48
-1.50
-1.52
-1.54
Energy
0.52
0.51
0.50
Flux quanta per unit cell

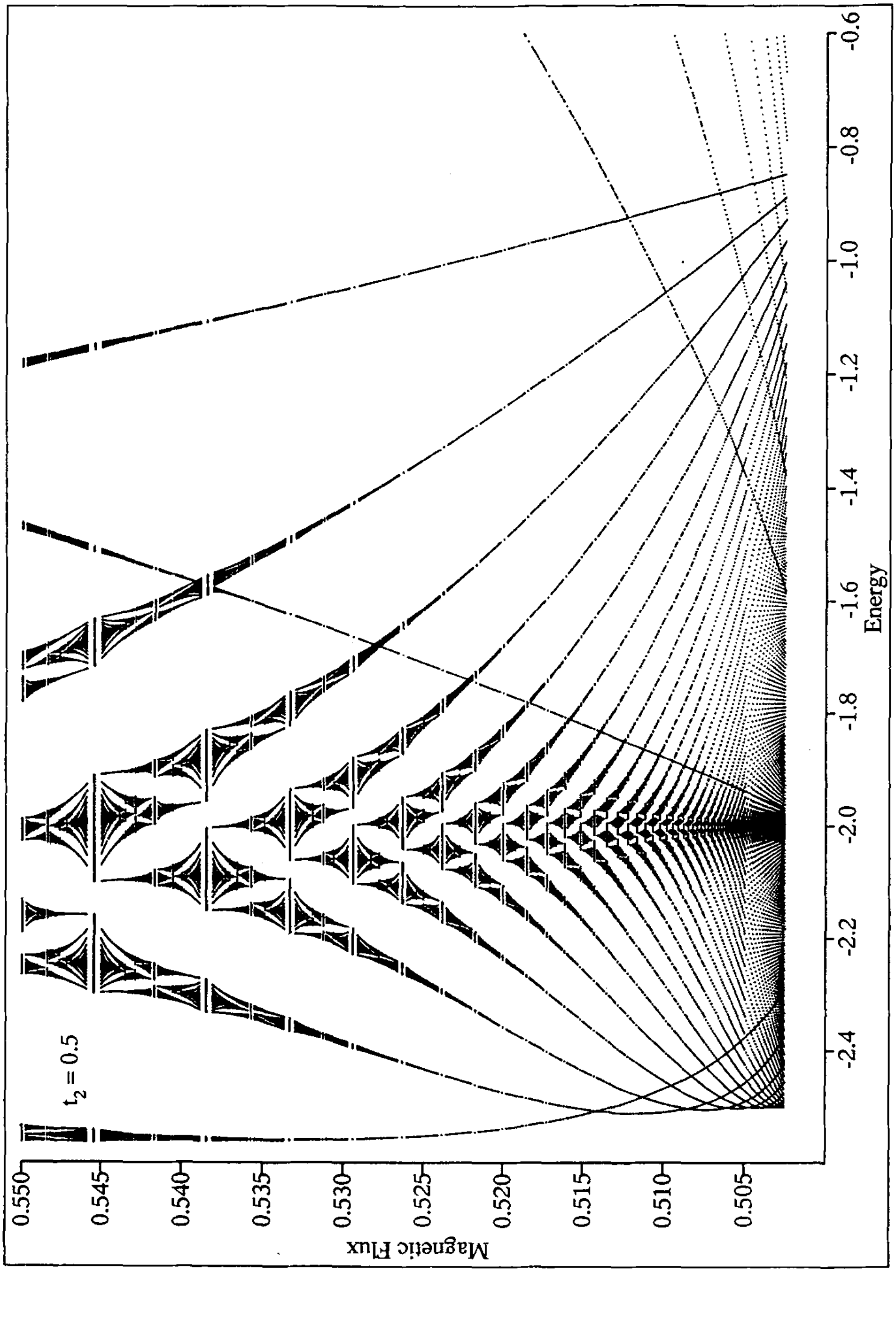


Energy
-2.5
-2.0
-1.5
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
5.5
Magnetic Flux
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9

