Chapter 12

Gap Labelling Theorems for Schrödinger Operators

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1. Introduction

Quantum Mechanics was born in 1900, when Planck [PL00] investigated the laws of black body radiation. He found the correct formula for the power spectrum in terms of the light frequency. Einstein's interpretation in 1905 by means of energy quanta [EI05] was confirmed by his interpretation of the photoelectric effect. However it took years before Quantum Mechanics became such a solid body of knowledge that it could not be avoided by any reasonable physicist. There is no doubt today that it is a fundamental theory of matter, and that it has changed daily life through new technology, in a way which has never been known before in human history.

My purpose in this introduction, is not to develop some philosophy about human society, but rather to reinterpret some basic facts in Quantum Mechanics, in the light of new sophisticated mathematical techniques which I have proposed to use a few years ago to get a gap labelling for quantum systems of aperiodic media.

Aperiodic media have been the focus of attention during the last twenty years in Solid State Physics: disorder in metals or semiconductors, charge density waves in quasi one dimensional organic conductors, superlattices, structure of glasses, quasicrystals, high temperature superconductors. They provide new materials with interesting unusual physical properties which are to be used in modern technology sooner or later. While periodic media are now well understood through Bloch theory, giving rise to band spectrum for electrons, the correct mathematical framework for aperiodic material is not completely developed yet. One proposal that I have attempted to give during the last few years is that Non Commutative Geometry and Topology, from the point of view developed by A. Connes, is the most accurate candidate for it. It is accurate both from a fundamental point of view, as I will try to show in this introduction, but also from a very practical point of view, through its efficiency in computing real things in real experiments.

In the present work I will explain only one piece in this game, namely how to obtain accurately the gap labelling for complicated band spectra, by computing K-groups of observable algebras appearing as natural objects associated to electrons motion.

However the reader must know that other pieces have been already worked at, such as the existence and quantization of plateaus for the Quantum Hall conductance [BE88a, BE88b, XI88, NB90], semiclassical calculations to explain the behavior of Bloch electrons in a uniform magnetic field [BR90] (a subject of interest in the field of high temperature superconductors), a semiclassical Birkhoff expansion and Nekhoroshev's type estimates in Quantum Mechanics [BV90], and electronic properties of scale invariant homogeneous media such as fractals or quasicrystals [SB90]. It seems also to be the correct tool to investigate rigorously properties of quantum systems which are classically chaotic, a field in which no rigorous result has been proved yet, even though physicists have accumulated an enormous body of knowledge [BE90b].

My main motivation in pushing toward Non Commutative techniques, comes from the fact that in many instances, physicists are using wave functions which are not well adapted in most problems encountered in modern Quantum Physics, essentially because they are defined up to a very troublesome phase factor. But if we are to abandon wave functions, we must explain how quantum interferences enter into the game. For indeed, each typical quantum phenomenon is due to quantum interferences: gaps in electron spectra, localization in disordered or aperiodic media, level repulsion in quantum chaos, phase quantization in Aharonov-Bohm effect, flux quantization in superconductors or in dirty metals, quantization of Hall conductance in the Quantum Hall effect, etc.

It is the purpose of this introduction to give a hint in this direction.

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1.1 Waves

The key idea in Einstein's explanation of black body radiation was that everything behaved as if the energy exchanged between matter and light were quantized as an integer multiple of a small unit. This small unit, called a quantum, is given by the famous Planck formula

$$E = h\nu \,, \tag{1.1.1}$$

where ν is the light frequency, and h a constant, Planck's constant.

In 1905, in the paper for which he was awarded the Nobel prize [EI05], Einstein used this result to explain the photoelectric effect. He claimed that Planck's quantum was actually the energy of a particle, the particle of light that he called 'photon' with momentum $p = h\sigma$, where σ was the number of waves per unit length. With such a simple idea, namely that light was actually made of particles, he could explain why there was no electric current as long as the light frequency was smaller than a threshold frequency ν_0, and why that current was proportional to the difference $\nu - \nu_0$, for $\nu > \nu_0$. Thanks to that theory, Planck's constant could be measured, and it appeared that it had the same value as the one given by black body radiation. Such an amazing result was an indication that Planck's formula had a universal meaning, namely that indeed light was made of photons as explained by Einstein.

In 1911, Lord Rutherford reported upon a series of experiments he had performed on α-ray diffraction patterns produced by various targets [RU11]. He concluded that atoms were not such simple objects as their name 'atom' could suggest. According to his picture, they were made of a heavy nucleus, supporting all the mass, around which electrons would move like planets around the sun. The attracting forces however, were Coulomb forces, namely purely electrostatic, in such a way that the nucleus had a positive charge $+ne$, if n was the number of electrons gravitating around, and e the electron charge.

It was immediately realized that such a structure was unstable from a purely classical point of view. For an accelerated charged particle must radiate some electromagnetic field, ending in a permanent loss of energy, which could be gained only in the collapse of the electrons on to the nucleus. Such a simple minded argument is questioned nowadays [GB82], because it does not take into account the possibility of regaining energy from the electromagnetic field floating around. But even with this, a classical treatment of atoms would be much too complicated, and would not explain the universality of Planck's constant, resulting in "the inadequacy of classical mechanics and electromagnetism to explain the inherent stability of atoms" [BO83].

By the time, a very elegant solution to this paradox was proposed a year later by N. Bohr in 1913 [BO13, BO14, BO15a, BO15b, BO18]. It started with the remark that Planck's constant had the dimension of an action. So that he was led to postulate that the electron classical phase space orbits γ giving rise to a stable motion, were such that their classical action was quantized according to the rule

$$\int_\gamma p \cdot dq = nh \,, n \in \mathbb{Z} \,. \tag{1.1.2}$$

For an electron on a circular orbit in a hydrogen atom, this simple rule together with the usual rules of classical mechanics, give quantization of energy levels according to $E_n = -E_0/n^2(n = 1, 2, \ldots)$, $E_0 = 2\pi^2 k^2 me^4/h^2 \approx 13.6\,\mathrm{eV}$.

In addition, whenever an electron jumps from the orbit n to the orbit m, the change in energy will be compensated by emitting or absorbing a photon of frequency $\nu_{n,m}$, given by Planck's law. It leads immediately to the famous Balmer formula

$$\nu_{n,m} = R \left| \frac{1}{n^2} - \frac{1}{m^2} \right| \,, \qquad R = \frac{E_0}{h} = \frac{2\pi^2 k^2 me^4}{h^3} \,, \tag{1.1.3}$$

which was known from the mid-nineteenth century by Balmer for $n = 2$, and then generalized by Rydberg for any pair n, m. It was used as a phenomenological formula to explain the ray spectrum emitted by hydrogen atoms. Amazingly enough the value of Rydberg's constant given by (1.1.3), agreed with the experimental to one part in 10^4! It could not be just a coincidence.

In the years following the First World War, more and more physicists were involved in the problem of understanding properties of atoms. The Stern-Gerlach experiments in 1922 [SG22] gave support to the idea of stationary states, while the Compton effect discovered in 1924 [CO23] confirmed the views of Einstein about photons.

As elegant and simple as it was, this argument did not explain yet why quantization of the electron action was required to get stability. And this is precisely the point where Quantum Mechanics had to be developed.

The key idea came in 1925 and was exposed in L. de Broglie's thesis [BR25]. Until Einstein's paper in 1905, and since the beginning of the 19th century, light was obviously a wave phenomenon. Interferences and diffraction had been successfully interpreted by physicists like Young, Fresnel, through Huygens's principle. By the 1860's, Maxwell had shown that they were actually electromagnetic waves of very short wavelength. However, Einstein had reintroduced also very successfully, an old concept, quite popular in the 17th century, namely that light was made of corpuscules.

L. de Broglie also pointed out that matter was obviously known to be made of particles. Chemists from the 19th century had used atoms and molecules to explain chemical reactions, and Mendeleev had also successfully classified elementary atoms. Moreover, several works on brownian motion in the first decade of the 20th century, gave some reality to the existence of atoms and molecules and the Avogadro number could be measured. By the end of the 19th century, J. J. Thompson had found that electric current was created by particles that he called electrons. Radioactivity gave also new particles like α particles. At last, Rutherford had confirmed the intricate relationships between these particles to constitute atoms.

There was clearly a dissymmetry between the Einstein treatment of radiations and the way particles like electrons were looked at. L. de Broglie claimed that the wave-corpuscule duality should be a universal principle: to each particle is associated a wave and vice-versa. This idea was soon convincingly confirmed by electron interference phenomena, a very widely accepted fact nowadays.

To give support to that idea, one has to go back to the very definition of a wave. The simplest example is provided by plane waves represented by a function proportional to

$$\psi(t, \mathbf{x}) = e^{\mathbf{i}(\omega t - \mathbf{kx})} , \tag{1.1.4}$$

where ω is the pulsation namely $\omega/2\pi$ is the number of waves per unit time, while $\mathbf{k}$ is the wave vector, with direction equal to the direction of the wave and $|\mathbf{k}|/2\pi$ is the number of waves per unit length. The phase is then constant on the planes $\mathbf{kx} = \omega t + \text{const.}$ resulting in a phase velocity $\mathbf{v}_\phi = \mathbf{k}\omega/|\mathbf{k}|^2$.

Now one can argue that plane waves are only an idealization of real ones even for free particles. Actually, a real wave is never pure, it is usually what one called at that time a 'wave packet', namely a superposition of plane waves in the form

$$\psi(t, \mathbf{x}) = \int d^3\mathbf{k} e^{\mathbf{i}(\omega_\mathbf{k} t - \mathbf{kx})} f(\mathbf{k}) , \tag{1.1.5}$$

with a $\mathbf{k}$-dependent pulsation (dispersion law). If f is a regular function decreasing rapidly at infinity in $\mathbf{k}$, so does $\psi(t, \mathbf{x})$, as a function of $\mathbf{x}$ at each time t. Its maximal value is reached at points $\mathbf{x}$'s for which the phase factor in the integral is stationary, namely

$$t\nabla_\mathbf{k}\omega = \mathbf{x} + \text{const.} , \tag{1.1.6}$$

resulting in a 'group velocity' $\mathbf{v}_g = \nabla_{\mathbf{k}}\omega$ if $\nabla_{\mathbf{k}}$ is the gradient with respect to $\mathbf{k}$. It is then natural to assume that the particle associated to this wave has a velocity given by the group velocity, and that its energy E is related to the pulsation ω through Planck's formula namely $E = \hbar\omega$ ($\hbar = h/2\pi$). We then remark that in classical mechanics, the energy E is represented by the Hamiltonian function $H(\mathbf{q},\mathbf{p})$ (where now $\mathbf{q}$ represents the position of the classical particle). Moreover the Hamilton equations of motion give for the velocity

$$\mathbf{v} = d\mathbf{q}/dt = \nabla_{\mathbf{p}} H \ . \tag{1.1.7}$$

The similarity between (1.1.6) for the group velocity and (1.1.7) for the particle velocity is striking and led de Broglie to supplement Planck's formula $E = h\nu = \hbar\omega$ by a similar expression for the momentum, namely

$$\mathbf{p} = \hbar\mathbf{k} \qquad \text{(de Broglie's formula)} \ , \tag{1.1.8}$$

extending the Einstein formula for the photon, to every free particle.

However the previous argument does not apply to particles submitted to forces, like for instance to electrons in the hydrogen atom. Clearly, free wave packets cannot represent such a particle. But we may extend the argument in the following way: on a very short distance $d\mathbf{x}$, and during a short amount of time dt, the wave can be approximated by plane waves, so that its phase factor increases by the amount

$$d\phi = \omega dt - \mathbf{k}d\mathbf{x} = \{H dt - \mathbf{p}d\mathbf{q}\}/\hbar \ . \tag{1.1.9}$$

Let us consider first a particle in a stationary state, namely such that its associated wave is monochromatic, which means that ω is fixed. Then $H(\mathbf{q},\mathbf{p}) = E = \text{const.}$ If γ represents its classical orbits in phase space, the total variation of the phase along this orbit will be

$$\Delta\phi = -\int_\gamma \mathbf{p}d\mathbf{q}/\hbar \ . \tag{1.1.10}$$

For closed orbits, the corresponding wave functions must be single-valued, resulting in the relation $\Delta\phi = 2\pi n\,(n \in \mathbb{Z})$ which is nothing but Bohr's quantization condition

$$\int_\gamma \mathbf{p}d\mathbf{q} = nh \ , \qquad (n \in \mathbb{Z}) \ . \tag{1.1.11}$$

Therefore stationary states are only stationary waves, giving a coherent scheme for their stability: *stability comes from constructive interferences for the wave associated to the particle.* Actually a real wave will be a superposition of pure waves. Going back to (1.1.9), and remarking that $H dt - \mathbf{p}d\mathbf{q} = -L(\mathbf{q}, d\mathbf{q}/dt)dt$ where L is the Lagrangian, a pure wave will have the form

$$(1.1.12)\quad \psi(t,\mathbf{x}) = \exp\left(-\frac{\mathbf{i}}{\hbar}\int_{t_0}^{t} ds L(\mathbf{q}(s),\mathbf{q}'(s))\right) , \quad \mathbf{q}(t_0) = \mathbf{x}_0 , \mathbf{q}(t) = \mathbf{x} ,$$

along a trajectory $s \to q(s)$ (here $\mathbf{q}' = d\mathbf{q}/ds$), which may or may not be a solution of the Hamilton equations. The importance of the Lagrangian in building up phases was emphasized in 1936 by Dirac. In his 1942 thesis [FE48], R.P. Feynman started from this expression to give an integral representation of the wave function in the formal form:

$$(1.1.13)\quad \psi(t,\mathbf{x}) = \int \prod_{s=0}^{s=t} D\mathbf{q}(s) \exp\left(-\frac{\mathbf{i}}{\hbar}\int_{t_0}^{t} ds L(\mathbf{q}(s),\mathbf{q}'(s))\right) \psi(0,\mathbf{q}(0)) ,$$

where one integrates over the set of all trajectories such that $\mathbf{q}(t) = \mathbf{x}$. This formal expression can be seen as a superposition of pure waves, like a wave packet is a superposition of plane waves.

Again, we can argue that the main contribution to this integral comes from trajectories which produce stationary phase factors. By the Maupertuis principle these trajectories are precisely the classical orbits of the corresponding classical system.

For a single particle in a potential, represented by a Hamiltonian $H = \mathbf{p}^2/2m + V(\mathbf{q})$, Feynman showed that the wave function in (1.1.13) is a solution of Schrödinger equation

$$(1.1.14)\qquad \mathbf{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V\psi ,$$

and conversely, he showed that every solution of Schrödinger equation can be written as the path integral (1.1.13), establishing a complete formal equivalence between the two formalisms. Notice that Schrödinger introduced this equation in 1926 [SR26] on the basis of a variational principle instead and the success was so high that this logical scheme was forgotten for a long time.

E. Nelson [NE64] proved in 1964 that one can give a mathematical rigorous meaning of (1.1.13) for most values of the mass m, whereas Albeverio et al. [AH76] developed a formalism based upon Fresnel integrals, which gives a mathematical status to (1.1.13) for potentials V which are Fourier transforms of positive measures. This result has been improved recently by Fujiwara [FU80, FU90]. Still, the mathematical status of the Feynman path integral is rather unclear. It will require quite a lot of improvements to make it a useful mathematical tool.

Even though (1.1.13) is quite formal, it has been used over and over, especially in Quantum Field Theory. Still today it is one of the most powerful tools for intuition, in dealing with proper definitions and properties of models both in Particle and Solid State Physics.

1.2 Particles

In the previous Section, we described the usual way of representing waves, leading to Schrödinger equation. However, it is to be noted that from a historical viewpoint, waves appeared later in the 1926 paper by E. Schrödinger [SR26], only a few months after the work of Heisenberg, Born and Jordan on what was called at the time 'Matrix Mechanics' [BH25].

The description of point-like particles had been the purpose of Classical Mechanics, which reached a very sophisticated level by the middle of the nineteenth century, with the works of Lagrange, Liouville, Hamilton, Jacobi. They had introduced a new function for the mechanical energy, the so-called 'Hamiltonian', giving rise to a symmetric treatment of momentum and position through the so-called 'phase space'. Moreover, the symplectic structure of the equations of motion was conserved by a special family of changes of coordinates called 'canonical transformations'. Instead of computing the solutions by various techniques, like perturbation theory, Jacobi proposed a new method: to compute a canonical change of coordinates transforming the equations of motion into a family of trivial ones. Liouville had introduced also the notion of action-angle variables [AR78], which had the property of being a universal choice of coordinates, at least locally in phase space, and one could define properly the notion of completely integrable system.

In the beginning of the twentieth century, this part of classical mechanics was taught in universities as the most sophisticated piece of knowledge.

No surprise then that Bohr pointed out the importance of action integrals in dealing with quantization. Soon enough after his 1913 seminal paper, A. Sommerfeld [SO15] had extended his quantization condition to get the two other quantum numbers for a complete description of stationary states of the hydrogen atom. In a 1917 paper [EI17], Einstein generalized the method to non-separable Hamiltonians, by means of Jacobi's method: through various canonical transformations, one expresses the classical Hamiltonian in terms of action variables only and then one replaces each action variable J_k by $n_k\hbar$, where n_k is an integer. In this way it became possible to treat more complicated problems like the emission spectrum of atoms and molecules [SR82, JR82, MA85]. The method was improved later on by Brillouin [BR26] and Keller [KE58] and is known nowadays as the EBK quantization scheme.

However, the method did not succeed at the time, in explaining details of atomic spectra: the case of the helium atom was especially emphasized in the beginning of the twenties. The exclusion principle partly responsible for the discrepancy was discovered only in 1925 by W. Pauli. Moreover, the Stern and Gerlach experiment in 1922 [SG22], coming after many difficulties in interpreting the Zeeman multiplets, showed the limits of the method in that this quantization condition was unable to explain why angular momenta in atoms required to introduce half-integers as quantum numbers. Sure enough there was something wrong about the old theory of quanta.

During the year 1925, W. Heisenberg using a method of 'systematic guessing' [VW67], was 'fabricating quantum mechanics' [HE25, HE85]. His first motivation was to compute the line intensities in the hydrogen spectrum. He used a generalization of Fourier analysis which proved quite hard and forced him to consider the simpler problem of a harmonic oscillator. That was enough to produce new rules, soon recognized as the rules of matrix multiplication. Immediately after his paper was published, M. Born conjectured that the basic equation in this theory was the 'commutation rule', and that was confirmed some days later by a calculation of his pupil P. Jordan who showed that within the Heisenberg rules, the canonical commutation relations $[q,p] = \mathbf{i}\hbar$ were correct [BJ25]. This led to the extraordinary 'three men paper' published in November 1925 by M. Born, W. Heisenberg and P. Jordan [BH25], in which the basic principles of quantum mechanics are settled in a way which requires no change in the light of later improvements. Moreover several difficulties were solved at once like formulæ for perturbation theory, the 'anomalous' Zeeman effect (quantization rules for the angular momentum), and they also recovered the Planck formula for black body radiation through using the statistical approach of P. Debye. A few weeks after, W. Pauli gave a treatment of the hydrogen atom, using this new mechanics [PA26].

The first step in Heisenberg's intuition is related to the fact that only observable quantities must enter in building up a theory. One of the main problem in the old theory comes from the fact that one is dealing with classical orbits which are obviously meaningless (this will be made more precise in 1927 [HE27, JA74], by means of the uncertainty principle). However, there was surely a notion of 'stationary states' which could be observed through spectral lines, and interpreted as energy levels. To compute line intensities, it was necessary to consider transitions from one stationary state labelled by n to another one labelled by $(n-l)$. In dealing with a time dependent quantity $x(t)$, there is a frequency $\nu(n, n-l) = \omega(n, n-l)/2\pi$ associated to such a transition: this is the frequency of the light emitted by the atom. The observed rule for transition is given by

$$\omega(n, n-l) + \omega(n-l, n-l-l') = \omega(n, n-l-l') \; , \tag{1.2.1}$$

which permits to write the transition frequency as

$$\omega(n, n-l) = \{W(n) - W(n-l)\}/\hbar \; , \tag{1.2.2}$$

where $W(n)$ is called a spectral term; according to Planck's formula, it is an energy (defined up to an additive constant term).

The second step is Bohr's correspondence principle according to which quantum mechanics must agree with classical laws for large quantum numbers n. Thus for fixed l's and large n's (2.2) gives

$$\omega(n, n-l) = l \cdot \omega(n) = l/\hbar \partial W/\partial n \; , \tag{1.2.3}$$

where $1/\hbar\partial W/\partial n$ can be understood as $\partial W/\partial J$ if $J = n\hbar$ is the classical action integral, in complete analogy with the classical calculation of particle frequencies.

The emission of electromagnetic waves is then classically governed by the laws of electrodynamics. For instance, the electric field at a distance $\mathbf{r}$ from the emitting electron is given by

$$\mathcal{E} \approx e/r^3\{\mathbf{r} \times (\mathbf{r} \times \mathbf{v}')\} , \tag{1.2.4}$$

where $\mathbf{v}' = d^2\mathbf{r}/dt^2$ is the electron acceleration. It implies that classically a quantity $x(t)$, like the electron position or its velocity, can be expanded in Fourier transform, in terms involving all possible transition frequencies. The quantum assumption made by Heisenberg is that the correspondence should be as follows

$$\begin{aligned} &\text{in quantum theory} \quad x(t) = \sum a(n, n-l)e^{i\omega(n,n-l)t} , \\ &\text{in classical theory} \quad x(t) = \sum a_l(n)e^{i\omega(n)\cdot l\cdot t} . \end{aligned} \tag{1.2.5}$$

He then addressed the question of how to compute quantities like $x(t)y(t)$, whenever x and y are two observable quantities. Using (1.2.1) and (1.2.4) he then found immediately

$$(x \cdot y)(n, n-l) = \sum_{l'} x(n, n-l')y(n-l', n-l) . \tag{1.2.6}$$

This law is nothing but matrix multiplication as M. Born realized soon after [VW67].

Differentiating $x(t)$ with respect to time, one gets the so-called Heisenberg equations of motion which are, in matrix language

$$dx/dt = (xW - Wx)/i\hbar , \tag{1.2.7}$$

where W is the diagonal matrix given by (1.2.2).