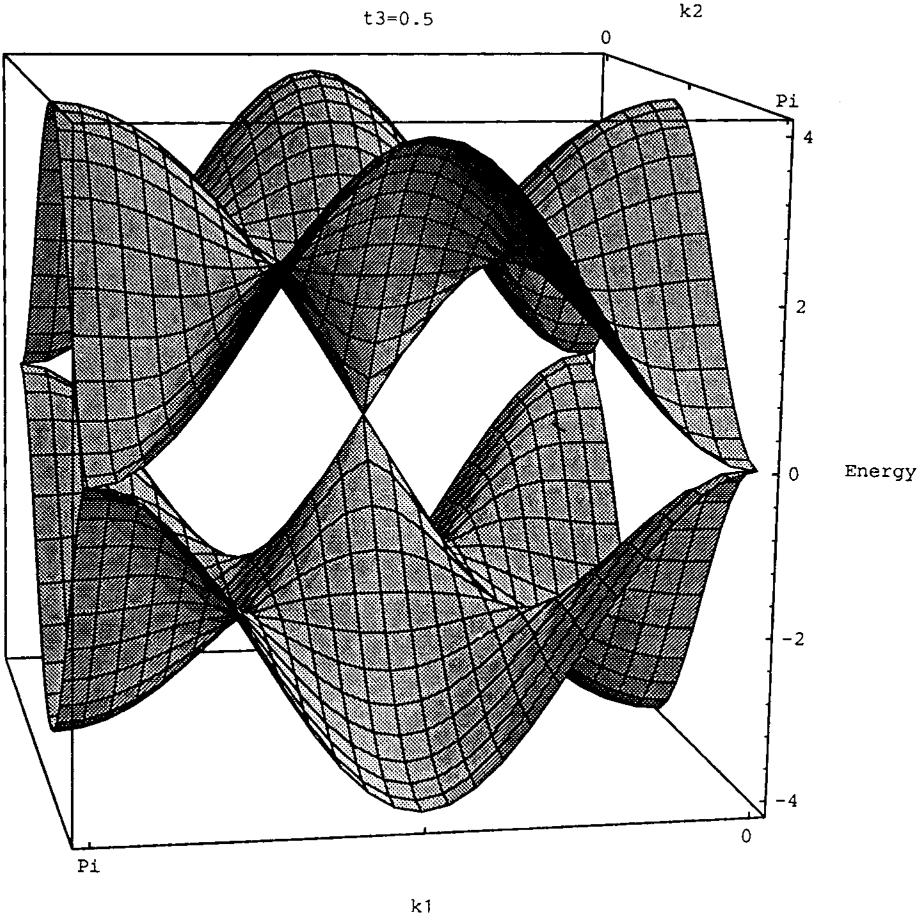


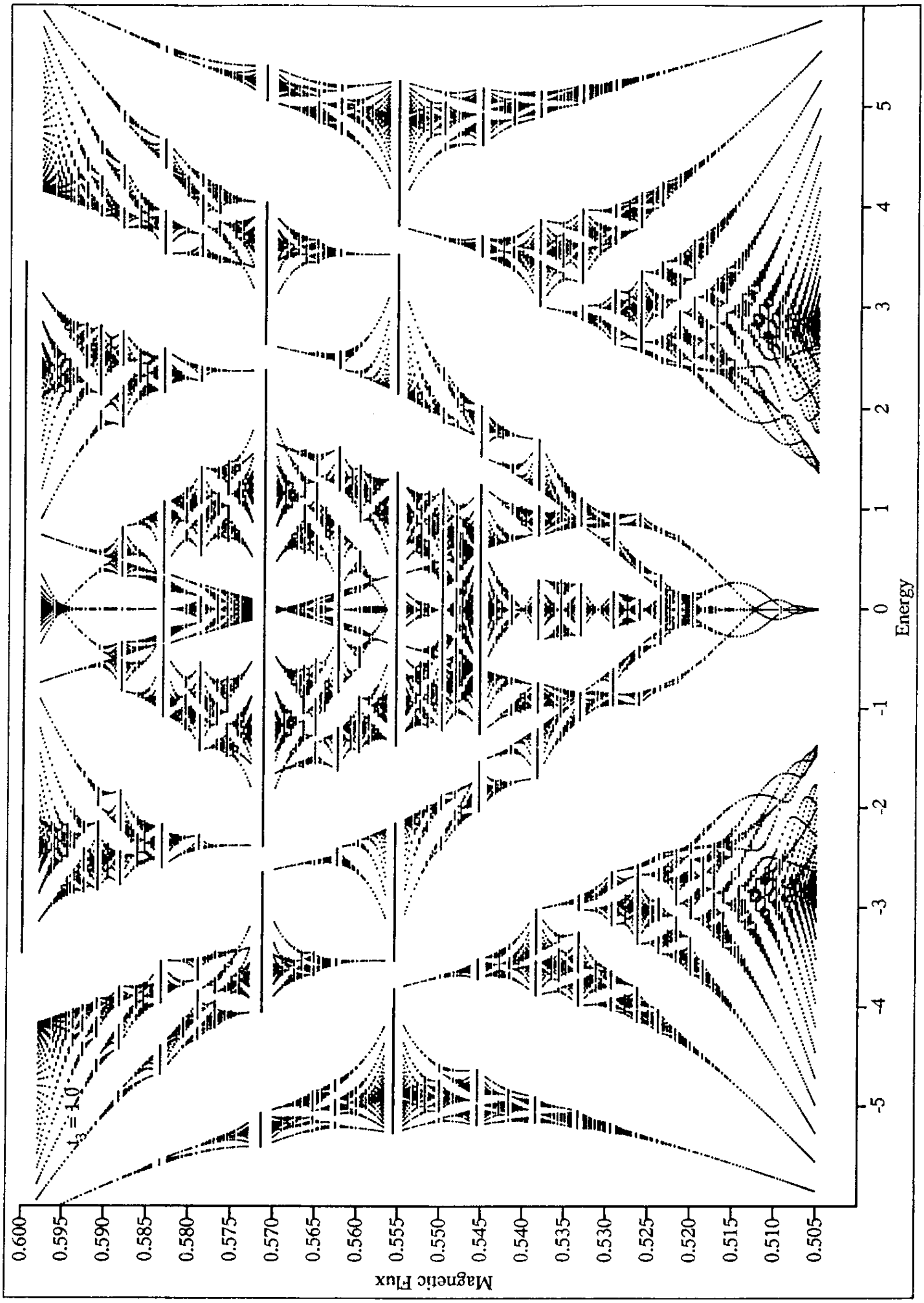
Magnetic Flux
0.40
0.39
0.38
0.37
0.36
0.35
0.34
0.33
0.32
0.31
0.30
0.29
0.28
0.27
0.26
Energy
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0