The third step is now to express the Bohr-Sommerfeld-Einstein quantization condition (1.1.2). In order to do so, let us consider a one dimensional particle, and let us write the action as

$$nh = \int_\gamma pdq = \int dtmv^2(t) = 2\pi m \sum_l |q_l(n)|^2 l^2 \omega(n) , \tag{1.2.8}$$

If one formally differentiates both sides with respect to n (!), one gets

$$\hbar = h/2\pi = m \sum_l l \cdot \partial/\partial n\{|q_l(n)|^2 l\omega(n)\} , \tag{1.2.9}$$

with its quantum equivalent [HE25]

$$\hbar = 2m \sum_{l>0} \{|q(n, n+l)|^2\omega(n, n+l) - |q(n, n-l)|^2\omega(n, n-l)\} . \tag{1.2.10}$$

In the matrix language, this is nothing but the diagonal part of the canonical commutation relation $\langle n|(q \cdot p - p \cdot q)|n\rangle = \mathbf{i}\hbar$, where $p = mdq/dt$ is the momentum. Using the Heisenberg equation of motion, Jordan could prove that the commutator $qp - pq$ is actually diagonal in such a way that

$$q \cdot p - p \cdot q = \mathbf{i}\hbar \,. \tag{1.2.11}$$

In his first 1925 paper, Heisenberg used this new mechanics to compute the spectrum of an anharmonic oscillator. He actually made a very important remark: in order to get the spectrum of a quantized harmonic oscillator, he imposed the existence of a 'ground state', namely he demanded that the energy be bounded from below. This can be expressed in modern language by asking the observable algebra to contain the notion of 'positive elements'.

1.3 Why is the Set of Observables a C^*-Algebra?

The Heisenberg arguments have justified the non commutative approach to quantum mechanics. The correspondence principle can be sharpened now in the following way:

(i) classical observable quantities are functions on the phase space. Quantum mechanically, they are replaced by elements of a non commutative algebra $\mathcal{A}$ over the complex numbers.

(ii) the observable algebra must admit an involution $A \to A^*$ such that $(A+B)^* = A^* + B^*$, $(AB)^* = B^*A^*$, (if $\lambda \in \mathbb{C}$, λ^* is the complex conjugate of λ). It expresses the fact that we need real numbers to measure physical quantities.

(iii) a measurement process is described by states, namely linear forms τ: $\mathcal{A} \to \mathbb{C}$, such that $\tau(A^*A) \geq 0$ for any A in $\mathcal{A}$ and $\tau(\mathbf{1}) = 1$.

The positivity property (iii) is actually crucial for quantization. Indeed one can exhibit examples of algebras, generated by elements $q = q^*$ and $p = p^*$, satisfying (1.2.11), for which the Hamiltonian $H = p^2 + q^2$ corresponding to a harmonic oscillator, would not admit quantized energy levels. In its first 1925 paper, Heisenberg found that the frequencies in this model satisfied $W(n) = \hbar\omega(n+\text{const.})$. To fix the arbitrary constant, he insists in having a lowest energy level, namely a ground state, which gives the famous $n + 1/2$.

If one insists that the observable algebra be entirely defined by measurement processes, one may use the Gelfand-Naimark-Segal construction [SA71, BR79, PE79, TA79] from any state τ, to get a Hilbert space $\mathcal{H}_\tau$, and a unit vector ζ_τ, for which any observable A is represented by an operator $\pi_\tau(A)$ such that $\pi_\tau(A^*) = \pi_\tau(A)^*$ and $\tau(A) = \langle\zeta_\tau|\pi_\tau(A)\zeta_\tau\rangle$.

Practical calculations will not be possible without permitting to take limits of sequences of observables. Otherwise it would be like ignoring real numbers and working only with rationals. The main problem is that there are many ways of defining a topology on such an algebra. The measurement process provides us with a way of defining a natural topology.

For technical simplicity, one can restrict oneself to the set of observables giving rise for each state, to bounded operators in the GNS representation, in such a way that if we set

$$\|A\| = \sup_{\tau} |\tau(A)| \ , \tag{1.3.1}$$

we get a norm on $\mathcal{A}$ satisfying

$$\|A^* A\| = \|A\|^2 \ . \tag{1.3.2}$$

Then one can include in $\mathcal{A}$ all possible elements obtained by a limiting procedure: this means that we ask $\mathcal{A}$ to be complete. Such an algebra is called a C^*-algebra:

Definition. 1) A *C^*-algebra* is an algebra over the complex field with an involution satisfying (ii), and a norm satisfying (1.3.2), for which it is complete.

2) A **-homomorphism* from the C^*-algebra $\mathcal{A}$ to the C^*-algebra $\mathcal{B}$ is a linear mapping $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\alpha(AB) = \alpha(A)\alpha(B)$ and $\alpha(A^*) = \alpha(A)^*$ for any $A, B \in \mathcal{A}$. It is a *-isomorphism if it is invertible, and a *-automorphism of $\mathcal{A}$ whenever it is an isomorphism from $\mathcal{A}$ to $\mathcal{A}$.

3) A *one-parameter group of *-automorphisms* is a family $\{\alpha_t; t \in \mathbb{R}\}$ of *-automorphisms of $\mathcal{A}$ such that $\alpha_{s+t} = \alpha_s \circ \alpha_t$ for any $s, t \in \mathbb{R}$; it is point-wise norm continuous whenever for any $A \in \mathcal{A}$ the mapping $t \in \mathbb{R} \to \alpha_t(A) \in \mathcal{A}$ is continuous in norm.

4) A **-derivation* on a C^*-algebra $\mathcal{A}$, is a linear map δ defined on a dense subalgebra $\mathcal{D}(\delta)$ in $\mathcal{A}$, and such that $\delta(AB) = \delta(A)B + A\delta(B)$, $\delta(A^*) = \delta(A)^*$ for any $A, B \in \mathcal{D}(\delta)$.

5) A *-derivation δ generates a one-parameter group of point-wise norm continuous *-automorphisms $\{\alpha_t; t \in \mathbb{R}\}$ if and only if for $A \in \mathcal{D}(\delta), \delta(A) = d\alpha_t(A)/dt$ at $t = 0$, namely if one can write $\alpha_t = \exp\{t\delta\}$.

As we see, C^*-algebras emerge as very natural objects from Heisenberg's construction. Moreover, the norm which has been constructed here has another canonical property: eq. (1.3.2) implies that the square of the norm of A is nothing but the spectral radius of A^*A; hence the topology given by a C^*-norm comes entirely from the algebraic structure (algebra and positivity)! In particular every *-homomorphism is automatically norm-continuous namely $\|\alpha(A)\| \leq \|A\|, A \in \mathcal{A}$, showing that the algebraic structure is sufficient to define the topology.

The restriction to bounded operators is not actually essential, in that 'good observables' can be computed through bounded operators, by means of resolvents (Green's functions), or any kind of functional calculus. However, as pointed out by von Neumann, unbounded observables raise a very difficult technical problem, due to the domain of definition which may lead to the impossibility of computing bounded functions. This is why he defined self-adjoint operators [FA75]. They are precisely those unbounded symmetric operators for which the Schrödinger equation $\mathbf{i}\partial\psi/\partial t = H\psi$ admits a unique solution for all time with a given initial data. Then the functional calculus and spectral theory exist for self-adjoint operators and permit to reduce their study to the case of bounded operators.

The correspondence principle is now supplemented by requiring that the classical symplectic structure should survive quantum mechanically. By analogy with the previous calculation, this is done by defining Poisson's brackets and canonical variables as follows

(iv) Poisson's brackets are given by [DI26]

$$\{A, B\} = [A, B]/\mathbf{i}\hbar\,, \qquad [A, B] = AB - BA\,. \tag{1.3.3}$$

Then one sees that the map $\mathcal{L}_H : A \to \{H, A\}$ satisfies all axioms of a derivation namely it is linear and

$$\mathcal{L}_H(AB) = \mathcal{L}_H(A)B + A\mathcal{L}_H(B)\,, \qquad \mathcal{L}_H(A)^* = \mathcal{L}_{H^*}(A^*)\,. \tag{1.3.4}$$

Quantum equations of motion will then be nothing but Hamilton-Jacobi's ones namely

$$dA/dt = \mathcal{L}_H(A)\,. \tag{1.3.5}$$

If $H = H^*$ is unbounded $\mathcal{L}_H$ is an unbounded derivation, which requires to define properly its domain of definition [BR79] in such a way that solutions of (1.3.5) be given by $A(t) = \exp\{t\mathcal{L}_H(A)\}$. More generally, any canonical transformation is generated by products of operators on $\mathcal{A}$ of the form $\exp\{\mathcal{L}_H\}$ with $H = H^*$. Going back to (1.3.3), one sees that they correspond formally to unitary transformations of the form $A \to SAS^{-1}$. In this way perturbation theory will be quite simple to develop in a completely similar way with classical mechanics [BV90].

(v) the observable algebra $\mathcal{A}$ can be constructed by means of a family of bounded functions of two families of elements $Q_k = Q_k^*$ and $P_k = P_k^*$ $(k = 1, \cdots, D)$ satisfying the canonical commutation relations

$$\{Q_k, Q_l\} = 0 = \{P_k, P_l\}\,, \{Q_k, P_l\} = \delta_{kl}\mathbf{1}\,. \tag{1.3.6}$$

As we will see there exist many different non isomorphic such algebras, depending upon the problem we want to investigate. The simplest one is generated by smooth fast decreasing functions of the P's and the Q's, through Weyl's quantization formula, and gives rise to the algebra $\mathcal{K}$ of compact operators, which is used in ordinary Quantum Mechanics. However, when dealing with periodic or aperiodic media we will get different algebras, giving rise to various kinds of spectra.

The connection with waves and Schrödinger's point of view was done in 1926 by Schrödinger and Pauli [JA74]. In modern language, wave functions provide a 'representation' of the observable algebra. For ordinary Quantum Mechanics, Weyl's theorem [RSII] shows that there is a unique (up to unitary equivalence) such representation, namely the Hilbert space is $L^2(\mathbb{R}^D)$ and Q_k

Fig. 1. Groupoid as transitions between stationary states

is the operator of multiplication by the k-th coordinate x_k, while P_k is the differential operator $-\mathbf{i}\hbar\partial/\partial x_k$. Therefore for a $3D$ particle in a potential $H = \mathbf{P}^2/2m + V(\mathbf{Q})$ is nothing but the Schrödinger operator (see eq. (1.1.14))

$$H = -\hbar^2/2m\Delta + V(\mathbf{x}) \, . \tag{1.3.7}$$

The three men paper [BH25] has been so cleanly written that most of the text-books in quantum mechanics have forgotten the original intuition of Heisenberg and usually introduce the operator aspect as a consequence of the Schrödinger equation, after having defined the notion of wave functions, of states as vectors in a Hilbert space, and then coming to the point where partial differential operators can be algebraically interpreted.

Actually, the original approach by Heisenberg contains much more structure than a first look may show. A lot of progress were made by mathematicians during the seventies to use this construction in building various C^*-algebras. The breakthrough went with the notion of a groupoid [HA78a, HA78b, RE80, CO79], which is nothing but the abstract generalization of the notion of transition between stationary states as defined by Bohr and Heisenberg.

A groupoid $\mathbf{G}$ is a family of two sets $\{G^{(0)}, G\}$ where $G^{(0)}$ is called the basis and generalizes the set of stationary states. An element γ of G is called an 'arrow' and represents a transition between two elements of $G^{(0)}$ its source $s(\gamma)$, and its range $r(\gamma)$. Giving two arrows γ and γ' such that $r(\gamma') = s(\gamma)$, one can define a new one $\gamma\circ\gamma'$ (see Fig. 1) with $r(\gamma\circ\gamma') = r(\gamma)$ and $s(\gamma\circ\gamma') = s(\gamma')$. Actually we demand that an element x of the basis be considered as a special arrow with $r(x) = s(x) = x$, $x \circ \gamma = \gamma$ for every arrow γ such that $r(\gamma) = x$, and $\gamma \circ x = \gamma$ for every arrow γ such that $s(\gamma) = x$; in other words, x is a unit. At last we demand that each arrow $\gamma \in G$ admits an inverse γ^{-1} with $r(\gamma^{-1}) = s(\gamma), s(\gamma^{-1}) = r(\gamma)$ and $\gamma^{-1} \circ \gamma = s(\gamma), \gamma \circ \gamma^{-1} = r(\gamma)$. Examples of groupoids are known:

1) if the basis has only one element, then $\mathbf{G}$ is a group.

2) On the other extreme, given a set X, a groupoid $\mathbf{G}(X)$ is defined with X as a basis and $G = X \times X$. Then $r(x,y) = x$, $s(x,y) = y$, and x is identified with the arrow (x,x); we define the product by $(x,y) \circ (y,z) = (x,z)$, and the inverse by $(x,y)^{-1} = (y,x)$.

3) Let now M be a compact space, and let Γ be a locally compact group. We assume that Γ acts on M through a group of homeomorphisms $g \in \Gamma \to f_g$ such that $f_{gg'} = f_g \circ f_{g'}$. Then we get a groupoid by letting M be the basis, and $G = M \times \Gamma$. Then $r(x,g) = x$, $s(x,g) = f_g^{-1}(x)$, $(x,g) \circ (f_g^{-1}(x), g') = (x, gg')$, $(x,g)^{-1} = (f_g^{-1}(x), g^{-1})$, and x will be identified with $(x,1)$ where 1 is the unit of Γ.

One can easily define the notion of measurable, topological, differentiable groupoid, by asking $G^{(0)}$ and G to be measurable spaces, topological spaces, or manifolds, and demanding that the maps defining the range, the source, the product and the inverse be measurable, continuous or smooth (see [CO79], for more details). In particular, foliations on a smooth compact manifold are described by differentiable groupoids.

Let $\mathbf{G}$ be a locally compact groupoid. We say that $\mathbf{G}$ is discrete whenever for every x in the basis, the set $G^{(x)} = \{\gamma \in G; r(\gamma) = x\}$ is discrete. In this case we consider the topological vector space $\mathcal{C}_c(G)$ of complex continuous functions with compact support on G. It becomes a *-algebra if we define the product and the adjoint as follows

$$(1.3.8) \qquad ff'(\gamma) = \sum_{\gamma' \in G^{(x)}} f(\gamma') f'(\gamma'^{-1} \circ \gamma) \,, \qquad f^*(\gamma) = f(\gamma^{-1})^* \,.$$

If $\mathbf{G}$ is given by the example 2) with $X = \{1, 2, \cdots, n\}$, a continuous function with compact support on G is nothing but a matrix $f = ((f(i,j)))$, and the product is nothing but the usual matrix multiplication, whereas the * is nothing but the usual hermitian conjugate.

For every x in the basis of $\mathbf{G}$, we then define the Hilbert space $H_x = l^2(G(x))$. A representation π_x of $\mathcal{C}_c(G)$ in this Hilbert space is provided by setting

$$(1.3.9) \qquad (\pi_x(f)\psi)(\gamma) = \sum_{\gamma' \in G^{(x)}} f(\gamma^{-1} \circ \gamma') \psi(\gamma') \,.$$

One can check that $\pi_x(ff') = \pi_x(f)\pi_x(f')$, and that $\pi_x(f)^* = \pi_x(f^*)$ for every f, f' in $\mathcal{C}_c(G)$. Therefore we get a C^*-norm by setting

$$(1.3.10) \qquad \|f\| = \sup_{x \in G^{(0)}} \|\pi_x(f)\| \,,$$

By completing $\mathcal{C}_c(G)$ with respect to this norm, we get a C^*-algebra $C^*(\mathbf{G})$. The generalization of this construction to every locally compact groupoid can be found in [RE80].

1.4 The Index Theorem Versus Wave-Particle Duality

In this Section we want to illustrate the wave-particle duality in a mathematical problem namely the Index Theorem for Tœplitz operators.

Let f be a complex continuous function on the real line $\mathbb{R}$, periodic of period 2π, which never vanishes. Then it is quite elementary to show that it can be written as

$$f(x) = \exp\{\mathbf{i}nx + \phi(x)\}\ , \tag{1.4.1}$$

where ϕ is a complex continuous function on the real line $\mathbb{R}$, periodic of period 2π, and n is an integer. Both n and ϕ are uniquely defined through (1.4.1). Indeed the integer n can be computed through a Cauchy formula:

$$n = \frac{1}{2\mathbf{i}\pi}\int_0^{2\pi} dx \frac{f'(x)}{f(x)} \tag{1.4.2}$$

showing that it can be seen as the winding number of the closed path $\gamma = f(\mathbb{T})$ in the complex plane given by the image of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ by f. Obviously, this winding number does not change under a continuous deformation of this path, namely, n is a homotopic invariant. Actually, f is homotopic to the function $e_n : x \to \exp\{\mathbf{i}nx\}$, simply by letting ϕ go to zero uniformly in (1.4.1). So, since a continuous function never vanishes if and only if it has a continuous inverse, the winding number n classifies the homotopy classes of invertible functions in the algebras $\mathcal{C}(\mathbb{T})$.

We then remark that n is defined by the requirement that the phase of f changes by a multiple of 2π whenever x varies through one period. This is therefore a condition very similar to the Bohr quantization rule (1.1.2). Moreover we see here that it is expressed as the evaluation of the closed 1-form $\eta = dz/2\mathbf{i}\pi z$ over the path $\gamma = f(\mathbb{T})$ defined by f

$$n = \int_\gamma \eta\ . \tag{1.4.3}$$

Since η is closed, it defines a de Rham cohomology class $[\eta]$ in the first cohomology group $H^1(\mathbb{C}\backslash\{0\}, \mathbb{C})$ over the pointed complex plane $\mathbb{C}\backslash\{0\}$. By Cauchy's formula the evaluation of $[\eta]$ on any closed path is an integer so $[\eta] \in H^1(\mathbb{C}\backslash\{0\}, \mathbb{Z})$.

On the other hand, γ is a closed path, and therefore it defines a homology class $[\gamma] = f^*[\mathbb{T}]$ in $H_1(\mathbb{C}\backslash\{0\}, \mathbb{Z})$. By the de Rham duality one can therefore write n in the following abstract way

$$n = \langle [\eta] | f^*[\mathbb{T}] \rangle \tag{1.4.4}$$

exhibiting the homological content of the 'wave' aspect of Bohr's quantization rule.

Let us now consider the Hardy space $\mathcal{H}(D)$ namely the space of holomorphic functions on the unit disc $D = \{z \in \mathbb{C}; |z| < 1\}$ with square integrable boundary value on the unit circle $S_1 = \{z \in \mathbb{C}; |z| = 1\} \approx \mathbb{T}$. $\mathcal{H}(D)$ is a closed subspace of the Hilbert space $L^2(\mathbb{T})$ namely the subspace of function having a Fourier series vanishing on negative frequencies. We will denote by P the orthogonal projection onto $\mathcal{H}(D)$.

For $f \in \mathcal{C}(\mathbb{T})$, we will denote by $T(f)$ the restriction to $\mathcal{H}(D)$ of the operator of multiplication by f. It is called the Tœplitz operator associated to f. The main property of $T(f)$ is the following

Proposition 1.4.1. *The Tœplitz operator $T(f)$ is Fredholm. More precisely, the operators $T(f)T(f^{-1}) - \mathbf{1}$ and $T(f^{-1})T(f) - \mathbf{1}$ are compact.*

Proof. First of all, it is easy to check that $\|T(f)\| \leq \|f\|_\infty$ where $\|f\|_\infty = \sup_{x\in\mathbb{T}}|f(x)|$. Therefore, by Stone-Weierstrass theorem, we can approximate f by a sequence of trigonometric polynomials. Since the set of compact operators is norm closed it is enough to prove the theorem for trigonometric polynomials.

We can write $T(f) = PfP$ where f denotes here the operator of multiplication by f in $L^2(\mathbb{T})$. Thus $T(f)T(f^{-1}) - \mathbf{1} = P[f,P]f^{-1}P$. It is enough to show that $[f,P]$ is compact. By linear combination, it is sufficient to consider the case for which $f(x) = \exp\{\mathbf{i}nx\}$. An elementary calculation shows that it is a finite rank operator. □

It is well known [RSIV, CF87] that a Fredholm operator T admits finite dimensional kernel and cokernel. The index is then defined as follows:

$$\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Coker}(T))\,. \tag{1.4.5}$$

It satisfies the following properties:

(i) Ind is norm continuous (homotopy invariance),
(ii) If K is any compact operator $\mathrm{Ind}(T+K) = \mathrm{Ind}(T)$,
(iii) $\mathrm{Ind}(T^*) = -\mathrm{Ind}(T)$.

The main result about Tœplitz operators is the following theorem

Index Theorem . *The index of the Tœplitz operator $T(f)$ is equal to the winding number of f namely:*

$$\mathrm{Ind}(T(f)) = -\langle[\eta]|f^*[\mathbb{T}]\rangle \tag{1.4.6}$$

where $\eta = dz/2\mathbf{i}\pi z$.

Proof. By the homotopy invariance of both sides, and thanks to the formula (1.4.1), it is sufficient to prove this formula for $f(x) = e_n(x) = \exp\{\mathbf{i}nx\}$. The formula (1.4.4) shows that the right-hand side is nothing but n. Let us consider $g \in \mathcal{H}(D)$. Then if $n \leq 0$, $T(e_n)g(z) = z^n g(z)$ showing that $\mathrm{Ker}(T(f)) = \{0\}$. If $n < 0$ $T(e_n)g(z) = \mathrm{Reg}(g(z)/z^{|n|})$ where $\mathrm{Reg}(h)$ denotes the regular part of the meromorphic function h; thus $T(e_n)g = 0$ if and only if g is a polynomial of degree $|n|-1$, namely $\dim\{\mathrm{Coker}(T(f))\} = |n| = -n$. Since $\mathrm{Coker}(T) = \mathrm{Ker}(T^*)$, and $T(e_n)^* = T(e_{-n})$ we get immediately the result. □

The Index Formula (1.4.6) is interesting in that, while the right hand side expresses in some way a property of 'waves', related to the necessity that the phase of f varies by a multiple of 2π as x varies on one period, the left hand side gives an expression of the winding number in term of an operator, which may be interpreted as the analog of the 'particle' interpretation in quantum mechanics.

1.5 The Sturm-Liouville Theory on an Interval

Let us consider now a Schrödinger equation on a finite interval, namely we seek for solutions of the following ordinary differential equation

$$\psi''(x) + (E - V(x))\psi(x) = 0 \qquad x \in (-L, L) \tag{1.5.1}$$

where we assume that V is a continuous non negative function on $(-L, L)$.

Let us consider first the point of view of waves. It is well known [AR74] that given x_0 in $(-L, L)$ and $A, B \in \mathbb{C}$, there is a unique solution such that $\psi(x_0) = A$ and $\psi'(x_0) = B$. This solution is of class $\mathcal{C}^2$. It is real whenever E, A and B are real. In particular for every value of E, we can find unique solutions $\psi_\pm$ such that

$$\psi_\pm(\pm L) = 0 \,, \qquad \pm\psi'_\pm(\pm L) = -1 \,, \tag{1.5.2}$$

They are real for real E's, and whenever $E \leq 0$, these two solutions are positive and convex everywhere on $(-L, L)$. Moreover they are analytic entire functions of E in the complex plane.

In what follows we will restrict ourselves to the solution ψ_- which will be denoted by ψ_E to specify the value of the energy. The case of ψ_+ can be treated in a similar way. The main result is contained in the following theorem

Theorem 1. *(i) The set of E's for which $\psi_E(L) = 0$ is an infinite discrete sequence of positive numbers $0 < E_0 < E_1 < \cdots < E_{n-1} < \cdots$ converging to ∞.*

(ii) If $E_{n-1} < E < E_n$, the solution ψ_E has exactly n simple, isolated zeroes $-L < x_1(E) < x_2(E) < \cdots < x_n(E) < L$ such that $dx_k(E)/dE < 0$ and $x_n(E)$ converges to L as E tends to E_{n-1} from above.

Proof. Let us define the Wronskian of two functions ψ, ϕ by $W_x(\phi, \psi) = \phi(x)\psi'(x) - \phi'(x)\psi(x)$. Differentiating and using the Schrödinger equation, we get the following identities

$$W_x(\psi_E, \psi_E^*) = 2\mathbf{i}\mathrm{Im}(E) \int_{[-L,x]} d\zeta |\psi_E(\zeta)|^2 \,, \tag{1.5.3a}$$

$$W_x(\partial_E\psi_E, \psi_E) = \int_{[-L,x]} d\zeta \psi_E(\zeta)^2 \,. \tag{1.5.3b}$$

The first identity applied to $x = L$ shows that if $\mathrm{Im}(E) \neq 0, \psi_E(L) \neq 0$. The second one shows that if $\psi_E(L) = 0$, then $\psi'_E(L) \neq 0$ (by the uniqueness theorem), and therefore $\partial_E \psi_E(L) \neq 0$. Thus any E such that $\psi_E(L) = 0$ is necessarily real and isolated. Since ψ_E is positive for $E \leq 0$, each such value is automatically positive. So we get a discrete set $0 < E_0 < E_1 < \cdots < E_{n-1} < \cdots$ corresponding to solutions of $\psi_E(L) = 0$. We will prove below that it is actually infinite and unbounded.

On the other hand, if for some x in $[-L, L]$ $\psi_E(x) = 0$, $\psi'_E(x)$ cannot vanish for otherwise, by the uniqueness theorem, we would have $\psi_E = 0$, contradicting the fact that $\psi'_E(-L) = 1$. Therefore, each zero of ψ_E is simple and isolated. Let $x(E)$ be such a zero. Differentiating the identity $\psi_E(x(E)) = 0$ with respect to E, and using (1.5.3b), we get (since for E real, ψ_E is real)

$$dx(E)/dE = -\{\psi'_E(x(E))\}^{-2} \int_{[-L,x(E)]} d\zeta \psi_E(\zeta)^2 < 0 \,. \tag{1.5.4}$$

The only way for a new zero to hold as E increases, is to appear at the extremity L of the interval namely for $E = E_n$ for some n. Otherwise, there would be a value $E' \neq E_n$ for all n's and a point x' in $(-L, L)$ such that for ε small enough and $E' - \varepsilon < E < E'$, ψ_E admits $-L < x_1(E) < x_2(E) < \cdots < x_k(E) < L$ as zeroes for some k, and for $E' < E < E' + \varepsilon$ there would be a new zero $x'(E)$ between say $x_j(E)$ and $x_{j+1}(E)$, converging to x' as E converges to E' from above. Since the function ψ_E is smooth in E, it converges point-wise to ψ'_E as $E \to E'$, and in particular $(-)^j \psi_{E'}(x) \geq 0$ for $x_j(E) < x < x_{j+1}(E)$. Thus x' would be a double zero of $\psi_{E'}$ which is impossible. Thus as E increases, one and only one zero appears every time E takes values in the set of E_n's.

Let us now introduce the real function $\theta_E(x)$ defined by $\tan(\theta_E(x)/2) = \sqrt{E}\psi_E(x)/\psi'_E(x)$ which may also be defined as the unique solution of

$$d\theta_E(x)/dx = 2\sqrt{E} - 2V(x)/\sqrt{E}\sin^2(\theta_E(x)/2) \,, \qquad \theta_E(-L) = 0 \,. \tag{1.5.5}$$

Since $V \geq 0$ we get $\sqrt{E}/\pi - \langle V \rangle/\pi\sqrt{E} \leq \theta_E(L)/4\pi L \leq \sqrt{E}/\pi$, with $\langle V \rangle = (1/2L)\int_{[-L,L]} dx V(x)$. On the other hand, $\theta_E(L)/2\pi = n$ for $E = E_{n-1}$ as one can easily see from the definition, so we get

$$n\pi/2L \leq E_{n-1}^{1/2} \leq n\pi/4L + \{n^2\pi^2/16L^2 + \langle V \rangle\}^{1/2} \,, \tag{1.5.6}$$

showing that the sequence $\{E_n; n \in N\}$ is infinite and unbounded. $\square$

Let us now consider the vector $\Psi_E(x) = (\psi_E(x), \psi'_E(x)/\sqrt{E}) \in \mathbb{R}^2$. Thanks to the uniqueness theorem, it never vanishes for $-L \leq x \leq L$. Thus it defines a unique line $\Delta_E(x)$ in the projective space $\mathbb{PR}(1)$. We will identify $\mathbb{PR}(1)$ with the unit circle through the stereographic projection so that $\Delta_E(x)$ is parametrized by the angle $\theta_E(x) \,(\mathrm{mod}\, 2\pi)$ defined above. Therefore denoting by η the closed 1-form $\eta = d\theta/2\pi$ on the unit circle, the previous theorem shows that

$$E_{n-1} < E < E_n \ , \qquad n = [\langle \eta | \Delta_E(-L, L) \rangle] \ , \tag{1.5.7}$$

where $[x]$ represents the integer part of x.

Let us now consider the operator point of view. Let H be the self adjoint operator defined by $H = H_0 + V$ where H_0 is defined as $-\partial^2/\partial x^2$ with Dirichlet boundary conditions on $[-L, L]$. By standard arguments, H_0 is a positive operator with a compact resolvent and therefore it has a discrete unbounded spectrum on the real positive axis. Since $V \geq 0$ the same is true for H. E is an eigenvalue of H if and only if the corresponding solutions $\psi_\pm$ both vanish on $\pm L$, in which case they are proportional. Thus the spectrum of H coincides with the family $\{E_n; n \in \mathbb{N}\}$ defined previously.

Let now P_E be the eigenprojection $\chi\{H \leq E\}$ of H onto the energies smaller than or equal to E. Then we get the following gap labelling theorem

Theorem 2 (The First Gap Labelling Theorem). *Let H be the self adjoint operator on $L^2(-L, L)$ defined by $H = -\partial^2/\partial x^2 + V$ with Dirichlet boundary conditions. We denote by $0 < E_0 < E_1 < \cdots < E_{n-1} \cdots$ the eigenvalues of H. For $E \in \mathbb{R}$, let P_E be the eigenprojection of H on energies smaller than or equal to E.*

Let also ψ_E be the unique solution of $-\psi_E''(x) + V(x)\psi_E(x) = E\psi_E(x)$ such that $\psi_E(-L) = 0$, $\psi_E'(-L) = 1$. Let $\Delta_E(x)$ be the line defined by the vector $(\psi_E(x), \psi_E'(x)/\sqrt{E})$ in $\mathbb{R}^2$, and let η be the canonical closed 1-form on $\mathbb{PR}(1)$.

Then if $E_{n-1} \leq E < E_n$ we get

$$\mathrm{Tr}(P_E) = [\langle \eta | \Delta_E(-L, L) \rangle] = n \ . \tag{1.5.8}$$

Remark. Here as in the index formula, we get a formula with two sides: the r.h.s. represents the operator, namely the particle point of view, while the l.h.s. represents the wave function point of view.

2. Homogeneous Media

2.1 Breaking the Translation Symmetry

This Section will be devoted to the description of the formalism required to treat more complicated problems arising in Physics, and especially in the physical properties of solids. The main tool in Solid State Physics is the Bloch theory valid for periodic crystals [MA76]. It gives rise to the theory of bands. Much effort has been devoted during the fifties and the sixties to the explicit calculation of bands and Bloch waves in real crystals. Considering the great number of crystal symmetries, one can imagine how difficult it was to exhaust all possible cases. The next class of problems solid state physicists were interested in, was the transport properties of these materials: electric conduction (metals, semiconductors, insulators, superconductors), thermal properties (phonons, heat capacity, diffusion constant). This requires the use of the Green-Kubo theory, which is not yet completely justified because of its conceptual difficulty, but which can be considered as a satisfactory and widely accepted phenomenological theory. Bloch's theory and Green-Kubo's theory are basic tools in dealing with the subject as long as we are not dealing with the many-body problem.

However, most materials are actually aperiodic. First of all, even though the periodic case is an excellent approximation, there is usually quite a lot of defects in real crystals. Moreover, temperature produces migration of atoms in a solid, leading to some randomness in the distribution of forces acting on the electrons. Roughness of interfaces may also produce random forces in $2D$ devices. However, defects and thermal fluctuations, as long as they are small, can be treated as a first order perturbation of band theory.

But aperiodicity may be produced on large scale for physical reasons. The oldest example is probably the effect of a uniform magnetic field on electronic properties of a crystal. It has been the focus of attention since the very early days of Solid State Physics with the works of Landau [LA30] and Peierls [PE33] in the thirties devoted to the electron diamagnetism of metals. As a matter of fact, a uniform magnetic field breaks the translation symmetry of the Bloch waves because it produces a non translation invariant phase factor. On the other hand, the response of a solid to a uniform magnetic field is one of the most useful tools for experiments to get information on the microscopic properties. The reason is that a magnetic field breaks the time reversal symmetry, and will enable to separate various effects. For instance the Hall effect permits to measure the sign and the density of charge carriers in a conductor. Moreover, de Haas-van Alfen oscillations of the magnetoresistance give precise informations on the shape of the Fermi surface. Still the calculation of the electronic energy spectrum in this case required the contribution of hundreds of the best physicists during the last forty years. One of the most spectacular results is probably the calculation by Hofstadter in 1976 [HO76, WI84, SO85] of the energy spectrum for a $2D$ electron in a square crystal submitted to a perpendicular magnetic field: it has a fractal structure which is still under study now by mathematicians [BS82b, EL82b, HS87, GH89, BR90].

Other materials are intrinsically aperiodic. The charge density wave in a one-dimensional chain submitted to a Peierls instability, is modulated at a frequency determined by the Fermi quasimomentum, giving rise to a quasiperiodic effective potential for electrons. Quasicrystals discovered in 1984 [SB84], have their atoms located on a quasiperiodic lattice. Amorphous materials, have their atoms located on a very aperiodic lattice, which nevertheless may be generated by a deterministic geometry [MS83]. All these states of matter are actually stable (or may be strongly metastable).

So there is a need for a mathematical framework liable to describe these situations, and to permit the calculation of physical quantities of interest such as the electronic spectrum, the density of states, thermodynamical quantities (e.g. the heat capacity), the transport coefficients. The main difficulty is that translation invariance is broken, in such a way that there is no Bloch decomposition of the problem.

The main tool we develop here is the description of a Non Commutative Manifold, namely the Brillouin zone of an aperiodic medium, in order to replace Bloch theory for aperiodic media!

2.2 Non-Commutative Geometry

An ordinary locally compact manifold M is usually described as a set with a topology which makes it a locally compact space. In addition, one usually defines a family of charts, namely mappings from open sets in $\mathbb{R}^n$ into M satisfying compatibility and smoothness conditions [BO67a]. It allows to define the space of $\mathcal{C}^\infty$ functions on M with compact support; it is a commutative Frechet *-algebra. The space $\mathcal{C}_0(M)$ of continuous functions vanishing to zero at infinity on M is the completion of the space $\mathcal{C}^\infty(M)$ under the sup-norm. It is a commutative C^*-algebra. Then smooth real vector fields with compact support are nothing but a family of *-derivations of this algebra generating a one-parameter group of point-wise norm continuous *-automorphisms.

Conversely, the manifold structure of M can be recovered from the data of $\mathcal{C}_0^\infty(M)$, namely from $\mathcal{C}_0(M)$ and the family of all smooth vector fields with compact support, the dense subalgebra given by the common domain of all polynomials in these vector fields being nothing but $\mathcal{C}_0^\infty(M)$.

Gelfand's theorem [SA71, BO67b] asserts that every commutative C^*-algebra $\mathcal{A}$ is the space of continuous functions vanishing to zero at infinity on a locally compact space M. In this latter case, M is constructed as the set of characters of $\mathcal{A}$ namely the set of *-homomorphisms from $\mathcal{A}$ to $\mathbb{C}$, and the identification between points x of M and a character χ of A is given through $\chi(f) = f(x)$ for all $f \in \mathcal{A}$.

A non-commutative locally compact space will be defined by analogy with the commutative case as the data of a non commutative C^*-algebra $\mathcal{A}$, which will represent the space of continuous functions vanishing to zero at infinity on a virtual object which will be the non commutative space itself [CO90]. The fact of the matter is that only functions on a non commutative space are defined, not the space itself at this level of generality. The game consists in expressing every geometrical property of an ordinary manifold M as an algebraic property on $\mathcal{C}_0(M)$, and in extending this last property as a definition in the non-commutative case. In this way, M becomes a smooth manifold whenever we have defined on $\mathcal{A}$ a family of *-derivations generating one-parameter groups of point-wise norm continuous *-automorphisms. In much the same way, a vector bundle $\mathcal{E}$ is defined through the data of its smooth sections, which is nothing but a module (in the algebraic sense) over the algebra $\mathcal{A}$. One can then associate to each derivation δ a connection ∇ on this bundle by means of a linear map $\nabla : \mathcal{E} \to \mathcal{E}$ such that $\nabla(f\eta) = \delta(f)\eta + f\nabla(\eta)$ for $f \in \mathcal{A}, \eta \in \mathcal{E}$. Such a connection is not unique: it is defined up to a module homomorphism.

In much the same way one can integrate functions. This can be done in the simplest cases by means of a trace on $\mathcal{A}$, namely a densely defined linear map τ on $\mathcal{A}$ such that $\tau(A^*A) = \tau(AA^*) \geq 0$ for any $A \in \mathcal{A}$. This is a natural generalization of the integral. However, there are C^*-algebras on which no non trivial trace exists, but as far as we will be concerned in this paper, all C^*-algebras we will consider will have a faithful trace. A trace is faithful if for each non zero element $A \in \mathcal{A}, \tau(AA^*) > 0$.

More generally, one can define a differential algebra $\Omega(\mathcal{A})$ as the linear space generated by symbols of the form $A_0 dA_1 \cdots dA_n, A_i \in \mathcal{A}$. This algebra becomes a *-algebra if we impose the relations $d(AB) = (dA)B + A(dB)$ and $d(A^*) = (dA)^*$. Moreover it is graded over $\mathbb{Z}$ if we define the degree of $A_0 dA_1 \cdots dA_n$ as n. The differential d is extended to $\Omega(\mathcal{A})$ by linearity and by imposing $d^2 = 0$. Then, a closed current is a trace on this graded algebra, namely a linear map $\tau : \Omega(\mathcal{A}) \to \mathbb{C}$ such that

$$\tau(\eta\eta') = (-1)^{\deg(\eta)\deg(\eta')}\tau(\eta'\eta), \qquad \tau(d\eta) = 0, \qquad \text{for all } \ \eta, \eta' \in \Omega(\mathcal{A}) \ .$$

The definition and the study of cyclic and de Rham cohomology defined on closed currents in this non commutative context, have been the subject of A. Connes's work and we will refer the reader to [CO82, CO83] for a complete description.

2.3 Periodic Media: Bloch Theory and the Brillouin Zone

Let us consider first a Schrödinger operator on $L^2(\mathbb{R}^n)$ of the form

$$H = P^2/2m + V \ ,$$

where V is a continuous and periodic function on $\mathbb{R}^n$ with a lattice of periods Γ. It means that H commutes with the operators $T(a)$ of translation by $a \in \Gamma$. Γ is a Bravais lattice namely a discrete subgroup of $\mathbb{R}^n$ such that the linear space it generates is equal to $\mathbb{R}^n$. We will set $\mathbf{V} = \mathbb{R}^n/\Gamma$; $\mathbf{V}$ can be represented by means of the unit cell of the lattice, which is a fundamental domain for Γ (the Voronoi cell).

The Bloch theory consists in diagonalizing simultaneously H and the $T(a)$'s. To do so, we introduce the reciprocal lattice Γ^* as the set of b's in $\mathbb{R}^n$ such that the scalar product $\langle b|a\rangle$ is an integer multiple of 2π for any $a \in \Gamma$. The Brillouin zone will be defined as the quotient $\mathbf{B} = \mathbb{R}^n/\Gamma^*$ and is isomorphic to a n-dimensional torus. Notice that this definition of the Brillouin zone is not exactly the same as the definition given by Solid State physicists [MA76].

Now, the generalized eigenfunctions of the $T(a)$'s are Bloch waves, namely functions $\psi_k(x)$ on $\mathbb{R}^n \times \mathbb{R}^n$ such that

$$\psi_{k+b}(x+a) = e^{\mathbf{i}\langle k|a\rangle}\psi_k(x) \ . \tag{2.3.1}$$

Here k is defined modulo Γ^* namely, $k \to \psi_k(x)$ can be seen as a function over $\mathbf{B}$. Moreover, for every pair $\psi_k(x), \phi_k(x)$ of Bloch waves, the map $x \in \mathbb{R}^n \to \psi_k(x)^*\phi_k(x)$ is Γ-periodic and can therefore be seen as a function over $\mathbf{V}$.

In order to get a mathematically clean picture we introduce the Hilbert space $\mathcal{W}$ built as the space of ψ's satisfying (2.3.1), and such that

$$\|\psi\|^2 = \int_{\mathbf{V}\times\mathbf{B}} d^n x d^n k |\psi_k(x)|^2 < \infty \tag{2.3.2}$$

$\mathcal{W}$ is actually isomorphic to $\mathcal{H} = L^2(\mathbb{R}^n)$ and a unitary transformation from $\mathcal{H}$ to $\mathcal{W}$ is defined by the following 'Wannier transform':

$$Wf(x,k) = \sum_{a\in\Gamma} f(x-a) e^{\mathbf{i}\langle k|a\rangle} , \quad f \in \mathcal{H} , \tag{2.3.3a}$$

$$W^*\psi(x) = (1/|\mathbf{B}|) \int_{\mathbf{B}} d^n k \psi_k(x) , \quad \psi \in \mathcal{W} , \tag{2.3.3b}$$

which satisfies $WW^* = \mathbf{1}_{\mathcal{W}}, W^*W = \mathbf{1}_{\mathcal{H}}$. In particular the norms are conserved.

We then introduce for each $k \in \mathbf{B}$ the Hilbert space $\mathcal{H}_k$ of functions u on $\mathbb{R}^n$, such that

$$u(x+a) = e^{\mathbf{i}\langle k|a\rangle} u(x) , \qquad \|u\|^2 = \int_{\mathbf{V}} d^n x |u(x)|^2 < \infty . \tag{2.3.4}$$

Then an element ψ of $\mathcal{W}$ defines a square integrable section $k \to \psi_k \in \mathcal{H}_k$ and conversely, each such section defines an element of $\mathcal{W}$, namely $\mathcal{W}$ can be identified with the direct integral [DI69]

$$\mathcal{W} = \int_{\mathbf{B}}^{\oplus} d^n k \mathcal{H}_k . \tag{2.3.5}$$

Then both $WT(a)W^*$ and WHW^* leave each fiber $\mathcal{H}_k$ invariant. $WT(a)W^*$ is nothing but the operator of multiplication by $e^{\mathbf{i}\langle k|a\rangle}$. Hence the C^*-algebra the $T(a)$'s generate, is nothing but the algebra $\mathcal{C}(\mathbf{B})$ of continuous functions over $\mathbf{B}$ acting on $\mathcal{W}$ by multiplication.

On the other hand, it is standard to check that the self-adjoint operator H is transformed by W into the family of partial differential operators

$$H_k = -(\hbar^2/2m) \sum_{i\in[1,n]} \partial^2/\partial x_i^2 + V(x) , \tag{2.3.6}$$

with domain $\mathcal{D}(H_k)$ given by the space of elements u's in $\mathcal{H}_k$ such that $\partial^2 u/\partial x_i^2 \in \mathcal{H}_k$. It is a standard result [RSII] that the H_k's have compact resolvents: this is because for $V = 0$, the spectrum of $\mathcal{H}_k$ is the discrete set of eigenvalues $\{(\hbar^2/2m)(k+b)^2; b \in \Gamma^*\}$, showing that its resolvent is indeed compact, whereas V is a bounded operator.

We can then identify $\mathcal{H}_k$ with $l^2(\Gamma^*)$ by means of the choice of the orthonormal basis $\{u_b(k); b \in \Gamma^*\}$, where

$$u_b(k) : x \in \mathbb{R}^n \to e^{\mathbf{i}\langle k+b|x\rangle} \in \mathbb{C} \ , \tag{2.3.7}$$

whereas the resolvent $(z\mathbf{1}-H_k)^{-1}$ becomes a compact operator $R_k(z)$ on $l^2(\Gamma^*)$ the matrix of which being $R_k(z)_{b,b'} = \langle u_b(k)|(z\mathbf{1}-H_k)^{-1}u_{b'}(k)\rangle$. If $L(b)$ is the translation by $b \in \Gamma^*$ in $l^2(\Gamma^*)$, we actually get

$$L(b)R_k(z)L(b)^{-1} = R_{k+b}(z) \ , \tag{2.3.8}$$

and the map $k \in \mathbb{R}^n \to R_k(z) \in \mathcal{K}(l^2(\Gamma^*))$ is norm continuous (here $\mathcal{K}(l^2(\Gamma^*))$ denotes the C^*-algebra of compact operators on $l^2(\Gamma^*)$). In other words, $R(z)$ defines a continuous map from $\mathbb{R}^n$ to the C^*-algebra $\mathcal{K}$ of compact operators, satisfying the covariance condition (2.3.8). Here we must notice that given two Hilbert spaces with a countable basis $\mathcal{H}$ and $\mathcal{H}'$, the C^*-algebras $\mathcal{K}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H}')$ are actually isomorphic, and this allows us to denote them by $\mathcal{K}$ without referring to the Hilbert spaces on which they act.