0
Pi
2
0
E
-2
0
Pi

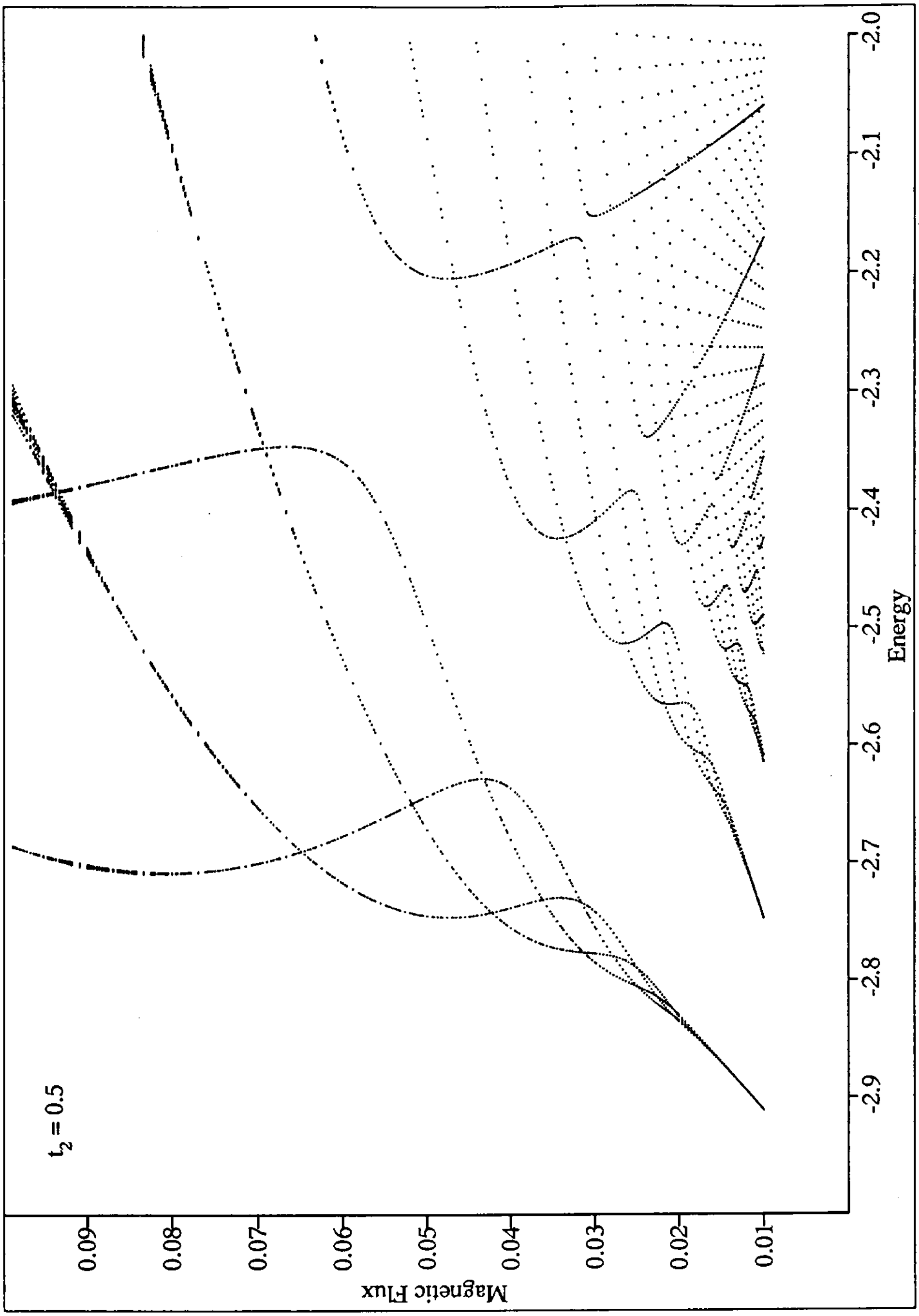


$t_2 = 0.5$
Energy
-2.4
-2.2
-2.0
-1.8
-1.6
-1.4
-1.2
-1.0
-0.8
-0.6
Magnetic Flux
0.550
0.545
0.540
0.535
0.530
0.525
0.520
0.515
0.510
0.505

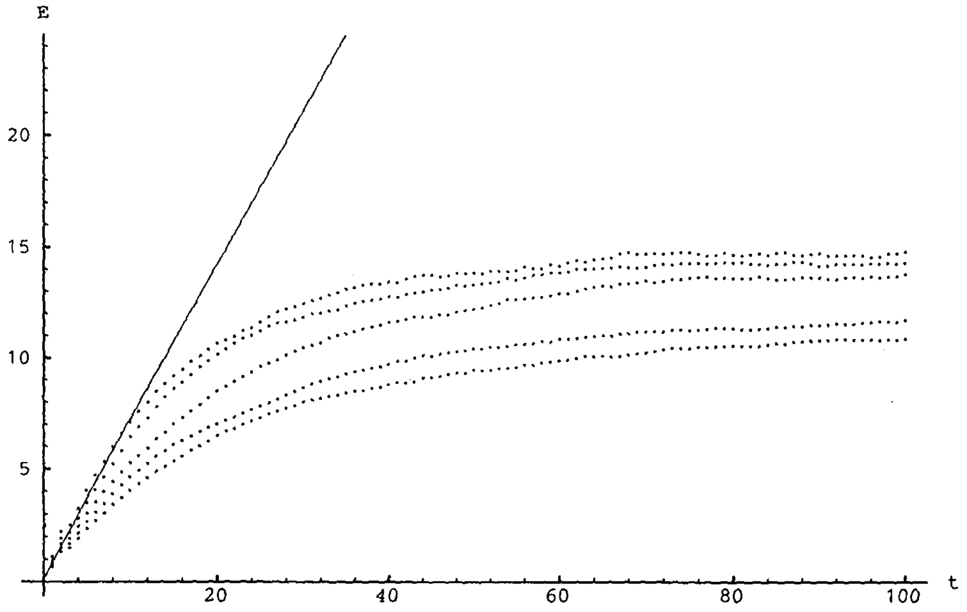

t3=0.5
k2
0
Pi
4
2
0
-2
-4
Energy
0
Pi
k1



Energy
Magnetic Flux
0.600
0.595
0.590
0.585
0.580
0.575
0.570
0.565
0.560
0.555
0.550
0.545
0.540
0.535
0.530
0.525
0.520
0.515
0.510
0.505
-5
-4
-3
-2
-1
0
1
2
3
4
5



t2 = 0.5
Energy
-2.9
-2.8
-2.7
-2.6
-2.5
-2.4
-2.3
-2.2
-2.1
-2.0
Magnetic Flux
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
0.09


E
20
15
10
5
20
40
60
80
100
t