A better representation will be given by the data of a family $\{\psi_j \in \mathcal{W}; j \in \mathbb{N}^n\}$, such that $\psi_j(k) : k \in \mathbf{B} \to \psi_j(.;k) \in L^2_{\text{loc}}(\mathbb{R}^n)$ is continuous, and

$$\int_V d^n x \psi_j(x,k)^* \psi_{j'}(x,k) = \delta_{j,j'} \ . \tag{2.3.9}$$

It is tedious but easy to prove that such a family exists and can be constructed in such a way to be $\mathcal{C}^\infty$ with respect to k. Clearly, $\{\psi_j(k); j \in \mathbb{N}^n\}$ gives an orthonormal basis of $\mathcal{H}_k$.

Let $b \in \Gamma^* \to n(b) \in \mathbb{N}^n$ be a bijection. We then denote by $S(k)$ the unitary operator from $l^2(\Gamma^*)$ into $l^2(\mathbb{N}^n)$ defined by

$$S(k) = \sum_b |\psi_{n(b)}(k)\rangle\langle u_b(k)| \ . \tag{2.3.10}$$

It satisfies $S(k)L(b) = S(k-b)$ for $b \in \Gamma^*$. Then $R'_k(z) = S(k)R_k(z)S(k)^*$ is norm continuous and is Γ^*-periodic in k, and defines therefore a continuous mapping from $\mathbf{B}$ into $\mathcal{K}$ namely an element of the C^*-algebra $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$.

The main result of this Section is the following

Theorem 3. *The C^*-algebra generated by the family $\{T(x)R(z)T(x)^{-1}; x \in \mathbb{R}^n\}$ of translated of the resolvent of H is isomorphic to $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$.*

This result is actually the intuitive key to understand the point of view developed later on. For indeed, the algebra $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$ can be identified with the algebra of the Brillouin zone.

Even though $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$ is already non commutative, its non commutative part comes from $\mathcal{K}$ which represents possible degeneracies. More precisely, given a complex vector bundle E over $\mathbf{B}$, we will denote by $\mathcal{E}$ the space of continuous sections of E: $\mathcal{E}$ is then a module over $\mathcal{C}(\mathbf{B})$ for the point-wise multiplication.

However, each section can be seen locally as a continuous map from $\mathbf{B}$ into $\mathbb{C}^N$ for some N, and therefore we can multiply it point-wise with a $N \times N$-matrix valued continuous function over $\mathbf{B}$, namely by an element of $\mathcal{C}(\mathbf{B}) \otimes M_N(\mathbb{C})$ which is canonically imbedded in $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$. Such an operator is a module homomorphism of $\mathcal{E}$, and we can see that each module homomorphism is of this type. Moreover, the Swann-Serre theorem [RI82] asserts that for every vector bundle E there is N big enough and a projection P in $\mathcal{C}(\mathbf{B}) \otimes M_N(\mathbb{C})$, such that its module of sections $\mathcal{E}$ is isomorphic to the module $P\mathcal{C}(\mathbf{B}) \otimes \mathbb{C}^N = \{f : k \in \mathbf{B} \to C^N; Pf = f\}$; conversely, each projection in $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$ gives rise to such a vector bundle. Hence $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$ contains not only the construction of $\mathbf{B}$ itself but also that of every vector bundle over $\mathbf{B}$.

This way of reasoning applies as well to every locally compact space M provided $\mathcal{C}_0(M) \otimes \mathcal{K}$ replaces $\mathcal{C}(\mathbf{B}) \otimes \mathcal{K}$. $\mathcal{C}_0(M) \otimes \mathcal{K}$ will be called the algebra of M; in particular, $\mathcal{K}$ represents the algebra of a point.

We say that a C^*-algebra $\mathcal{A}$ is stable whenever it is isomorphic to $\mathcal{A} \otimes \mathcal{K}$. It is a standard result that $\mathcal{K}$ is stable [BL86]. Therefore for any C^*-algebra $\mathcal{A}$, $\mathcal{A} \otimes \mathcal{K}$ is always stable, and is called the stabilized of $\mathcal{A}$. Two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are stably isomorphic whenever $\mathcal{A} \otimes \mathcal{K}$ is isomorphic to $\mathcal{B} \otimes \mathcal{K}$.

The Theorem 3 shows that if H is a periodic Schrödinger operator, the algebra of the Brillouin zone is nothing but the algebra $C^*(H)$ generated by the family of all translated of the resolvent of the Hamiltonian H. But the construction of $C^*(H)$ does not require the periodicity of the potential V, and can be done for an aperiodic potential as well. Therefore, $C^*(H)$ will be called the 'Non Commutative Brillouin zone' or the 'Brillouin zone' of H.

2.4 Homogeneous Schrödinger Operators [BE86]

We see that two ingredients are necessary to define a Non-Commutative Brillouin zone, namely the energy operator and the action of the translation group on H. Actually, this is not so surprising if we deal with a homogeneous medium. For indeed, a homogeneous medium with infinite volume looks translation invariant at a macroscopic scale, even though translation invariance may be microscopically broken. For this reason, there is no natural choice of an origin in space. In particular if H is a Hamiltonian describing one particle in this medium, we can choose to replace it by any of its translated $H_a = T(a)HT(a)^{-1}$ $(a \in \mathbb{R}^n)$ and the physics will be the same. This choice is entirely arbitrary, so that the smallest possible set of observables must contain at least the full family $\{H_a; a \in \mathbb{R}^n\}$.

Actually H is not a bounded operator in general, so that calculations are made easier if we consider its resolvent instead. On the other hand as has already been argued in Section 1.3, the set of observables must be a C^*-algebra. So we must deal with the algebra $C^*(H)$ anyway. Let us define precisely what we mean by 'homogeneity' of the medium described by H.

Definition. Let $\mathcal{H}$ be a Hilbert space with a countable basis. Let $\mathbf{G}$ be a locally compact group (for instance $\mathbb{R}^n$ or $\mathbb{Z}^n$). Let U be a unitary projective representation of $\mathbf{G}$ namely for each $a \in \mathbf{G}$, there is a unitary operator $U(a)$ acting on $\mathcal{H}$ such that the family $U = \{U(a); a \in \mathbf{G}\}$ satisfies the following properties:

(i) $U(a)U(b) = U(a+b)e^{\mathbf{i}\phi(a,b)}$ for all $a, b \in \mathbf{G}$, where $\phi(a,b)$ is some phase factor.

(ii) For each $\psi \in \mathcal{H}$, the map $a \in \mathbf{G} \to U(a)\psi \in \mathcal{H}$ is continuous.

Then a self adjoint operator H on $\mathcal{H}$ is *homogeneous* with respect to $\mathbf{G}$ if the family $S = \{R_a(z) = U(a)(z\mathbf{1} - H)^{-1}U(a)^{-1}; a \in \mathbf{G}\}$ admits a compact strong closure.

Remark. A sequence A_n of bounded operators on $\mathcal{H}$ converges strongly to the bounded operator A if for every $\psi \in \mathcal{H}$, the sequence $\{A_n\psi\}$ of vectors in $\mathcal{H}$ converges in norm to $A\psi$. So the set S has a strong compact closure if given $\varepsilon > 0$, and a finite set $\psi_1, \cdots, \psi_N$ of vectors in $\mathcal{H}$, there is a finite set $a_1, \cdots, a_n$ in $\mathbf{G}$ such that for every a in $\mathbf{G}$ and every $1 \le j \le N$ there is $1 \le i \le n$ such that $\|(R_a(z) - R_{a_i}(z))\psi_j\| \le \varepsilon$. In other words, the full family of translated of $R(z)$ being well approximated on vectors by a finite number of them, repeats itself infinitely many times up to infinity.

The virtue of this definition comes from the construction of the 'hull'. Indeed let z be in the resolvent set $\rho(H)$ of H, and let H be homogeneous. Then let $\Omega(z)$ be the strong closure of the family $\{R_a(z) = U(a)(z\mathbf{1} - H)^{-1}U(a)^{-1}; a \in \mathbf{G}\}$. It is therefore a compact space, which is metrizable since the Hilbert space $\mathcal{H}$ has a countable basis. Moreover, it is endowed with an action of the group $\mathbf{G}$ by means of the representation U. This action defines a group of homeomorphisms of $\Omega(z)$.

We first remark that if z' is another point in $\rho(H)$ the spaces $\Omega(z)$ and $\Omega(z')$ are actually homeomorphic. For indeed, we have for $a \in \mathbf{G}$

$$\begin{aligned} R_a(z') &= \{\mathbf{1} + (z' - z)R_a(z)\}^{-1}R_a(z) \, , \\ \{\mathbf{1} + (z' - z)R_a(z)\}^{-1} &= \{\mathbf{1} - (z' - z)R_a(z')\} \, , \end{aligned} \tag{2.4.1}$$

so that $\{R_{a_i}(z); i \ge 0\}$ converges strongly to R if and only if $\{R_{a_i}(z'); i \ge 0\}$ converges strongly to some R' and the map $R \in \Omega(z) \to R' \in \Omega(z')$ is an homeomorphism. Identifying them gives rise to an abstract compact metrizable space Ω endowed with an action of $\mathbf{G}$ by a group of homeomorphisms. If $\omega \in \Omega$ and $a \in \mathbf{G}$ we will denote by $T^a.\omega$ the result of the action of a on ω, and by $R_\omega(z)$ the representative of ω in $\Omega(z)$. Then one gets

$$U(a)R_\omega(z)U(a)^* = R_{T^a.w}(z) \, , \tag{2.4.2}$$

$$R_\omega(z') - R_\omega(z) = (z - z')R_\omega w(z')R_\omega(z) = (z - z')R_\omega(z)R_\omega(z') \, . \tag{2.4.3}$$

In addition, $z \to R_\omega(z)$ is norm-holomorphic in $\rho(H)$ for every $\omega \in \Omega$, and $\omega \to R_\omega(z)$ is strongly continuous.

Definition. Let H be a homogeneous operator on the Hilbert space $\mathcal{H}$ with respect to the representation U of the locally compact group $\mathbf{G}$. Then the *hull* of H is the dynamical system $(\Omega, \mathbf{G}, U)$ where Ω is the compact space given by the strong closure of the family $\{R_a(z) = U(a)(z\mathbf{1} - H)^{-1}U(a)^{-1}; a \in \mathbf{G}\}$, and $\mathbf{G}$ acts on Ω through U.

The equation (2.4.3) is not sufficient in general to insure that $R_\omega(z)$ is the resolvent of some self-adjoint operator H_ω, for indeed one may have $R_\omega(z) = 0$ if no additional assumption is demanded. A sufficient condition is that H be given by $H_0 + V$ where H_0 is self-adjoint and $\mathbf{G}$-invariant, whereas V is relatively bounded with respect to H_0, i.e. $\|(z - H_0)^{-1}V\| < \infty$, and $\lim_{|z|\to\infty}\|(z - H_0)^{-1}V\| = 0$. Then,

$$R_\omega(z) = \{\mathbf{1} - (z - H_0)^{-1}V_\omega\}^{-1}(z - H_0)^{-1}$$

where $(z - H_0)^{-1}V_\omega$ is defined as the strong limit of $(z - H_0)^{-1}V_{a_i}$, which obviously exists. So $R_\omega(z)$ is the resolvent of $H_0 + V_\omega$.

In the case of Schrödinger operator, the situation becomes simpler. Let us consider the case of a particle in $\mathbb{R}^n$ submitted to a bounded potential V and a uniform magnetic field with vector potential A, such that

$$\partial_\mu A_\nu - \partial_\nu A_\mu = B_{\mu,\nu} = \text{const.} \tag{2.4.4}$$

The Schrödinger operator is given by

$$H = (1/2m) \sum_{\mu\in[1,n]} (P_\mu - eA_\mu)^2 + V = H_0 + V\ . \tag{2.4.5}$$

The unperturbed part H_0 is actually translation invariant provided one uses magnetic translations [ZA64] defined by (if $a \in \mathbb{R}^n$, $\psi \in L^2(\mathbb{R}^n)$)

$$U(a)\psi(x) = \exp\left\{(\mathbf{i}e/\hbar)\int_{[x-a,x]} dx'^\mu A_\mu(x')\right\}\psi(x-a)\ . \tag{2.4.6}$$

It is easy to check that the $U(a)$'s give a projective representation of the translation group. The main result in this case is given by

Theorem 4. *Let H be given by* (2.4.4) *and* (2.4.5) *with V a measurable essentially bounded function over $\mathbb{R}^n$. Let $\mathcal{B}_s\{L^2(\mathbb{R}^n)\}$ represent the space of bounded linear operators on $L^2(\mathbb{R}^n)$ with the strong topology, let $L^\infty_\mathbb{R}(\mathbb{R}^n)$ be the space of measurable essentially bounded real functions over $\mathbb{R}^n$ with the weak topology of $L^1(\mathbb{R}^n)$, and let z be a complex number with non zero imaginary part. Then, the map $V \in L^\infty_\mathbb{R}(\mathbb{R}^n) \to \{z - H_0 - V\}^{-1} \in \mathcal{B}_s\{L^2(\mathbb{R}^n)\}$ is continuous.*

The proof of this theorem can be found in [NB90, Appendix]. As a consequence we get

Corollary 2.4.1. *Let H be given by* (2.4.4) *and* (2.4.5) *with V a real, measurable, essentially bounded function over $\mathbb{R}^n$. Then H is homogeneous with respect to the representation U (Eq. (2.4.6)) of the translation group.*

Proof. Indeed, any ball in $L^\infty_{\mathbb{R}}(\mathbb{R}^n)$ is compact for the weak topology of $L^1(\mathbb{R}^n)$. Moreover, $U(a)VU(a)^* = V_a$ where $V_a(x) = V(x-a)$ almost surely. So V_a belongs to the ball $\{V' \in L^\infty_{\mathbb{R}}(\mathbb{R}^n); \|V'\| \le \|V\|\}$, and the weak closure of the family $\{V_a; a \in \mathbb{R}^n\}$ in $L^\infty_{\mathbb{R}}(\mathbb{R}^n)$ is compact. Thanks to Theorem 4, it follows that the strong closure of the family $\{(z - H_0 - V_a)^{-1}; a \in \mathbb{R}^n\}$ is the direct image of a compact set by a continuous function, and is therefore compact. □

Another consequence of this result is given by the following characterization of the hull

Corollary 2.4.2. *Let H be as in Corollary 2.4.1. Then the hull of H is homeomorphic to the hull of V namely the weak closure of the family $\Omega = \{V_a; a \in \mathbb{R}^n\}$ in $L^\infty_{\mathbb{R}}(\mathbb{R}^n)$. Moreover, there is a Borelian function v on Ω such that $V_\omega(x) = v(T^{-x}\omega)$ for almost every $x \in \mathbb{R}^n$ and all $\omega \in \Omega$. If in addition V is uniformly continuous and bounded, v is continuous.*

Proof. Let ρ_k be non negative functions on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} d^n x \rho_k(x) = 1$ and that for each $\delta > 0$ $\lim_{k\to\infty} \int_{|x|>\delta} d^n x \rho_k(x) = 0$. V_ω is an element of $L^\infty_{\mathbb{R}}(\mathbb{R}^n)$. Let $v_k(\omega)$ be defined by $v_k(\omega) = \int_{\mathbb{R}^n} d^n x V_\omega(x)\rho_k(x)$. By definition of the hull, this is a sequence of continuous functions on Ω. We set $v(\omega) = \lim_{k\to\infty} v_k(\omega)$ if the limit exists. This is a Borelian function because if I is a closed interval in $\mathbb{R}$, the set $\sum(I) = \{\omega \in \Omega; v(\omega) \in I\}$ is given as $\sum(I) = \cap_{n\ge1} \cup_{k\ge1} \cap_{p\ge k} \{\omega \in \Omega;\, v_p(\omega) \in I^{(1/n)}\}$, where $I^{(\varepsilon)}$ is the set of points in $\mathbb{R}$ within the distance ε from I. Since v_k is continuous, $\sum(I)$ is a Borel set. If now $F \in L^1(\mathbb{R}^n)$, one gets $\int_{\mathbb{R}^n} d^n x v(T^{-x}\omega)F(x) = \lim_{k\to\infty} \int_{\mathbb{R}^{2n}} d^n x d^n y V_{T^{-x}\omega}(y)\rho_k(y)F(x)$, which by the covariance property is nothing but $\lim_{k\to\infty} \int_{\mathbb{R}^{2n}} d^n x d^n y V_\omega(x)\rho_k(y)F(x-y)$. Since the convolution $\lim_{k\to\infty} \int_{\mathbb{R}^n} d^n y \rho_k(y)F(x-y)$ converges to F in $L^1(\mathbb{R}^n)$, it follows that $v(T^{-x}\omega) = V_\omega(x)$ for almost all x's, and all $\omega \in \Omega$.

Let V be uniformly continuous and bounded on $\mathbb{R}^n$. Then V_ω exists as an element of $L^\infty_{\mathbb{R}}(\mathbb{R}^n)$. We claim that $(v_k)_{k\ge0}$ is a Cauchy sequence for the uniform topology. For indeed, by definition of the hull, for each $\omega \in \Omega$, there is a sequence $\{a_i\}$ in $\mathbb{R}^n$, such that $\int_{\mathbb{R}^n} d^n x V_\omega(x)F(x) = \lim_{i\to\infty} \int_{\mathbb{R}^n} d^n x V(x-a_i)F(x)$, for every $F \in L^1(\mathbb{R}^n)$. In particular, we get $|v_k(\omega) - v_{k'}(\omega)| \le \lim_{i\to\infty} \int_{\mathbb{R}^2} d^n x d^n y |V(x-a_i) - V(y-a_i)|\rho_k(x)\rho_{k'}(y)$. Since V is uniformly continuous, given $\varepsilon > 0$, there is $\delta > 0$ such that if $|x-y| < \delta \Rightarrow |V(x) - V(y)| < \varepsilon/2$. Therefore, using $|x-y| > \delta \Rightarrow |x| > \delta/2$ or $|y| > \delta/2$, we get whenever $k < k'$

$$\int_{\mathbb{R}^{2n}} d^n x d^n y |V(x-a_i) - V(y-a_i)|\rho_k(x)\rho_{k'}(y)$$
$$\le \varepsilon/2 + 2\|V\|_\infty \left\{ \int_{|x|>\delta/2} d^n x \rho_k(x) + \int_{|x|>\delta/2} d^n x \rho_{k'}(x) \right\} .$$

Choosing N big enough, for $k, k' > N$ the right hand side is dominated by ε, proving the claim. Therefore, the sequence v_k converges uniformly to a continuous function v. □

The situation is very similar and actually technically simpler for a discrete Schrödinger operator on a lattice. Let us consider a Bravais lattice, that will be identified with $\mathbb{Z}^n$, and let us consider the Hilbert space $l^2(\mathbb{Z}^n)$ of square summable sequences indexed by n-uples $x = \{x_1, \cdots, x_n\} \in \mathbb{Z}^n$. Let us consider the operator H acting on $l^2(\mathbb{Z}^n)$ as follows

$$H\psi(x) = \sum_{x' \in \mathbb{Z}^n} H(x, x')\psi(x') \; , \qquad \psi \in l^2(\mathbb{Z}^n) \; , \tag{2.4.7}$$

where the sequence $\{H(x, x'); x' \in \mathbb{Z}^n\}$ satisfies $|H(x, x')| \leq f(x - x')$ with $\sum_{a \in \mathbb{Z}^n} f(a) < \infty$. Therefore H is bounded, and there is no need to consider resolvents anymore. Then we get the following theorem, where we let now U be the unitary representation of the translation group $\mathbb{Z}^n$ given by

$$U(a)\psi(x) = \psi(x - a) \; , \qquad \psi \in l^2(\in \mathbb{Z}^n) \; . \tag{2.4.8}$$

Theorem 5. *Let H be given by* (2.4.7)*. Then H is homogeneous with respect to the representation U (Eq.* (2.4.8)*) of the translation group.*

Moreover, if Ω is the hull of H, there is a continuous function vanishing at infinity h on $\Omega \times \mathbb{Z}^n$, such that $H_\omega(x, x') = h(T^{-x}\omega, x' - x)$, for every pair $(x, x') \in \mathbb{Z}^n$ and $\omega \in \Omega$.

Proof. Let $D(a)$ be the disc in the complex plane centered at zero with radius $f(a)$. Let Ω_0 be the product space $\prod_{(x,x') \in \mathbb{Z}^n \times \mathbb{Z}^n} D(x - x')$. By Tychonov's theorem, it is a compact space for the product topology. The sequence $\underline{H}(a) = \{H(x - a, x' - a); (x, x') \in \mathbb{Z}^n \times \mathbb{Z}^n\}$ belongs to Ω_0 for every $a \in \mathbb{Z}^n$. Therefore the closure Ω of the family $\{\underline{H}(a); a \in \mathbb{Z}^n\}$ is compact. For $\omega \in \Omega$ let $H_\omega(x, x')$ be the (x, x') component of ω. For the product topology the projections on components being continuous, the map $\omega \in \Omega \to H_\omega(x, x') \in \mathbb{C}$ is continuous for each pair $(x, x') \in \mathbb{Z}^n \times \mathbb{Z}^n$. Moreover we easily get $|H_\omega(x, x')| \leq f(x - x')$ for all pairs $(x, x') \in \mathbb{Z}^n \times \mathbb{Z}^n$. Let now H_ω be the operator defined by (2.4.7) with $\underline{H}$ replaced by $\underline{H}_\omega$. Its norm is bounded by $\|H_\omega\| \leq \sum_{a \in \mathbb{Z}^n} f(a)$, and the map $\omega \in \Omega \to H_\omega$ is weakly continuous, and therefore strongly continuous. In particular, the family $\{H_\omega; \omega \in \Omega\}$ is strongly compact. We also remark that if ω is the limit point of the sequence $\underline{H}(a_i)$ in Ω_0 then H_ω is the strong limit of the sequence $U(a_i)HU(a_i)^*$. At last, let T^a be the action of the translation by a in Ω_0 defined by $T^a\underline{H}' = \{H'(x - a, x' - a); (x, x') \in \mathbb{Z}^n \times \mathbb{Z}^n\}$ whenever $\underline{H}' \in \Omega_0$ then one has

$$U(a)H_\omega U(a)^* = H_{T^a_\omega} \qquad a \in \mathbb{Z}^n, \omega \in \Omega \; . \tag{2.4.9}$$

So that H is indeed homogeneous, and its hull is precisely Ω. Moreover, (2.4.9) implies that $H_\omega(x - a, x' - a) = H_{T^a_\omega}(x, x')$, namely, $H_\omega(x, x') = H_{T^{-x}\omega}(0, x' - x)$. Thus $h(\omega, x) = H_\omega(0, x)$. □

Some examples of Hamiltonians for which the hull can be explicitly computed will be especially studied in Section 3.

2.5 The Non-Commutative Brillouin Zone

Let us consider now a topological compact space Ω with a $\mathbb{R}^n$-action by a group $\{T^a; a \in \mathbb{R}^n\}$ of homeomorphisms. Given a uniform magnetic field $B = \{B_{\mu,\nu}\}$ we can associate to this dynamical system a C^*-algebra $C^*(\Omega \times \mathbb{R}^n, B)$ defined as follows. We first consider the topological vector space $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$ of continuous functions with compact support on $\Omega \times \mathbb{R}^n$. It is endowed with the following structure of *-algebra

$$\begin{aligned} fg(\omega, x) &= \int_{\mathbb{R}^n} d^n y f(\omega, y) g(T^{-y}\omega, x-y) e^{\mathrm{i}\pi(e/h)B.x\wedge y} , \\ f^*(\omega, x) &= f(T^{-x}\omega, -x)^* , \end{aligned} \tag{2.5.1}$$

where $B.x\wedge y = B_{\mu,\nu}x_\mu y_\nu$, $f, g \in \mathcal{C}_K(\Omega \times \mathbb{R}^n)$, and $\omega \in \Omega$, $x \in \mathbb{R}^n$. Here e is the electric charge of the particle and $h = 2\pi\hbar$ is Planck's constant. Let us remark here that this construction is very similar to the one given for the algebra of a groupoid (see Section 1.3). Here the groupoid is the set $\Gamma = \Omega \times \mathbb{R}^n$, its basis is $\Gamma^{(0)} = \Omega$, and the laws are given by: $r(\omega, x) = \omega$, $s(\omega, x) = T^{-x}\omega$, $(\omega, x) = (\omega, y) \circ (T^{-y}\omega, x-y)$, $(\omega, x)^{-1} = (T^{-x}\omega, -x)$.

This *-algebra is represented on $L^2(\mathbb{R}^n)$ by the family of representations $\{\pi_\omega; \omega \in \Omega\}$ given by

$$\pi_\omega(f)\psi(x) = \int_{\mathbb{R}^n} d^n y f(T^{-x}\omega, y-x) e^{\mathrm{i}\pi(e/h)B.y\wedge x}\psi(y) , \quad \psi \in L^2(\mathbb{R}^n) \tag{2.5.2}$$

namely, π_ω is linear and $\pi_\omega(fg) = \pi_\omega(f)\pi_\omega(g)$ and $\pi_\omega(f)^* = \pi_\omega(f^*)$. In addition, $\pi_\omega(f)$ is a bounded operator, for $\|\pi_\omega(f)\| \leq \|f\|_{\infty,1}$ where

$$\|f\|_{\infty,1} = \mathrm{Max}\{\sup_{\omega\in\Omega} \int_{\mathbb{R}^n} d^n y|f(\omega, y)|, \sup_{\omega\in\Omega} \int_{\mathbb{R}^n} d^n y|f^*(\omega, y)|\} . \tag{2.5.3}$$

It is a norm such that $\|fg\|_{\infty,1} \leq \|f\|_{\infty,1}$, $\|f\|_{\infty,1} = \|f^*\|_{\infty,1}$, and we will denote by $L^{\infty,1}(\Omega \times \mathbb{R}^n; B)$ the completion of $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$ under this norm. We then remark that these representations are related by the covariance condition:

$$U(a)\pi_\omega(f)U(a)^{-1} = \pi_{T^a\omega}(f) . \tag{2.5.4}$$

Now we set

$$\|f\| = \sup_{\omega\in\Omega}\|\pi_\omega(f)\| , \tag{2.5.5}$$

which defines a C^*-norm on $L^{\infty,1}(\Omega \times \mathbb{R}^n; B)$, and permits to define $\mathcal{C}^*(\Omega \times \mathbb{R}^n, B)$ as the completion of $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$ or of $L^{\infty,1}(\Omega \times \mathbb{R}^n; B)$ under this norm.

The main result of this Section is summarized in the:

Theorem 6. *Let A_μ be the vector potential of a uniform magnetic field B, and let V be in $L^\infty(\mathbb{R}^n)$. Let H be the operator*

$$H = (1/2m) \sum_{\mu \in [1,n]} (P_\mu - eA_\mu)^2 + V = H_B + V$$

and we denote by Ω its hull. Then for each z in the resolvent set of H, and for every $x \in \mathbb{R}^n$ there is an element $r(z;x) \in C^*(\Omega \times \mathbb{R}^n, B)$, such that for each $\omega \in \Omega, \pi_\omega(r(z;x)) = (z - H_{T^{-x}\omega})^{-1}$.

Proof. Since the magnetic field B is uniform, it defines a real antisymmetric $N \times N$ matrix. By Cartan's theorem, there is an orthonormal basis in $\mathbb{R}^n \{e_1, e_2, \cdots, e_{2l-1}, e_{2l}, e_{2l+1}, \cdots, e_n\}$ and positive real numbers $B_j (1 \le j \le l \le n/2)$ such that

$$\text{(2.5.6)} \qquad \begin{array}{l} \begin{array}{r} Be_{2j-1} = B_j e_{2j} \ , \\ Be_{2j} = -B_j e_{2j-1} \ . \end{array} \quad \text{if } 1 \le j \le l, \\ Be_m = 0, \quad \text{if } m \ge 2l+1 \ , \end{array}$$

Let A_μ be the vector potential in the symmetric gauge, namely, $A_\mu = -(1/2)B_{\mu\nu}x_\nu$. Then, the operators $K_\mu = -\mathbf{i}\partial_\mu - 2\pi e/hA_\mu$ obey the following commutation relations

$$\text{(2.5.7)} \qquad \begin{array}{ll} [K_{2j-1}, K_{2j}] = \mathbf{i}2\pi e/hB_j \ , & 1 \le j \le l \ , \\ [K_\mu, K_\nu] = 0 \text{ otherwise} \ . & \end{array}$$

Therefore at $V = 0$, the Schrödinger operator H_B will satisfy

$$\text{(2.5.8)} \qquad \begin{array}{rl} \exp\{-tH_B\} = & \displaystyle\prod_{j \in [1,l]} \exp\left\{-t(h^2/8\pi m)(K_{2j-1}^2 + K_{2j}^2)\right\} \\ & \displaystyle\prod_{m \in [2j+1,n]} \exp\left\{-t(h^2/8\pi m)K_m^2\right\} \ . \end{array}$$

In particular, it acts on $\psi \in L^2(\mathbb{R}^n)$ through

$$\text{(2.5.9)} \qquad \exp\{-tH_B\}(x) = \int_{\mathbb{R}^n} d^n y f_B(x-y;t) e^{\mathbf{i}\pi(e/h)B.y\wedge x}\psi(y)$$

where $f_B(x;t)) = \prod_{j \in [1,l]} f_2(\mathbf{x}_j;t;B_j) \prod_{m \in [2j+1,n]} f_1(x_m;t)$, with the following notations

(i) if $x = (x_1, \cdots, x_n)$, then $\mathbf{x}_j = (x_{2j-1}, x_{2j}) \in \mathbb{R}^2$ for $1 \le j \le l$,
(ii) $f_1(x;t) = (t.h/2\pi m)^{-1}\exp\{-2\pi m x^2/t.h^2\}$ if $x \in \mathbb{R}$,
(iii) $f_2(\mathbf{x};t;B) = (eB/h)\,(2\sinh(t.eBh/4\pi m))^{-1}$
$\exp\{-\pi eB\mathbf{x}^2/(2h.\tanh(t.eBh/4\pi m))\}$.

In particular, the kernel $f_B(x;t)$ is smooth and fast decreasing in x. In view of (2.5.2) and (2.5.3), we see that $f_B(t) \in L^{\infty,1}(\Omega \times \mathbb{R}^n; B)$ and that $\pi_\omega(f_B(t)) = \exp\{-tH_B\}$ whatever Ω and $\omega \in \Omega$.

Now by Dyson's expansion, we get

$$e^{-t(H_B+V)} = e^{-tH_B} + \sum_{n=0}^{\infty}(-)^n \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_{n-1}} ds_n e^{-(t-s_1)H_B} V e^{-(s_1-s_2)H_B} \ldots V e^{-s_n H_B} .$$

Thus it is enough to show that $V(t,s) = \exp\{-tH_B\}V \exp\{-sH_B\}$ has the form $\pi_\omega(g)$ for some $g \in L^{\infty,1}(\Omega\times\mathbb{R}^n; B)$. For indeed the Dyson series converges in norm since V is bounded. Thanks to (2.5.8), we get $V(t,s) = \pi_\omega(v(t,s))$ with

$$v(t,s)(\omega,x) = \int_{\mathbb{R}^n} d^n y f_B(y;t)V_\omega(y)f_B(x-y;s)e^{\mathbf{i}\pi(e/h)B.y\wedge x} . \tag{2.5.10}$$

Now we recall that V_ω is the weak limit in $L^\infty(\mathbb{R}^n)$ of a sequence of the form $V(.-a_i)$ as $i \to \infty$. Since f_B is smooth and fast decreasing, it follows that the left hand side of (2.5.10) is continuous in both ω and x, and that its $L^{\infty,1}(\Omega\times\mathbb{R}^n; B)$-norm is estimated by

$$\|v(t,s)\|_{\infty,1} \le \|V\|_\infty \|f_B(t)\|_{\infty,1}\|f_B(s)\|_{\infty,1} \le \text{ const. } \|V\|_\infty , \tag{2.5.11}$$

uniformly in s,t. Therefore the Dyson expansion converges in $L^{\infty,1}(\Omega\times\mathbb{R}^n; B)$, and $\exp\{-t(H_B+V_\omega)\} = \pi_\omega(f_{B,V}(t))$ for some $f_{B,V}(t) \in L^{\infty,1}(\Omega\times\mathbb{R}^n; B)$. It follows from (2.5.5) that if $V(x) \ge V_-$ almost surely, we have

$$\|f_{B,V}(t)\| \le e^{-tV_-} . \tag{2.5.12}$$

In particular, since $(z-H_\omega)^{-1} = \int_{[0,\infty]} dt \exp\{-t(H_B+V_\omega-z)\}$ provided $\mathrm{Re}(z) < V_-$, the theorem is proved. □

Problem. When is the C^*algebra $\mathcal{A}$ generated by the Hamiltonian and the family of its translated, identical to $C^*(\Omega\times\mathbb{R}^n, B)$? In most examples, one can prove the equality. But is there a general condition on the potential implying it?

2.6 Non-Commutative Calculus

In the previous Section we constructed the non commutative Brillouin zone, which is identified with the C^*-algebra $\mathcal{A} = C^*(\Omega\times\mathbb{R}^n, B)$. We now want to show that it is indeed a manifold. In order to do so we will first describe the integration theory and then we will define a differential structure.

We have already seen in Section 2.2 that one can indeed integrate functions by means of a trace on $\mathcal{A}$. Let $\mathbf{P}$ be now a probability measure on Ω, invariant and ergodic under the action of $\mathbb{R}^n$. We associate to $\mathbf{P}$ a normalized trace τ on $\mathcal{A} = C^*(\Omega\times\mathbb{R}^n, B)$ as follows:

$$\tau(f) = \int_\Omega \mathbf{P}(d\omega)f(\omega,0) , \qquad f \in \mathcal{C}_K(\Omega\times\mathbb{R}^n) . \tag{2.6.1}$$

One can easily check that this formula defines a trace on the dense subalgebra $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$. If $p \geq 1$, we denote by $L^p(\mathcal{A}, \tau)$ the completion of $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$ under the norm:

$$\|f\|_{L^p} = \tau(\{ff^*\}^{p/2}) . \tag{2.6.2}$$

In particular, one can check that the space $L^2(\mathcal{A}, \tau)$ is a Hilbert space (GNS construction) identical to $L^2(\Omega \times \mathbb{R}^n)$. The map $\phi \in L^2(\mathcal{A}, \tau) \to f\phi \in L^2(\mathcal{A}, \tau)$ defines a representation π_{GNS} of $\mathcal{A}$. The weak closure $\mathcal{W} = \pi_{\mathrm{GNS}}(\mathcal{A})''$, denoted by $W^*(\Omega, \mathbb{R}^n, B, \mathbf{P})$, is a Von Neumann algebra. By construction the trace τ extends to a trace on this algebra. We remark that if H is a self-adjoint element of $\mathcal{A}$, its eigenprojections are in general elements of the Von Neumann algebra $\mathcal{W}$.

The main result is provided by the following theorem

Theorem 7. *Let f be an element of $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$. Then its trace can be obtained as the trace per unit volume of the operator $\pi_\omega(f)$, namely for $\mathbf{P}$-almost all ω's*

$$\tau(f) = \lim_{\Lambda \uparrow \mathbb{R}^n} 1/|\Lambda| \mathrm{Tr}_\Lambda(\pi_\omega(f)) , \tag{2.6.3}$$

where Tr_Λ is the restriction of the usual trace in $L^2(\mathbb{R}^n)$ onto $L^2(\Lambda)$.

Remark. In the limit $\Lambda \uparrow \mathbb{R}^n$ the subsets Λ are measurable, their union adds up to $\mathbb{R}^n$, and $\lim_{\Lambda \uparrow \mathbb{R}^n} |\Lambda \Delta (\Lambda + a)|/|\Lambda| = 0$, where Δ represents the symmetric difference (Følner sequence [GR69]) .

Proof. Since f belongs to $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$, the operator $\pi_\omega(f)$ admits a smooth kernel given by $F_\omega(x, y) = f(T^{-x}\omega, y - x)e^{\mathbf{i}B.x\wedge y}$ (see Eq. (2.5.2)). By Fredholm theory [SI79], the trace of this operator restricted to Λ is given by:

$$\mathrm{Tr}_\Lambda(\pi_\omega(f)) = \int_\Lambda d^n x F_\omega(x, x) = \int_\Lambda d^n x f(T^{-x}\omega, 0)) .$$

By Birkhoff's ergodic theorem, we get for $\mathbf{P}$-almost all ω's

$$\lim_{\Lambda \uparrow \mathbb{R}^n} 1/|\Lambda| \mathrm{Tr}_\Lambda(\pi_\omega(f)) = \lim_{\Lambda \uparrow \mathbb{R}^n} 1/|\Lambda| \int_\Lambda d^n x f(T^{-x}\omega, 0) = \int_\Lambda \mathbf{P}(d\omega) f(\omega, 0) ,$$

which proves the result. □

To define the differential structure, we denote by ∂_μ the linear map from $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$ into $\mathcal{C}_K(\Omega \times \mathbb{R}^n)$ defined by:

$$\partial_\mu f(\omega, x) = \mathbf{i} x_\mu f(\omega, x) . \tag{2.6.4}$$

It is quite easy to check that it is a family of mutually commuting *-derivations (see Definition in Section 1.3), which generates an n-parameter group of *-automorphisms namely:

$$\rho_\zeta(f)(\omega,x) = e^{\mathbf{i}\langle\zeta,x\rangle} f(\omega,x) \, . \tag{2.6.5}$$

This group extends by continuity to the C^*-algebra $\mathcal{A}$, and therefore the ∂'s are generators. It is an elementary calculation to show that

$$\pi_\omega(\partial_\mu f) = \mathbf{i}[X_\mu, \pi_\omega(f)] \, , \tag{2.6.6}$$

where X_μ is the position operator namely the multiplication by x_μ in $L^2(\mathbb{R}^n)$.

In the periodic case (see Section 2.3), let $k \in \mathbb{R}^n \to f_k$ be the compact operator valued map associated with f in the Wannier representation (Eq. (2.3.6)), such that $L(b)f_kL(b)^{-1} = f_{k+b}$ for $b \in \Gamma^*$. Then it is easy to check that $\{\mathbf{i}[X_\mu, f]\}_k = \partial f_k/\partial k_\mu$, showing that ∂_μ is nothing but the derivation with respect to k in the Brillouin zone. In such a way that the differential structure defined by the ∂'s is a natural generalization of the differential structure on the Brillouin zone in the non commutative case.

3. The Lattice Case: Physical Examples

In Section 2 we have introduced the Non-Commutative Brillouin zone of an electron in a potential submitted to a uniform magnetic field. The relation between the Schrödinger operator in $\mathbb{R}^n$ and the lattice problem has been discussed over and over again and is known in textbooks as the 'tight binding representation'. Recently, several mathematical studies have been proposed to justify such a representation [BE88b, GM90]. One can see this representation in many ways: the usual one consists in starting from Bloch's theory for the periodic crystal. Then one reduces the study to the bands lying in the vicinity of the Fermi level, since only electrons with energy close to the Fermi energy up to thermal fluctuations, contribute to physical effects like electric conductivity. In the roughest approximation, only one band contributes, but in general a finite number of them has to be considered. We then describe the corresponding Hilbert subspace selected in this way by using a basis of Wannier functions. Then adding either a weak magnetic field, or impurities in the crystal, or even the Coulomb potential between electrons results in getting a lattice operator, with short range interaction, where the lattice sites actually label the Wannier basis. In the case of a weak magnetic field, we obtain the main approximation for the effective Hamiltonian by means of the so-called 'Peierls substitution', namely if $E(k)$ is the energy band function as a function of the quasimomentum $k \in \mathbf{B}$, we simply replace k by the operator $K = (P - eA)/\hbar$, where $P = -\mathbf{i}\hbar\nabla$ is the momentum operator, and A is the magnetic vector potential. Since $E(k)$ is periodic in k, it can be expanded in Fourier series, it is enough to define by means of a Weyl quantization, the operators $\exp\{\mathbf{i}\langle K|b\rangle\}$ where $b \in \Gamma^*$. These operators are called 'magnetic translations' [ZA64]. In Section 3.1, we will focus on this question.

Fig. 2. The square lattice in two dimensions. U_1 and U_2 are the magnetic translations, ϕ is the flux through the unit cell, $\phi_0 = h/e$ is the flux quantum, and $\alpha = \phi/\phi_0$.

3.1 Two-Dimensional Lattice Electrons in a Uniform Magnetic Field

We consider a 2D lattice, that will be identified with $\mathbb{Z}^2$. Examples of such lattices with a symmetry group are the square lattice, the triangular lattice, the honeycomb lattice (see Fig. 2). We can also consider lattices like the rectangular one or even a rhombic lattice. We assume that this lattice is imbedded in a plane of the real 3D space, and is submitted to a perpendicular uniform magnetic field B. The Hilbert space of states is identified with $l^2(\mathbb{Z}^2)$, and the most important class of operators acting on it is the set of magnetic translations denoted by $W(m)$, $m \in \mathbb{Z}^2$, acting as follows:

$$(3.1.1) \qquad W(m)\psi(x) = e^{\mathbf{i}\pi\alpha x\wedge m}\psi(x-m), \quad x \in \mathbb{Z}^2, \psi \in l^2(\mathbb{Z}^2) \,,$$

where $\alpha = \phi/\phi_0$, $\phi_0 = h/e$ is the quantum of flux, whereas ϕ is the magnetic flux through a unit cell. They are unitaries and satisfy the following commutation rule:

$$(3.1.2) \qquad W(m)W(m') = e^{\mathbf{i}\pi\alpha m\wedge m'}W(m+m') \,.$$

Fig. 3. The triangular lattice with the three magnetic translations and the two different normalized fluxes η and $\alpha - \eta$.

If we set $U_1 = W(1,0)$ and $U_2 = W(0,1)$, we get $W(m) = U_1^{m_1} U_2^{m_2} e^{\mathbf{i}\pi\alpha m_1 m_2}$ whereas they obey the following commutation rule:

$$U_1 U_2 = e^{2\mathbf{i}\pi\alpha} U_2 U_1 \; . \tag{3.1.3}$$

It turns out that all models constructed so far to represent a lattice electron in a magnetic field, are given by a Hamiltonian belonging to the C^*-algebra $\mathcal{A}_\alpha$ generated by U_1 and U_2. The simplest example is provided by the Harper model: if the energy band of the electron is given by $E(k) = 2t\{\cos a_1 k_1 + \cos a_2 k_2\}$ (square lattice) the Peierls substitution gives

$$H_S = t\{U_1 + U_1^* + U_2 + U_2^*\} \; . \tag{3.1.4}$$

After the gauge transformation $\psi(m_1, m_2) = e^{-\mathbf{i}\pi\alpha m_1 m_2}\phi(m_1, m_2 - 1)$ the corresponding Schrödinger equation $H\psi = E\psi$ leads to the so-called ‘Harper equation’ [HA55]

$$\begin{aligned} &\phi(m_1+1, m_2) + \phi(m_1-1, m_2) + e^{-2\mathbf{i}\pi\alpha m_1}\phi(m_1, m_2+1) \\ &\quad + e^{2\mathbf{i}\pi\alpha m_1}\phi(m_1, m_2-1) = E\phi(m_1, m_2) \; . \end{aligned} \tag{3.1.5}$$

Fig. 4. The honeycomb lattice with its two sublattices.

In much the same way one can describe the nearest neighbour model for a triangular or hexagonal lattice as follows: one introduces the unitary operator U_3 defined by

$$U_1 U_2 U_3 = e^{2\mathbf{i}\pi\eta}\mathbf{1} \ . \tag{3.1.6}$$

These three operators will represent the translation in the three directions of the triangular lattice, in which we suppose that the flux through a triangle with vertex up is given by η (in units of the flux quantum), whereas the flux through triangles with vertex down is $\alpha - \eta$ (see Fig. 3). The nearest neighbour Hamiltonian will simply be

$$H_T = t\{U_1 + U_1^* + U_2 + U_2^* + U_3 + U_3^*\} \ . \tag{3.1.7}$$

The honeycomb lattice can be decomposed into two sublattices Γ_A and Γ_B (see Fig. 4), so that we can decompose the Hilbert space into the direct sum of the two subspaces $\mathcal{H}_I = l^2(\Gamma_I), I = A, B$, and the Hamiltonian can be written in matrix form as

$$H_H = \begin{vmatrix} 0 & U_1 + U_2 + U_3 \\ U_1^* + U_2^* + U_3^* & 0 \end{vmatrix} \ , \tag{3.1.8}$$

Fig. 5. The Hofstadter spectrum, namely the spectrum of $H = U_1 + U_1^* + U_2 + U_2^*$ as a function of α [HO76].

where the U's refer to an underlying triangular lattice in which α has to be replaced by $\alpha/3$ and η by $\alpha/6$.

More generally one can easily take into account interactions with any sites by adding to the Hamiltonian other monomials in the U's. One can also easily describe a model in which the unit cell contains different sites, occupied by different species of ions by introducing a decomposition of the Hilbert space similar to the one of the honeycomb lattice.

The non-commutative manifold associated to $\mathcal{A}_\alpha$ is a non-commutative torus. For indeed we have two non commuting unitary generators. This algebra was introduced and studied by M. Rieffel [RI81] in 1978, and has been called the 'rotation algebra'. For indeed, it can also be constructed as the crossed product of the algebra of continuous functions on the unit circle, by the action of $\mathbb{Z}$ given by the rotation by α on the circle. Then U_1 is nothing but the map $x \in \mathbb{T} \to e^{\mathbf{i}x} \in \mathbb{C}$, whereas U_2 is the rotation by α.

The trace on $\mathcal{A}_\alpha$ is defined by analogy with the Haar measure on $\mathbb{T}^2$ by means of the Fourier expansion namely:

$$(3.1.9) \qquad \tau(W(m)) = \delta_{m,0}\ , \qquad m \in \mathbb{Z}^2\ .$$

Again this trace is also equal to the trace per unit volume in the representation of $\mathcal{A}_\alpha$ defined by Eq. (3.1.1) on $l^2(\mathbb{Z}^2)$.

On the other hand the differential structure is defined by the two commuting derivations ∂_1 and ∂_2 given by:

$$(3.1.10) \qquad \partial_\mu U_\nu = \mathbf{i}\delta_{\mu,\nu} U_\nu\ .$$

In much the same way, if $X_\mu(\mu = 1, 2)$ represents the (discrete) position operator on $l^2(\mathbb{Z}^2)$, we also get:

$$(3.1.11) \qquad \partial_\mu f = \mathbf{i}[X_\mu, f]\ .$$

Let us mention the work of A. Connes and M. Rieffel [CR87, RI90] which classifies the fiber bundles on this non commutative torus, and also the moduli space of connections.

One of the remarkable facts about the previous models, and also about most smooth self adjoint elements of this C^*-algebra (namely at least of class $\mathcal{C}^k$ for $k > 2$), is that their spectrum is a Cantor set provided α is irrational. One sees in Fig. 5 above the famous Hofstadter spectrum representing the spectrum of the Harper Hamiltonian as a function of α. One sees a remarkable fractal structure which has been investigated by many physicists [HO76, CW78, CW79, CW81, WI84, RA85, SO85, BK90] and mathematicians [BS82b, EL82b, HS87, GH89, BR90], without having been completely understood quantitatively so far. We refer the reader to [BE89] for a review.

3.2 Quasicrystals

In 1984, in a famous paper, Shechtman et al. [SB84] announced the discovery of a new type of crystalline phase in rapidly cooled alloys of Aluminium and Manganese, for which the diffraction pattern was point like but with a fivefold symmetry. This is forbidden by theorems on crystalline groups in 3 dimensions. However, if one breaks the translation invariance it is possible to get such a diffraction pattern provided the lattice of sites where the ions lie is quasiperiodic.

This discovery created an enormous amount of interest. Actually aperiodic tilings of the space had already been studied since the end of the seventies. The first example was provided by Penrose in 1979 [PN79] who gave a tiling of the plane with two kinds of tiles, having a fivefold symmetry. This was followed by several works concerning a systematic construction of such tilings in particular by Conway [GA77], de Bruijn [BR81]. Mackay [MK82] and then Mosseri and Sadoc [MS83, NS85] gave arguments for the relevance of such tilings in solid state physics. A 3-dimensional generalization of the Penrose tiling was theoretically described by Kramer and Neri [KN84]. Experimentally, the diffraction pattern created by such tilings was observed empirically by Mackay,

Fig. 6. The cut and projection method for a one dimensional quasicrystal.

whereas Levine and Steinhardt [LS84] suggested to consider quasiperiodicity as a basis to get a quasicrystal with icosahedral symmetry.

In this Section we will rather use the projection method introduced independently by Elser [EL85] and Duneau-Katz [DK85] (see Fig. 6). In this method, a quasicrystal is seen as the projection of a lattice $\mathbb{Z}^\nu$ of higher dimension contained in $\mathbb{R}^\nu$, on a subspace $E \approx \mathbb{R}^n$ irrationally oriented with respect to the canonical basis defining $\mathbb{Z}^\nu$ which will be identified with the physical space. We then call $E_\perp$ the subspace perpendicular to E. We will denote by Π and $\Pi_\perp$ the orthogonal projections onto E and $E_\perp$. Let $\varepsilon(\mu)\,(1 \leq \mu \leq \nu)$ be the unit vectors of the canonical basis of $\mathbb{R}^\nu$. We set $\varepsilon(\mu + j\nu) = (-)^j \varepsilon(\mu)$, for $j \in \mathbb{Z}$. Then we set $e(\mu) = \Pi(\varepsilon(\mu))$ and $e'(\mu) = \Pi_\perp(\varepsilon(\mu))$. Let now Δ be the (half-closed) unit cube $[0,1]^\nu$ of $\mathbb{R}^\nu, \zeta \in \mathbb{R}^\nu$, and let $\sum(\zeta)$ be the strip obtained by translating $\Delta + \zeta$ along E, namely $\sum(\zeta) = \Delta + E + \zeta$. In the sequel we shall drop the reference to ζ. We denote by $\mathcal{S} = \sum \cap \mathbb{R}^\nu$ the set of lattice points contained in the strip $\sum$. The quasilattice $\mathcal{L}$ will be obtained as the orthogonal projection of $\mathcal{S}$ on E (see Fig. 7). It is a discrete subset of E, and there is a one-to-one correspondence between $\mathcal{L}$ and $\mathcal{S}$ [DK85]. The 'acceptance zone' is the intersection $\Omega = \sum \cap E_\perp$ of $\sum$ with the perpendicular space (see Fig. 8).

Fig. 7. The octagonal lattice, obtained as a projection of $\mathbb{Z}^4$ on $\mathbb{R}^2$ in such a way as to conserve the 8-fold symmetry. The projection of the canonical basis of $\mathbb{Z}^4$ on $\mathbb{R}^2$ is shown above.

It is a polyhedron in a $(\nu - n)$-dimensional space. The projection of $\sum$ on Ω is discrete if E is rationally oriented and dense if it is irrationally oriented. If ν is high enough one can also get intermediate situations where this projection is dense in a union of polyhedra of lower dimension. A substrip $\sum'$ of $\sum$ is a subset of $\sum$ invariant by translation along E. Such a substrip will be identified with its projection $\Omega' = \Pi_\perp\{\sum'\}$ in the acceptance zone. Ω' will be called the 'acceptance zone' of $\sum'$.

To describe the quantum mechanics of a particle on $\mathcal{L}$, we first introduce the large Hilbert space $\mathcal{K} = l^2(\mathbb{Z}^\nu)$ and the physical Hilbert space $\mathcal{H} = l^2(\mathcal{L})$. $\mathcal{H}$ can be seen as a subspace of $\mathcal{K}$ if we identify $\mathcal{L}$ with $\mathcal{S}$. Denoting by χ_Ω

the characteristic function of $\mathcal{S}$ in $\mathbb{Z}^\nu$ we can identify the orthogonal projection onto $\mathcal{H}$ in $\mathcal{K}$ with the operator of multiplication by χ_Ω. More generally, if $\sum'$ is a substrip of $\sum$, the orthogonal projection in $\mathcal{K}$ onto $l^2(\sum' \cap \mathbb{Z}^\nu)$ will be denoted by $\chi_{\Omega'}$, if Ω' is the acceptance zone of $\sum'$.

Fig. 8. The acceptance zone of the octagonal lattice. The various subsets correspond to the projections of points in the octagonal lattice having a given local environment [SB90].

A free particle on the lattice $\mathbb{Z}^\nu$ is usually described by the algebra generated by the translation operators $T_\mu (1 \le \mu \le \nu)$. It is isomorphic to the algebra $\mathcal{C}(\mathbb{T}^\nu)$ of continuous functions on the ν-dimensional torus, namely the Brillouin zone in this case. For convenience, we will set $T_{\mu+\nu} = T_\mu^* = T_\mu^{-1}$, and $T_{\mu+2\nu} = T_\mu$. By analogy, we shall consider the algebra $\mathcal{Q}(\nu, E)$ generated by the restrictions $S_\mu = \chi_\Omega T_\mu \chi_\Omega$ of the translation operator to $\mathcal{H}$. For instance, the Laplacian $\Delta_{\mathcal{L}}$ on the lattice $\mathcal{L}$ is nothing but the Hamiltonian given by

$$\Delta_{\mathcal{L}} = \sum_{\mu=1}^{2\nu} (S_\mu - \mathbf{1}) \; . \tag{3.2.1}$$

The operators S_μ are no longer unitary, but they are partial isometries, and they do not commute anymore. Thus $\mathcal{Q}(\nu, E)$ describes the set of continuous functions on a non commutative compact space (the identity belongs to $\mathcal{Q}(\nu, E)$).The commutation rules are given as follows:

$$S_\mu S_\mu^* = \chi_{\Omega \cap \{\Omega + e'(\mu)\}} \, , \qquad S_\mu^* S_\mu = \chi_{\Omega \cap \{\Omega - e'(\mu)\}} \, . \tag{3.2.2}$$

Actually we get $S_\mu = \chi_{\Omega \cap \{\Omega + e'(\mu)\}} T_\mu \chi_{\Omega \cap \{\Omega - e'(\mu)\}}$. More generally, if $x \in \mathbb{Z}^\nu$, $T(x)$ will denote the translation by x in $\mathcal{K}$. Then we set $S(x) = \chi_\Omega T(x) \chi_\Omega$, in particular $S_\mu = S(\varepsilon(\mu))$. It follows immediately that:

$$S(\varepsilon(\mu_1))S(\varepsilon(\mu_2))\ldots S(\varepsilon(\mu_N)) = \chi_{\Omega(\underline{\mu})} T(\varepsilon(\mu_1) + \varepsilon(\mu_2) + \ldots + \varepsilon(\mu_N)) \tag{3.2.3}$$

where $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$, and

$$\Omega(\underline{\mu}) = \Omega \cap \{\Omega + e'(\mu_1)\} \cap \ldots \cap \{\Omega + e'(\mu_1) + \ldots + e'(\mu_N)\} \, . \tag{3.2.4}$$

This latter set is nothing but the acceptance zone of the set of points x in $\sum$ such that the path $\gamma(x) = (x_0, \ldots, x_N)$ is entirely included in $\sum$, whenever $x_0 = x$, $x_k = x_{k-1} + \varepsilon(\mu_k)$. In particular, any element f of the algebra $\mathcal{Q}(\nu, E)$ can be approximated in norm, by a sequence of 'trigonometric polynomials' namely finite sums of the form

$$f = \sum_{\underline{\mu}} c(\underline{\mu}) \chi_{\Omega(\underline{\mu})} T(\varepsilon(\mu_1) + \varepsilon(\mu_2) + \ldots + \varepsilon(\mu_N)) \, , \tag{3.2.5}$$

where the $c(\underline{\mu})$'s are complex coefficients.

For $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_N)$ such that $1 \leq \mu_i \leq 2\nu$, we set $x(\underline{\mu}) = \varepsilon(\mu_1) + \varepsilon(\mu_2) + \ldots + \varepsilon(\mu_N)$, and for $a \in \mathbb{Z}^\nu$, we denote by $I(a)$ the set of $\underline{\mu}$'s such that $x(\underline{\mu}) = a$. A linear form on $\mathcal{Q}(\nu, E)$ is then defined by linearity as follows:

$$\tau\{\chi_{\Omega_{\underline{\mu}}} T(x(\underline{\mu}))\} = \delta_{x(\underline{\mu}),0} |\Omega(\underline{\mu})| \, , \tag{3.2.6}$$

where $|.|$ denotes the Lebesgue measure in $E_\perp$. Since the Lebesgue measure is translation invariant, one can easily check that if f, g are two trigonometric polynomials, $\tau(fg) = \tau(gf)$ and that $\tau(ff^*) > 0$ as soon as f is non zero in $\mathcal{Q}(\nu, E)$. At last, $\tau(\mathbf{1}) = 1$ and therefore τ extends uniquely to a normalized trace on $\mathcal{Q}(\nu, E)$.

As in the previous Section, one defines the differential structure on $\mathcal{Q}(\nu, E)$ by means of the group of *-automorphisms η_a ($a \in \mathbb{T}^\nu$) defined by linearity by:

$$\eta_a\{\chi_{\Omega(\underline{\mu})} S(x(\underline{\mu}))\} = e^{\mathbf{i}\langle a | x(\underline{\mu})\rangle} \chi_{\Omega(\underline{\mu})} S(x(\underline{\mu})) \, , \tag{3.2.7}$$

and the derivation ∂_μ is given by:

$$\partial_\mu f = d\eta_a(f)/da_\mu|_{a=0} \, . \tag{3.2.8}$$

A special case concerns the 1D quasicrystals. Then $\nu = 2$, and E is a line in $\mathbb{R}^2$ (see Fig. 6). Let ω be the slope of E and we set $\alpha = \omega/(\omega + 1)$. It is sufficient to choose ζ of the form $\zeta = (0, n_0 + \eta)$ with $n_0 \in \mathbb{Z}$, and $-1 \leq \eta < \omega$. The lattice $\mathcal{L}$ is then a sublattice of E, and it is in one-to-one correspondence with $\mathbb{Z}$. More precisely, if $x = (m, n) \in \mathcal{S}$, one set, $l(x) = m + n - n_0$. The map $x \in \mathcal{S} \to l(x) \in \mathbb{Z}$, is one-to-one, with inverse given by $n = n_0 + 1 + [l\alpha - \theta]$, $m = l - 1 - [l\alpha - \theta]$, provided $\theta = \alpha - (1 - \alpha)\eta \in [0, 1]$ (here, $[r]$ denotes the integer part of r). In particular, $\mathcal{H}$ is naturally isomorphic to $l^2(\mathbb{Z})$.

Fig. 9. The acceptance zone for the one dimensional quasilattice shown in Fig. 6. The subintervals correspond to the projections of the lattice points with indicated environment, c stands for horizontal bonds whereas s stands for the vertical ones.

Then setting $T = S_1 + S_2$, T is nothing but the translation by one in $\mathcal{H}$. On the other hand, we can easily check that $\chi_\alpha = \chi_{\Omega \cap \{\Omega - e'(2)\}} = S_2^* S_2$ (see Fig. 9) is identified with the operator $\chi_{\alpha,\theta}$ of multiplication by $\chi_{[1-\alpha,1)}(l\alpha - \theta)$ in $\mathcal{H}$ where $\chi_{[1-\alpha,1)}$ is the characteristic function of $[1-\alpha, 1)$ in the unit circle. Hence, the algebra $\mathcal{Q}\alpha$ is generated by T and χ_α. By the previous remark, the Abelian C^*-algebra generated by the family $\{\chi_{\alpha,n} = T^n \chi_\alpha T^{-n}\}$ is naturally isomorphic to the algebra $\mathcal{C}_\alpha$ generated by the characteristic functions $\chi_{[1+(n-1)\alpha, 1+n\alpha)}$ of the interval $[1 + (n-1)\alpha, 1 + n\alpha)$ of the unit circle. Thus $\mathcal{Q}_\alpha = \mathcal{Q}(2, E)$ appears as the cross product of $\mathcal{C}_\alpha$ by the rotation by α on the unit circle. This algebra was already studied by J. Cuntz and shown to be isomorphic to the UHF algebra constructed by Pimsner and Voiculescu in the study of the rotation algebra. In particular, when α is irrational, $\mathcal{C}_\alpha$ contains all continuous functions on the unit circle, and therefore, $\mathcal{Q}_\alpha$ contains the irrational rotation algebra $\mathcal{A}_\alpha$.

It will be convenient to define a 'universal' C^*-algebra formally given by $\mathcal{Q} = \cup_{\alpha \in [0,1]} \mathcal{Q}_\alpha$. In order to do so, let χ_n be the map on $\mathbb{T}^2$ defined by $\chi_n(\alpha, x) = \chi_{[1+(n-1)\alpha, 1+n\alpha)}(x)$. We shall denote by $\mathcal{C}$ the Abelian C^*-algebra generated by the χ_n's. If f is the diffeomorphism of $\mathbb{T}^2$ defined by $f(\alpha, x) = (\alpha, x - \alpha)$, then $\mathcal{Q}$ appears as the cross product of $\mathcal{C}$ by f. For an irrational, the set $\mathcal{I}_\alpha$ of elements of $\mathcal{Q}$ vanishing at α is a closed two-sided ideal. Then, $\mathcal{Q}_\alpha$ can be identified with the quotient $\mathcal{Q}/\mathcal{I}_\alpha$. The quotient map will be denoted by ρ_α.

One model has been the focus of attention in this latter case, namely the 'Kohmoto model' [KK83, OP83, KO84, OK85, CA86, KL86, LP86, SU87, BI89, LE89, SU89, BI90], the Hamiltonian of which acting on $l^2(\mathbb{Z})$ as follows:

$$H_{(\alpha,\theta)} = T + T^* + V\chi_{(\alpha,\theta)} = S_1 + S_2 + S_1^* + S_2^* + VS_2^*S_2 \,, \tag{3.2.9}$$

where $V \geq 0$ is a coupling constant. It describes a 1D Schrödinger operator with a potential taking two values in a quasiperiodic way with quasiperiods 1 and α. This operator and operators related to it have been also proposed to describe the behavior of 1D electrons in a Charge Density Wave (CDW) once the linear chain is submitted to a Peierls instability. The energy spectrum of the Kohmoto Hamiltonian has been computed first by Ostlund and Kim [OK85]. It has a beautiful fractal structure which can be investigated by means of a renormalization group analysis [KK83, OP83, KO84, SO85, KL86, CA86, LP86, SU87, BE89, BI89, LE89, SU89]. More precisely one has

Proposition 3.2.1 [BI90]. *Let H be a self adjoint element in $\mathcal{Q}$. Then the gap boundaries of the spectrum of $\rho_\alpha(H)$ are continuous with respect to α at any irrational number.*

Proposition 3.2.2 [BI89]. *For any number α in $[0,1]$, the spectrum $\sum(\alpha)$ of the operator $H_{(\alpha,\theta)}$ acting on $l^2(\mathbb{Z})$ and given by Eq. (3.2.9), is independent of $\theta \in [0,1)$.*

Proposition 3.2.3 [BI89]. *For any irrational number α in $[0,1]$, and any θ in [0,1), the operator $H_{(\alpha,\theta)}$ acting on $l^2(\mathbb{Z})$ and given by Eq. (3.2.9), has a Cantor spectrum of zero Lebesgue measure, and its spectral measure is singular continuous.* (See Fig. 10)

In dimension $D > 1$, the character of the spectrum is not known rigorously yet. However, several theoretical works have been performed in 2D and they indicate that two regimes can be obtained, depending upon the values of the Fourier coefficients of the Hamiltonian under consideration: an 'insulator' regime, for which the spectrum looks like a Cantor set, presumably of zero Lebesgue measure, and a 'metallic' regime for which the spectrum has very few gaps and a continuous (but fractal) density of states.

The oldest works [see for reviews BE89 and SO87, KS86, ON86, TF86, HK87], consist in restricting the lattice to a finite box Λ, and to diagonalize numerically by brute force the finite dimensional matrix obtained by restricting the Hamiltonian to Λ. The main difficulty with this method comes from surface states. For small boxes (with about 2000 sites), it is estimated that surface states may contribute up to 20 % of the spectrum. It is then necessary to eliminate the surface state contribution from the spectrum to get a good approximation of the infinite volume limit. To solve this difficulty one may approximate the lattice $\mathcal{L}$ by a periodic lattice with large period, and use Bloch's theory to compute the spectrum numerically. This can be done by replacing the space E by a 'rational' one close to it [MO88, JD88].

More recently a solvable model called the 'labyrinth model', has been found by C. Sire [SI89], using a Cartesian product of two 1D chains. It exhibits a

Fig. 10. The spectrum of the Kohmoto Hamiltonian (eq. 3.2.9) as a function of α [OK85].

'metal-insulator' transition. In the insulating regime this Hamiltonian admits a Cantor spectrum of zero Lebesgue measure.

At last a Renormalization Group approach has been proposed to compute numerically the spectrum in the infinite volume limit [SB90]. Even though it is in principle rigorous, the practical calculation on the computer requires approximations. The cases of Penrose and octagonal lattices in 2D with nearest neighbour interactions have been performed, and also give a Cantor spectrum in the insulating regime [BS 91].

3.3 Superlattices and Automatic Sequences

Superlattices are made of two species of doped semiconductors piled in on top of each other to produce a 1D chain of quantum wells. A qualitative model to describe electronic wave functions is given by a 1D discrete Schrödinger operator of the form

$$H\psi(n) = \psi(n+1) + \psi(n-1) + \lambda V(n)\psi(n) \ , \psi \in l^2(\mathbb{Z}) \ , \tag{3.3.1}$$

where $V(n)$ represents the effect of each quantum well, and λ is a positive parameter which will play the role of a coupling constant. An interesting situation consists in building up the superlattice by means of a deterministic rule. The simplest one is given by a periodic array, in which we alternate the two species in a periodic way. But in general the rule will be aperiodic. One widely studied example is the Fibonacci sequence: given two letters a and b, one substitutes the word ab to a, and the word a to b. Starting from a one generates an infinite sequence $abaababaabaababa\ldots$ in which the frequency of a's is given by the golden mean $(\sqrt{5}-1)/2$. Another example (the oldest one actually) of such a substitution is the Thue-Morse sequence [TU06, MO21, QU87]. It is obtained through the substitution $a \to ab, b \to ba$, and gives $abbabaabbaababba\ldots$ Also if $S_2(n)$ is the sum of the digits of n in a dyadic expansion, then the sequence $(u(n))_{n\geq 0}$ defined by setting $u(n) = a$ if $S_2(n)$ is even, and $u(n) = b$ if $S_2(n)$ is odd, is the Thue-Morse sequence again.

To define a substitution, we start with the data of a finite set A called an 'alphabet'. Elements of A are called 'letters'. Elements of the Cartesian product A^k are called 'words of length k'. The disjoint union $A^* = \cup_{k\geq 1} A^k$, is the set of words. The length of a word w is denoted by $|w|$. A substitution is a map ζ from A to A^*, which associates to each letter $a \in A$ a word $\zeta(a)$. We extend ζ into a map from A^* into A^* by concatenation namely $\zeta(a_0a_1\ldots a_n) = \zeta(a_0)\zeta(a_1)\ldots\zeta(a_n)$. In particular one can define the iterates ζ^n of ζ. We shall assume the following [see QU87 p. 89]:

(S1) $\lim_{n\to\infty} |\zeta^n(a)| = +\infty$ for every $a \in A$.
(S2) there is a letter $0 \in A$ such that $\zeta(0)$ begins by 0.
(S3) for every $a \in A$, there is $k \geq 0$ such that $\zeta^k(a)$ contains 0.

The first condition insures the existence of an infinite sequence. The second is actually always fulfilled [QU87 p. 88], so that the 'substitution sequence' $u = \lim_{n\to\infty} \zeta^n(0)$ exists in the infinite product space $A^{\mathbb{N}}$ such that $\zeta(u) = u$. Let T be the shift in $A^{\mathbb{N}}$, namely $(Tw)_k = w_{k+1}$, and let Ω be the compact subspace of $A^{\mathbb{N}}$ given as the closure of the family $O(u) = \{T^n u;\ n \geq 0\}$. Then (Ω, T) is a topological dynamical system, and the condition (S3) insures that it is minimal, namely that every orbit is dense [QU87, Theorem V.2]. By P. Michel's theorem [QU87, Theorem V.13], it is therefore uniquely ergodic namely it admits a unique ergodic invariant probability measure that will be denoted by $\mathbf{P}$.

In much the same way, defining $u'_{-k} = u'_{k-1} = u_{k-1}$, for $k \geq 1$, one gets a two-sided sequence in $A^{\mathbb{Z}}$. Ω' is the closure of the orbit of u' under the two-sided shift T' defined similarly on $A^{\mathbb{Z}}$. It is then easy to check that the dynamical system (Ω', T') is still uniquely ergodic. Moreover, the unique invariant probability measure $\mathbf{P}'$ is uniquely determined by $\mathbf{P}$. Thus each non empty open set of Ω' has a positive measure, implying that (Ω', T') is minimal (see Proposition IV.5 in [QU87]).

A sequence $\{V(n); n \in \mathbb{Z}\}$ of real numbers is a 'substitution potential' whenever there is a substitution fulfilling conditions (S1–3) and a mapping $\mathcal{V}$ from A to $\mathbb{R}$ such that

$$V(n) = \mathcal{V}(u_n) = V(-n-1)\,, \qquad \text{for } n \geq 0\,. \tag{3.3.2}$$

It will be convenient to extend $\mathcal{V}$ to Ω' by setting $v(w) = \mathcal{V}(w_0)$, for $w \in \Omega'$. Then for $w \in \Omega$, we define the potential V_w by:

$$V_w(n) \equiv v(T^n w)\,, \Rightarrow V = V_u\,. \tag{3.3.3}$$

Following the method described in Section 2, we deduce that the Hamiltonian given in Eq. (3.3.1) belongs to the C^*-Algebra $C^*(\Omega', T')$ which is the crossed-product of the space $\mathcal{C}(\Omega')$ by the action of $\mathbb{Z}$ defined by T'. An element F in this algebra is described by a kernel $F(w,n)$, which is for each $n \in \mathbb{Z}$, a continuous function of w on Ω'. For each $w \in \Omega'$ one gets a representation $(\pi_w, \mathcal{H}_w)$ of this algebra, such that $\mathcal{H}_w = l^2(\mathbb{Z})$, and if U denotes the translation by 1 in $l^2(\mathbb{Z})$, $f \in \mathcal{C}(\Omega')$:

$$\pi_w(T') = U^* \qquad \pi_w(f) = \text{Multiplication by } f(T'^n w)\,. \tag{3.3.4}$$

This is equivalent to saying that the matrix elements of $\pi_w(F)$ are:

$$\langle n|\pi_w(F)|m\rangle = F(T'^n w, m-n)\,. \tag{3.3.5}$$

We immediately get the 'covariance property' $U^{-1}\pi_w(F)U = \pi_{T'w}(F)$. Since (Ω', T') is minimal, it follows that the spectrum of $\pi_w(F)$ (as a set) does not depend on $w \in \Omega'$.

A trace τ on this algebra is defined as the trace per unit volume. Using Birkhoff's theorem [HA56], and Eq. (3.3.5), it can easily be shown that it is given as follows:

$$\tau(F) \equiv \lim_{N\to\infty} N^{-1} \sum_{0\leq n<N} \langle n|\pi_w(F)|n\rangle = \int_{\Omega'} \mathbf{P}'(dw) F(w,0)\,. \tag{3.3.6}$$

In much the same way as in the previous Sections, a differential structure is defined by one derivation ∂ such that:

$$\partial F(w,n) \equiv \mathbf{i} n F(w,n)\,, \quad \Rightarrow \pi_w(\partial F) = -\mathbf{i}[X, \pi_w(F)]\,, \tag{3.3.7}$$

where X is the position operator namely $X\psi(n) = n\psi(n)$ for $\psi \in l^2(\mathbb{Z}), n \in \mathbb{Z}$.

Very few results have been obtained yet on the spectral properties of the Hamiltonian H above. The Fibonacci sequence is actually a special case of a 1D quasicrystal, and gives rise to Kohmoto's model.

J.M. Luck [LU90], gave a series of theoretical arguments to show that several non almost-periodic examples of substitutions give rise to a Cantor spectrum. The width of the gaps follows a scaling behaviour in λ as $\lambda \to 0$. This has been rigorously proved in two cases [AA86, AP88, BE90a, DP90, BB90]: the Thue-Morse (given by $a \to ab$, $b \to ba$) and the period-doubling sequence (given by $a \to ab$, $b \to aa$). More precisely one has:

Proposition 3.3.1. *Let H be given by* (3.3.1) *above, where V is defined either by the Thue-Morse or by the period doubling sequence. Then H has a Cantor spectrum of zero Lebesgue measure. Its spectral measure is singular continuous. Moreover as $\lambda \approx 0$, the gap widths W obey to the following asymptotics:*

(i) for the Thue-Morse sequence: $W \approx \mathrm{const.}\, \lambda^{\sigma} b(\lambda)$, with $\sigma = Ln4/Ln3$, and b is a continuous function bounded away from zero and such that $b(\lambda) = b(\lambda/3)$;

(ii) for the period doubling sequence, we get two families of gaps with

– either $W \approx \mathrm{const.}\, \lambda$,

– or $W \approx \mathrm{const.}\, \exp\{3Ln2/\lambda\}\lambda^{Ln2}$.

4. K-Theory and Gap Labelling

This Section is devoted to the exposition of the general theory of gap labelling. The operator side of the formula is given by a natural countable group associated to each algebra, the group K_0 [AT67, KA78, CU82, BL86] which serves to label gaps of an aperiodic Hamiltonian in a stable way under perturbations. In Section 4.1, we introduce the Integrated Density of States (IDS) [CF87], using the trace per unit volume, to count the number of states below some energy level. In Sections 4.2 and 4.3, we will give the general properties of the K_0-group, generalizing in this way the homotopy invariant aspect of the index of a Fredholm operator. The wave aspect of the gap labelling will be investigated in the next Sections. Most of the content of this Section can be found in [BE86], already. We will postpone the study of examples to Section 5.

4.1 Integrated Density of States and Shubin's Formula

The operators we have studied up to now model the motion of a single particle in a solid with infinitely many fixed ions producing the potential. This is the 'one-body' approximation. However in a solid with infinitely many ions, there is an infinite number of electrons. This is not a problem as long as we can neglect the electron-electron interaction, and also the deformation of the ionic potential by the electrons themselves (electron-phonon interaction). For then the total energy is simply the sum of the individual single electron energies. If we take into account the Pauli exclusion principle, electrons being fermions, the ground state energy of the electron gas will simply be given by adding up all filled individual energy states, each of them being occupied by one electron only. The main technical difficulty comes from having to distribute an infinite number of fermions on a continuum of energy levels. The key fact about it comes from the homogeneity of the crystal we are investigating: electrons in far apart areas tend to ignore each other; moreover, spectral properties will tend to be almost translation invariant, at least if we compare large regions. Therefore it will be enough to consider the number of available energy levels per unit volume. This is how the integrated density of states IDS is defined.

Let $\mathcal{H}$ be the Hilbert space $L^2(\mathbb{R}^n)$, and let us consider the Schrödinger operator given by (2.4.5) $H = (1/2m)\sum_{\mu\in[1,n]}(P_\mu - eA_\mu)^2 + V = H_B + V$, where A is the vector potential associated to a uniform magnetic field B, and V is a real measurable essentially bounded function on $\mathbb{R}^n$. For any rectangular box Λ, we denote by H_Λ the restriction of H to Λ with some boundary conditions (for instance, Dirichlet or periodic boundary conditions). Then H_Λ has a discrete spectrum bounded from below. Let $N_\Lambda(E)$ be the number of eigenvalues of H_Λ smaller than or equal to E. Because of the homogeneity condition, if we translate Λ by ζ, we expect $N_{\Lambda+\zeta}(E) = N_\Lambda(E) + o(|\Lambda|)$ (where $|.|$ denotes the Lebesgue measure), and if Λ and Λ' are two large non intersecting boxes, $N_{\Lambda\cup\Lambda'}(E) = N_\Lambda(E) + N_{\Lambda'}(E) + o(|\Lambda\cup\Lambda'|)$. In other words, we expect $N_\Lambda(E)$ to increase with Λ as $|\Lambda|$. So we define the IDS as:

$$\mathcal{N}(E) = \lim_{\Lambda\uparrow\mathbb{R}^n} N_\Lambda(E)/|\Lambda| \,. \tag{4.1.1}$$

Here the limit $\Lambda \uparrow \mathbb{R}^n$ is understood in the following sense: we consider a 'Følner sequence' [GR69] namely a sequence Λ_k of bounded open sets, with piecewise smooth boundary, with union $\mathbb{R}^n$, and such that $|\Lambda_k \Delta(\Lambda_k + a)|/|\Lambda_k|$ converges to zero as $k \to \infty$ (Δ is the symmetric difference of sets).

The very same definition can apply to the case of a Hamiltonian on a lattice (Section 2.4, Eq. (2.4.7)) and we can also extend this definition to the case where the lattice is a homogeneous sublattice of $\mathbb{R}^n$. We will leave to the reader the obvious extension to this case.

One remarks that $N_\Lambda(E)$ is also the trace of the eigenprojection $\chi(H_\Lambda \leq E)$ onto eigenstates of H_Λ with energy less than or equal to E. So that the IDS appears as

$$\mathcal{N}(E) = \lim_{\Lambda\uparrow\mathbb{R}^n} \mathrm{Tr}\{\chi(H_\Lambda \leq E)\}/|\Lambda| \,. \tag{4.1.2}$$

The first rigorous work on the IDS goes back to Benderskii-Pastur [BP70], who proved the existence of the limit for a one-dimensional Schrödinger operator on a lattice with a random potential. Then the existence and smoothness properties of the derivative $d\mathcal{N}$ as a Stieljes-Lebesgue measure, were proved by different methods with an increasing degree of generality for the Schrödinger operator with random potential by Pastur [PA73], Nakao [NA77], Kirsch and Martinelli [KM82]. The algebraic approach goes back to the work of Shubin [SH79] inspired by the Index theory of Coburn, Moyer and Singer [CM73] on uniformly elliptic operators with almost periodic coefficients. The extension to more general coefficients is elementary and has been given in the discrete case by the author in [BE86], and for the continuum case it is given below [BL85].

The formula (4.1.2) is very reminiscent to the formula defining the trace per unit volume of the eigenprojector $\chi(H \leq E)$ of the infinite volume limit. Notice that this projector does not belong in general to the C^*-algebra $\mathcal{A}$ of H but to the Von Neumann algebra $L^\infty(\mathcal{A}, \tau)$ associated to the trace per unit volume by means of the GNS construction.

Definition : Shubin's Formula.

We say that H obeys to *Shubin's formula* whenever the IDS satisfies:

(4.1.3)
$$\mathcal{N}(E) = \tau\{\chi(H \leq E)\} ,$$

where τ is the trace per unit volume in the C^*-algebra of H.

To establish Shubin's formula, we have to compare $\mathrm{Tr}\{\chi(H_\Lambda \leq E)\}/|\Lambda|$ with $\mathrm{Tr}\{\chi_\Lambda \chi(H \leq E)\}/|\Lambda|$ as $\Lambda \uparrow \mathbb{R}^D$, where χ_Λ is the characteristic function of Λ , and to show that they are equal. The first result in this respect was provided by Shubin [SH79] whenever H is a uniformly elliptic partial differential operator with almost periodic coefficients.

Actually, Shubin's proof can be extended to a more general situation. Let us formulate the result. Let Ω be a compact space with a $\mathbb{R}^D$-action by a group $\{T^a;\, a \in \mathbb{R}^D\}$ of homeomorphisms. We assume that there is $\omega_0 \in \Omega$ the orbit of which being dense in Ω. Let $U^\infty(\Omega, \mathbb{R}^n)$ be the space of smooth functions f on $\mathbb{R}^n$, with bounded derivatives of every order, such that there exists a continuous function F on Ω for which $f(x) = F(T^{-x}\omega_0)$. We set $f_\omega(x) = F(T^{-x}\omega)$. Let H_ω be a uniformly elliptic formally self-adjoint operator of the form:

(4.1.4)
$$H_\omega = \sum_{|\alpha| \leq m} h_\omega^{(\alpha)}(x) D^\alpha ,$$

where $h^{(\alpha)} \in U^\infty(\Omega, \mathbb{R}^n)$, $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, whereas $D^\alpha = \prod_{1 \leq \mu \leq n} \{-\mathbf{i}\partial_\mu\}^{\alpha_\mu}$. By uniformly elliptic we mean that the principal symbol $h_\omega^{(m)}$ satisfies:

(4.1.5)
$$h_\omega^{(m)}(x, \zeta) = \sum_{|\alpha| = m} h_\omega^{(\alpha)}(x) \zeta^\alpha \geq \varepsilon |\zeta|^m , \quad \varepsilon > 0 \,; x, \zeta \in \mathbb{R}^n .$$

By formally self-adjoint, we mean that for any pair ϕ, ψ of functions in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ we have:

(4.1.6)
$$\langle \phi | H_\omega \psi \rangle = \langle H_\omega \phi | \psi \rangle .$$

Then it is well-known that H_ω admits a unique self-adjoint extension, still denoted by H_ω and that $\mathcal{S}(\mathbb{R}^n)$ is a core [SH79, HÖ85]. Moreover, it satisfies the covariance condition

$$U(a)H_\omega U(a)^* = H_{T^a\omega} \ . \tag{4.1.7}$$

Now for any Λ in the Følner sequence defined above, we consider the operator $H_{\omega\Lambda}$ defined by (4.1.4), with domain $\mathcal{D}(\Lambda)$ given by the space of functions ψ in the Sobolev space $\mathcal{H}^m(\Lambda)$ satisfying the boundary conditions $B_j\psi|_{\partial\Lambda} = 0\,(1 \leq j \leq m/2)$; we assume the operators B_j to have order not exceeding $m-1$, and to be self-adjoint. It is known that $H_{\omega,\Lambda}$ is self-adjoint and has a discrete spectrum, bounded from below. Therefore $N_\Lambda(E)$ exists and is finite. The main result of Shubin [SH79, Theorem 2.1] can be rephrased as follows:

Theorem 8. *If H_ω is a uniformly elliptic self-adjoint operator with coefficients in $U^\infty(\Omega, \mathbb{R}^n)$, then:*

(i) there is $r(z) \in C^*(\Omega \times \mathbb{R}^n, B = 0)$ such that $\pi_\omega(r(z)) = \{z - H_\omega\}^{-1}$, for every complex z in the resolvent set of H_ω;

(ii) Shubin's formula holds.

Sketch of the proof. Let us only indicate the punch line of the proof. First of all, standard results show that the elementary solution of the heat equation $\partial u/\partial t = -H_\omega u$ admits a regular kernel $G_\omega(t; x, y)$ which depends smoothly upon x and y, and satisfies for any multi-indices α, β, γ the following type of estimate:

$$|\partial_x^\alpha \partial_y^\beta \partial_t^\gamma G_\omega(t; x, y)| \leq C t^{-(n+|\alpha|+|\beta|+m\gamma)/m} \exp\{-\varepsilon(|x-y|^m/t)^{1/m}\} \ , \tag{4.1.8}$$

where $0 < t < T$, and the constant $C > 0$ depends only on α, β, γ and T. The covariance condition shows that if we set $G(t; \omega, x) = G_\omega(t; 0, x)$, then $G_\omega(t; x, y) = G(t; T^{-x}\omega, y - x)$. Thus $G(t; \omega, x)$ defines for each $t > 0$, an element $g(t)$ of $L^{\infty,1}(\Omega \times \mathbb{R}^n; B = 0)$, showing that $e^{-tH_\omega} = \pi_\omega(g(t))$. It follows from the covariance condition, the strong continuity of $\pi_\omega(g(t))$ with respect to ω, and the density of the orbit of ω_0 in Ω, that the spectrum of H_ω is contained in the spectrum of H. The resolvent is given by $(H_\omega - z)^{-1} = \int_{[0,\infty]} \mathrm{dt} \exp\{-t(H_\omega - z)\}$ for $\mathrm{Re}(z)$ sufficiently negative, showing that $r(z)$ exists.

On the other hand, $\mathcal{N}(E)$ is a non decreasing non negative function of E. Its derivative exists as a Stieljes-Lebesgue measure. It is thus sufficient to show that its Laplace transform $N(t) = \int \mathcal{N}(dE)e^{-Et}$ is equal to $\tau\{\exp(-tH)\}$. If we call $f_{\omega,\Lambda}$ the kernel of the operator $\exp\{-tH_\omega\} - \exp\{-tH_{\omega,\Lambda}\}$, one has the following estimate:

$$|\partial_x^\alpha \partial_t^\gamma f_{\omega,\Lambda}(t; x, y)| \leq C t^{-(n+|\alpha|+m\gamma)/m} \exp\{-\varepsilon[(|x-y| + \mathrm{dist}(y, \partial\Lambda))^m/t]^{1/m}\} \ , \tag{4.1.9}$$

showing that $\lim_{\Lambda\uparrow\mathbb{R}^n} \mathrm{Tr}\{f_{\omega,\Lambda}\}/|\Lambda| = 0$, and therefore the Shubin formula holds. $\square$

In the discrete case the situation is much simpler, from a technical point of view. Let G be a (non necessarily Abelian) countable discrete amenable group [GR69]. By amenable we mean that there is a Følner sequence, namely an increasing sequence $(\Lambda_n)_{n\geq 0}$ of finite subsets of G, with union equal to G, and such that $\lim_{n\to\infty} |\Lambda_n \Delta(a^{-1}\Lambda_n)|/|\Lambda_n| = 0$. In practice, G will be $\mathbb{Z}^n$, but it may perfectly contain non Abelian parts, corresponding to discrete symmetries. Let Ω be a compact space endowed with a G-action by homeomorphisms, and let $\mathbf{P}$ be a G-invariant ergodic probability measure. The set $\Gamma = \Omega \times G$ is a topological groupoid if we set $r(\omega, x) = \omega, s(\omega, x) = x^{-1}\omega, (\omega, y) \circ (y^{-1}\omega, y^{-1}x) = (\omega, x), (\omega, x)^{-1} = (x^{-1}\omega, x^{-1})$. Let $\Gamma^{(2)}$ denote the set of pairs (γ_1, γ_2) with $r(\gamma_2) = s(\gamma_1)$. A unitary cocycle is a mapping $\delta : \Gamma^{(2)} \to S_1$, where S_1 is the unit circle, such that $\delta(\gamma_1, \gamma_2)\delta(\gamma_1 \circ \gamma_2, \gamma_3) = \delta(\gamma_1, \gamma_2 \circ \gamma_3)\delta(\gamma_2, \gamma_3)$. We can then construct the C^*-algebra $C^*(\Omega, G, \delta)$ by means of the method of Section 2.5 [RE80], where now G replaces $\mathbb{R}^n$, the Haar measure on G (here the discrete sum) replacing Lebesgue's measure, and δ replacing $e^{\mathrm{i}\pi(e/h)B.x\wedge y}$. We also get a family π_ω of representations on $l^2(G)$ in a similar way. Then [BE86 Appendix]:

Theorem 9. *Let H be a self-adjoint element of $C^*(\Omega, G, \delta)$. Then the Shubin formula holds for H.*

In what follows, the spectrum of H is understood as the spectrum in the C^*-algebra, namely

$$\mathrm{Sp}(H) = \bigcup_{\omega\in\Omega} \sigma\{\pi_\omega(H)\} . \tag{4.1.10}$$

The main properties of the IDS are listed below. From Shubin's formula it is easy to get:

Proposition 4.1.1. *Let H be a homogeneous self-adjoint Hamiltonian, and let $\mathcal{A}$ be the C^*-algebra it generates by translation. Let τ be a translation invariant trace, for which H obeys to Shubin's formula.*

Then its IDS *is a non negative, non decreasing function of E, which is constant on each gap of* $\mathrm{Sp}(H)$.

The IDS also depends upon the choice of a translation invariant ergodic probability measure $\mathbf{P}$ on the hull of the Hamiltonian. We say that a trace τ on a C^*-algebra $\mathcal{A}$ is faithful if any f in $L^1(\mathcal{A}, \tau) \cap \mathcal{A}$ with $\tau(f^*f) = 0$ necessarily vanishes. Then

Proposition 4.1.2. *Let H be as in Proposition* 4.1.1. *If τ is faithful, the spectrum of H coincides with the set of points $E \in \mathbb{R}$ in the vicinity of which the IDS is* *not constant.*

Proof. Let E be a point in the spectrum of H (the spectrum is the set of $E \in \mathbb{R}$ for which $(E\mathbf{1}-H)$ has no inverse in $\mathcal{A}$). Then for any $\delta > 0$ the eigenprojection $\chi(E-\delta < H \leq E+\delta)$ is a non zero positive element of the Von Neumann algebra $L^\infty(\mathcal{A},\tau)$. Since τ is faithful, we have

$$\mathcal{N}(E+\delta) - \mathcal{N}(E-\delta) = \tau(\chi(E-\delta < H \leq E+\delta)) > 0 \; ,$$

proving the result. □

Remark. Suppose we consider the operator $H = -\Delta + V$ on $\mathbb{R}^n$ where V decays at infinity. Then the hull of H is the one-point compactification of $\mathbb{R}^n$. The only translation invariant ergodic probability measure on the hull $\Omega = \mathbb{R}^n \cup \{\infty\}$ is the Dirac measure at ∞. Therefore, the trace per unit volume cannot be faithful. In particular the IDS does not take into account the discrete spectrum of H, since the density of such eigenvalues is zero.

Another example of such phenomena is provided by strong limits when $\alpha \to p/q$ of the Kohmoto model (see Section 3.2), which may exhibit simple isolated eigenvalues in the gaps of the spectrum of the periodic model [BI90].

Proposition 4.1.3. *With the assumption of Proposition* 4.1.1, *any discontinuity point of the IDS is an eigenvalue of H with an infinite multiplicity.*

Proof. Let E be such a discontinuity point. It means that there is $\varepsilon > 0$, such that for every $\delta > 0$, $\mathcal{N}(E+\delta) - \mathcal{N}(E-\delta) = \tau(\chi(E-\delta < H \leq E+\delta)) \geq \varepsilon$. Taking the limit $\delta \to 0$, we get a non zero eigenprojection $\chi(H = E)$ corresponding to the eigenvalue E. Since $\tau(\chi(H=E)) \geq \varepsilon > 0$, it follows that the multiplicity per unit volume is bigger than or equal to ε, and therefore the multiplicity of E is infinite. □

Remark. Many examples of physical models have given such eigenvalues. For instance, the discrete Laplace operator on a Sierpinski lattice [RA84] has a pure-point spectrum containing an infinite family of isolated eigenvalues of infinite multiplicity, the other eigenvalues being limit points and having infinite multiplicity as well. So that the IDS has no continuity point on the spectrum. Another example was provided by Kohmoto et al.[KS86], namely the discrete Laplace operator on a Penrose lattice in 2D, for which $E = 0$ is an isolated eigenvalue with a non zero multiplicity per unit area.

Actually one has the following result proved by Craig and Simon [CS83] and in a very elegant way, by Delyon and Souillard [DS84]:

Proposition 4.1.4. *Let $H = \Delta + V$ be self adjoint, bounded and homogeneous on $\mathbb{Z}^n$. Then its IDS is continuous.*

Sketch of the proof. Taking Λ a finite parallelepipedic box, the eigenvalues of H_Λ have multiplicity not bigger than $O(|\partial\Lambda|)$. This is due to the special geometry of the hypercubic lattice $\mathbb{Z}^n$. Therefore the multiplicity per unit volume of any eigenvalue must vanish. □

Remark. 1) Such a general argument is not available yet in the continuum. However with reasonable assumptions on the potential one may expect such a result.

2) Craig and Simon have actually proved that the IDS is Log-Hölder continuous [CS83].

3) The extension of this result to Hamiltonians with short range interactions should be possible.

4.2 Gap Labelling and the Group $\mathbf{K}_0$

In the previous Section, we have introduced the IDS, which is constant on each gap of the spectrum. A label of a spectral gap $\{g\}$ is given by the value $\mathcal{N}(g)$ taken by the IDS on this gap. This is a real number which has been recognized to be very rigid under perturbations of the Hamiltonian. In this Section we will label gaps in a different way, which will explain such a rigidity.

For indeed, one can associate to $\{g\}$ the eigenprojection $P(g)$ of the Hamiltonian H, on the interval $(-\infty, E]$ of energies where E is any point in $\{g\}$. Clearly, $P(g)$ does not depend upon the choice of E in $\{g\}$. Moreover, if f is any smooth function on $\mathbb{R}$ such that $0 \leq f \leq 1$, $f(E) = 1$ for $E \leq \inf\{g\}$, and $f(E) = 0$ for $E \geq \sup\{g\}$, then $P(g) = f(H)$. In particular, $P(g)$ belongs to the C^*-algebra $\mathcal{A}$ of H . However, the data of $P(g)$ contained not only an information about the spectral gap $\{g\}$, but also about the nature of the spectral measure of H. Changing H by a unitary transformation, or more generally by an algebraic automorphism will not change the spectrum as a set. So that it is sufficient to label $\{g\}$ by means of the equivalence class of $P(g)$ under unitary transformations.

This is a highly non trivial fact, for the set of equivalence classes of projections of $\mathcal{A}$ is usually a rather small set. To illustrate this claim, let $\mathcal{K}$ be the algebra of compact operators on a separable Hilbert space. Then a projection is compact if and only if it is finite dimensional. Moreover, two such projections are unitarily equivalent if and only if they have the same dimension. So, up to unitary equivalence, the set of projections in $\mathcal{K}$ is nothing but the set $\mathbb{N}$ of integers.

Actually we will use the Von Neumann definition of equivalence [PE79], namely:

Definition Two projections P and Q of a C^*-algebra $\mathcal{A}$ are *equivalent* if there is $U \in \mathcal{A}$ such that $UU^* = P$ and $U^*U = Q$. We will then write $P \approx Q$.

We will now give a few lemmas.

Lemma 4.2.1 [PE79]. *Let P and Q be two projections in a C^*-algebra $\mathcal{A}$ such that $\|P-Q\| < 1$. Then they are equivalent. In particular any norm continuous path $t \in \mathbb{R} \to P(t) \in \mathrm{Proj}(\mathcal{A})$ is made of mutually equivalent projections.*

Proof. We consider $F = PQ$ in $\mathcal{A}$. We claim that its square FF^* is invertible in the subalgebra $P\mathcal{A}P$. Actually, since P is the unit in $P\mathcal{A}P$, we get $\|FF^* - P\| \leq \|Q - P\| < 1$. Therefore the series $\sum_{n\geq 0} c_n(P - FF^*)^n$ converges absolutely in norm provides $\limsup_{n\to\infty}|c_n|^{1/n} \leq 1$. In particular $(FF^*)^{-1/2} = (P - (P - FF^*))^{-1/2}$ can be constructed as the sum of such a norm convergent series in $P\mathcal{A}P$. If we now set $U = (FF^*)^{-1/2}F$ then we get an element in $\mathcal{A}$ which satisfies $UU^* = P$. On the other hand $U^*U = F^*\{FF^*\}^{-1}F$ is the sum of the series $\sum_{n\geq 0} F^*(P - FF^*)^n F = \sum_{n\geq 0}(Q - F^*F)^n F^*F = Q$. Thus $P \approx Q$. □

Let $\mathcal{A}$ be a separable C^*-algebra. We will denote by $\mathcal{P}(\mathcal{A})$ the set of equivalence classes of projections in $\mathcal{A}$, and by $[P]$ the equivalence class of P. The next important property is

Lemma 4.2.2 [PE79]. *Let $\mathcal{A}$ be a separable C^*-algebra. The set $\mathcal{P}(\mathcal{A})$ of equivalence classes of projections in $\mathcal{A}$ is countable.*

Proof. Since $\mathcal{A}$ is separable, there is a countable set $(A_n)_{n\geq 0}$ in $\mathcal{A}$ which is norm dense in the unit ball. Given P a projection, and $\varepsilon < 1/2$, there is an $n \in \mathbb{N}$ such that $\|P - A_n\| \leq \varepsilon$. Replacing A_n by $(A_n + A_n^*)/2$ does not change this estimate, so that we can assume A_n to be self-adjoint. Thus its spectrum is contained into the union of two closed real intervals of width ε, the first centered at zero, the other at one. If the eigenprojection corresponding to the latter piece of the spectrum of A_n is denoted by P_n, we get $\|P_n - A_n\| \leq \varepsilon$, showing that $\|P - P_n\| \leq 1$, namely $P \approx P_n$. Therefore the set of equivalence classes of projections contains at most a countable family of elements. □

Two projections P and Q in $\mathcal{A}$ are orthogonal whenever $PQ = QP = 0$. In this latter case, the operator sum $P+Q$ is a new projection called the direct sum of P and Q, and it is denoted by $P \oplus Q$. We then get:

Lemma 4.2.3. *Let P and Q be two orthogonal projections in the C^*-algebra $\mathcal{A}$. Then the equivalence class of their direct sum depends only upon the equivalence classes of P and of Q . In particular, a sum is defined on the set $\mathcal{E}$ of pairs $([P],[Q])$ in $\mathcal{P}(\mathcal{A})$ such that there are $P' \approx P$ and $Q' \approx Q$ with $P'Q' = Q'P' = 0$, by $[P]+[Q] \equiv [P' \oplus Q']$. This composition law is commutative and associative.*

Proof. Suppose that there are projections $P' \approx P$ and $Q' \approx Q$ such that $P'Q' = Q'P' = 0$. This means that there are U and V in $\mathcal{A}$ such that $UU^* = P$, $U^*U = P'$, $VV^* = Q$, $V^*V = Q'$. One immediately checks that $(\mathbf{1} - P)U = 0$, and similar relations with U^*, V, V^*. It implies in particular that $VU^* = UV^* = V^*U = U^*V = 0$. Thus if we set $W = U + V$, we get $WW^* = P \oplus Q$ and $W^*W = P' \oplus Q'$. Hence, $P \oplus Q \approx P' \oplus Q'$. So that the sum is defined on the set $\mathcal{E}$ of pairs $([P],[Q])$ in $\mathcal{P}(\mathcal{A})$ such that there are $P' \approx P$ and $Q' \approx Q$ with $P'Q' = Q'P' = 0$, by $[P] + [Q] \equiv [P' \oplus Q']$. Since two orthogonal projections commute, the sum is commutative. Because the direct sum comes from the operator sum it is associative as well. □

The main problem is that $\mathcal{E}$ can be rather small, and our first task is to enlarge this definition in order to define the sum everywhere. The main idea consists in replacing the algebra $\mathcal{A}$ by the stabilized algebra $\mathcal{A}\otimes\mathcal{K}$. We have seen that such an algebra is naturally associated to a non commutative manifold in that it gives the set of multipliers of sections on any fiber bundle. We will imbed $\mathcal{A}$ into $\mathcal{A}\otimes\mathcal{K}$ by identifying $A \in \mathcal{A}$ with the matrix $\{i(A)\}_{m,n} = A\delta_{m,0}\delta_{n,0}$, if $\mathcal{K}$ is identified with the algebra generated by finite dimensional matrices indexed by $\mathbb{N}$. Actually we get:

Lemma 4.2.4. *Given any pair P and Q of projections in the C^*-algebra $\mathcal{A}\otimes\mathcal{K}$, there is always a pair P', Q' of mutually orthogonal projections in $\mathcal{A}\otimes\mathcal{K}$ such that $P' \approx P$ and $Q' \approx Q$. In particular the sum $[P]+[Q] \equiv [P' \oplus Q']$ is always defined.*

Proof. By definition any element in $\mathcal{A}\otimes\mathcal{K}$ can be approximated in norm by a finite matrix with elements in $\mathcal{A}$, namely by an element of $\mathcal{A}\otimes M_N(C)$ for some $N \in \mathbb{N}$. In particular if P is a projection in $A\otimes\mathcal{K}$, given $\varepsilon < 1/2$, one can find $A = A^*$ in $\mathcal{A}\otimes M_N(C)$ such that $\|P - A\| \leq \varepsilon$. Thus the spectrum of A is contained in the union of two disks of radius ε one centered at the origin, the other one centered at $z = 1$. The spectral projection P' corresponding to the latter one satisfies $\|P' - A\| \leq \varepsilon$, and $P' \in \mathcal{A}\otimes M_N(C)$. We get that $\|P - P'\| < 1$, namely $P \approx P'$. Thus one can always suppose that P is in $\mathcal{A}\otimes M_N(C)$. If now Q is another projection in $\mathcal{A}\otimes M_N(C)$, Q is equivalent to the matrix Q' in $\mathcal{A}\otimes M_{2N}(C)$ with $\{Q'\}_{m,n} = 0$ for $m,n \notin [N, 2N-1]$ and $\{Q'\}_{N+m,N+n} = \{Q\}_{m,n}$ if $m,n \in [0, N-1]$. In particular, $PQ' = Q'P = 0$. Therefore in $\mathcal{A}\otimes\mathcal{K}$ one can always add two equivalence classes. □

In other words, if $\mathcal{A}$ is a stable algebra, the set $\mathcal{P}(\mathcal{A})$ of equivalence classes of projections is an Abelian monoid with neutral element given by the class of the zero projection. If $\mathcal{A}$ is not stable, we will still denote by $\mathcal{P}(\mathcal{A})$ the set of equivalence classes of projections of the stabilized algebra $\mathcal{A}\otimes\mathcal{K}$.

There is a standard way to construct a group from such a monoid [CU82], generalizing the construction of $\mathbb{Z}$ from $\mathbb{N}$. We consider on the set of pairs $([P],[Q]) \in \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$, the equivalence relation

$$(4.2.1)\quad ([P],[Q])\mathcal{R}([P'],[Q']) \Leftrightarrow \exists [R] \in \mathcal{P}(\mathcal{A})\ [P]+[Q']+[R] = [P']+[Q]+[R]$$

Roughly speaking it means that $[P] - [Q] = [P'] - [Q']$. We get

Theorem 10. *(i) For a separable C^*-algebra $\mathcal{A}$, the set $\mathrm{K}_0(\mathcal{A}) = \mathcal{P}(\mathcal{A}\otimes\mathcal{K}) \times \mathcal{P}(\mathcal{A}\otimes\mathcal{K})/\mathcal{R}$ is countable and has a natural structure of Abelian group.*

(ii) Any real valued map ϕ defined on the set of projections of $\mathcal{A}$ such that $P \approx Q \Rightarrow \phi(P) = \phi(Q)$, and $\phi(P \oplus Q) = \phi(P) + \phi(Q)$, defines canonically a group homomorphism ϕ^* from $\mathrm{K}_0(\mathcal{A})$ into $\mathbb{R}$.

(iii) Any trace τ on $\mathcal{A}$ defines in a unique way a group homomorphism τ^* such that if P is a projection on $\mathcal{A}, \tau(P) = \tau^*([P])$ where $[P]$ is the class of P in $\mathrm{K}_0(\mathcal{A})$.

Proof. (i) Since $\mathcal{A} \otimes \mathcal{K}$ is separable the set $\mathcal{P}(\mathcal{A})$ is countable and so is $\mathrm{K}_0(\mathcal{A})$. Moreover, the equivalence relation $\mathcal{R}$ is compatible with the addition in $\mathcal{P}(\mathcal{A})$, namely if $([P_1],[Q_1])\mathcal{R}([P_1'],[Q_1'])$ and $([P_2],[Q_2])\mathcal{R}([P_2'],[Q_2'])$ then from (4.2.1) there is an $[R]$ such that

$$[P_1] + [P_2] + [Q_1'] + [Q_2'] + [R] = [P_1'] + [P_2'] + [Q_1] + [Q_2] + [R] ,$$

namely $([P_1] + [P_2], [Q_1] + [Q_2])\mathcal{R}([P_1'] + [P_2'], [Q_1'] + [Q_2'])$, showing that the addition on $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$ defines on the quotient space a structure of Abelian monoid. The neutral element is the class of $[0] \equiv ([0],[0])$, namely the set of $([R],[R])$'s with $[R]$ in $\mathcal{P}(\mathcal{A})$. Thus for any element $([P],[Q])$ in $\mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{A})$ the sum $([P],[Q]) + ([Q],[P])$ is equivalent to $[0]$, showing that in the quotient space any element has an opposite element: so it is a group. Clearly $\mathcal{P}(\mathcal{A})$ is imbedded in this group through the map $[P] \longrightarrow ([P],[0])$ and the quotient map, giving a homomorphism of monoid.

From now on, we will identify $[P]$ with its image in $\mathrm{K}_0(\mathcal{A})$. The assertions (ii) and (iii) are direct consequences of the definition of $\mathrm{K}_0(\mathcal{A})$. □

Theorem 11 (The Abstract Gap Labelling Theorem). *Let H be a homogeneous self-adjoint operator satisfying Shubin's formula (4.1.3). Let $\mathcal{A}$ be the C^*-algebra it generates together with the translation group (its non commutative Brillouin zone). Let $\mathrm{Sp}(H)$ be its spectrum in $\mathcal{A}$. Then for any gap $\{g\}$ of $\mathrm{Sp}(H)$,*

(i) the value of the IDS *of H on $\{g\}$ belongs to the countable set of real numbers $[0, \tau(\mathbf{1})] \cap \tau^*(\mathrm{K}_0(\mathcal{A}))$.*

(ii) the equivalence class $n(g) = [P(g)] \in \mathrm{K}_0(\mathcal{A})$, gives a labelling invariant under small perturbations of the Hamiltonian H within $\mathcal{A}$.

(iii) If S is a subset of $\mathbb{R}$ which is closed and open in $\mathrm{Sp}(H)$, then $n(S) = [P_S] \in \mathrm{K}_0(\mathcal{A})$ where P_S is the eigenprojection of H corresponding to S, is a labelling for each such part of the spectrum.

As a consequence, we get the stability of the gap labelling namely the property (ii) above which is immediate from the Lemma 4.2.1. Let us make this point more precise:

Proposition 4.2.5 (Stability and Sum Rules for the Labelling). *Let $t \in \mathbb{R} \rightarrow H(t)$ be a continuous family (in the norm-resolvent sense) of self-adjoint operators with resolvent in a separable C^*-algebra $\mathcal{A}$.*

(i) invariance: *the gap edges of H are continuous and the labelling of a gap $\{g(t)\}$ is independent of t as long as the gap does not close.*

(ii) sum rule: *suppose that for $t \in [t_0, t_1]$, the spectrum of $H(t)$ contains an open-closed subset $S(t)$ such that $S(t_0) = S_+ \cup S_-$ and $S(t_1) = S'_+ \cup S'_-$ where $S_\pm$ and $S'_\pm$ are open-closed in* $\mathrm{Sp}(H(t_0))$, $\mathrm{Sp}(H(t_1))$; *then $n(S_+) + n(S_-) = n(S'_+) + n(S'_-)$.*

4.3 Properties of the K-Groups

In Section 4.2, we have defined the group $\mathrm{K}_0(\mathcal{A})$ in an abstract way, and we have shown how it can be used to label the gaps or the open-closed subsets of the spectrum of a homogeneous Hamiltonian. Obviously we will need to compute more explicitly the group $\mathrm{K}_0(\mathcal{A})$ and its image under the trace per unit volume in order to get a labelling of the gaps. In this Section let us give some general rules for such a computation, which we will use in the last Section to get the result in various classes of examples. The proofs of these properties are too long to be reproduced here, so that we will invite the reader to look into the references if he wants to have the proofs.

Let us first define the group $\mathrm{K}_1(\mathcal{A})$. Let $\mathcal{A}$ be a separable C^*-algebra with a unit. $\mathrm{GL}_n(\mathcal{A})$ is the group of invertible elements of the algebra $M_n(\mathcal{A})$. One can consider $\mathrm{GL}_n(\mathcal{A})$ as the subgroup of $\mathrm{GL}_{n+1}(\mathcal{A})$ made of matrices X with coefficients in $\mathcal{A}$, such that $X_{n+1,m} = X_{m,n+1} = \delta_{n+1,m}$. $\mathrm{GL}(\mathcal{A})$ is then the inductive limit of this family, namely the norm closure of their union. Let $\mathrm{GL}(\mathcal{A})_0$ be the connected component of the identity in $\mathrm{GL}(\mathcal{A})$. One set:

$$\mathrm{K}_1(\mathcal{A}) = \pi_0(\mathrm{GL}(\mathcal{A})) = \mathrm{GL}(\mathcal{A})/\mathrm{GL}(\mathcal{A})_0 \,. \tag{4.3.1}$$

Here $\pi_0(M)$ denotes the set of connected components of the topological space M. If A has no unit, one sets $\mathrm{K}_1(\mathcal{A}) = \mathrm{K}_1(\mathcal{A}^\sim)$.

So $\mathrm{K}_1(\mathcal{A})$ is an Abelian group. For indeed the path $B(t)$ defined by

$$B(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & \mathbf{1} \end{bmatrix} \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

permits to connect the following matrices in $\mathrm{GL}(\mathcal{A})$

$$\begin{bmatrix} AB & 0 \\ 0 & \mathbf{1} \end{bmatrix} \approx \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \approx \begin{bmatrix} BA & 0 \\ 0 & \mathbf{1} \end{bmatrix} .$$

Since $\mathcal{A}$ is separable, it is easy to check that $\mathrm{K}_1(\mathcal{A})$ is countable.

The main general properties of $\mathrm{K}_0(\mathcal{A})$ and $\mathrm{K}_1(\mathcal{A})$ are summarized in the following theorem

Theorem 12. *Let $\mathcal{A}, \mathcal{B}$ be separable C^*-algebras* *(i) If $f : \mathcal{A} \to \mathcal{B}$ is a *-homomorphism, then f defines in a unique way groups homomorphisms f_* from $\mathrm{K}_i(\mathcal{A})$ into $\mathrm{K}_i(\mathcal{B})$ $(i = 0, 1)$. They satisfy $\mathrm{id}_* = \mathrm{id}$, and $(f \circ g)_* = f_* \circ g_*$.*

(ii) $\mathrm{K}_i(\mathcal{A} \oplus \mathcal{B})$ is isomorphic to $\mathrm{K}_i(\mathcal{A}) \oplus \mathrm{K}_i(\mathcal{B})$.

(iii) If $\mathcal{A}$ is the inductive limit of a sequence $(\mathcal{A}_n)_{n\geq 0}$ of C^*-algebras then $\mathrm{K}_i(\mathcal{A})$ is the inductive limit of the groups $\mathrm{K}_i(\mathcal{A}_n)$.

(iv) If $\mathcal{C}_0(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$ vanishing at infinity, $\mathrm{K}_i(\mathcal{A})$ is isomorphic to $\mathrm{K}_{i+1}(\mathcal{C}_0(\mathbb{R}) \otimes \mathcal{A})$ (where $i + 2 \equiv i$).

(v) $\mathrm{K}_0(\mathcal{A})$ is isomorphic to the group $\pi_1(\mathrm{GL}(\mathcal{A}))$ of homotopy classes of closed paths in $\mathrm{GL}(\mathcal{A})$. If τ is a trace on $\mathcal{A}$ and if $t \in [0,1] \to U(t)$ is a closed path in $\mathrm{GL}(\mathcal{A})$ we get [CO81]:

$$\tau_*([U]) = 1/2\mathbf{i}\pi \int_{[0,1]} \mathrm{d}t\tau(U(t)^{-1}U'(t)) \tag{4.3.2}$$

(vi) If $\phi : \mathcal{J} \to \mathcal{A}$, and $\psi : \mathcal{A} \to \mathcal{B}$ are *-homomorphisms such that the sequence $0 \to \mathcal{J} \to \mathcal{A} \to \mathcal{B} \to 0$ is exact, then the following six-terms sequence is exact [CU82, BL86]*:*

$$\begin{tikzcd}
\mathrm{K}_0(\mathcal{J}) \arrow[r, "\phi^*"] & \mathrm{K}_0(\mathcal{A}) \arrow[r, "\psi^*"] & \mathrm{K}_0(\mathcal{B}) \arrow[d, "\mathrm{Exp}"] \\
\mathrm{K}_1(\mathcal{B}) \arrow[u, "\mathrm{Ind}"] & \mathrm{K}_1(\mathcal{A}) \arrow[l, "\psi^*"'] & \mathrm{K}_1(\mathcal{J}) \arrow[l, "\phi^*"']
\end{tikzcd}$$

In the previous theorem, Ind and Exp are the 'connection' automorphisms defined as follows assuming that $\mathcal{A}$ has a unit. Let P be a projection in $\mathcal{B} \otimes \mathcal{K}$, and let A be a self-adjoint element of $\mathcal{A} \otimes \mathcal{K}$ such that $\psi \otimes \mathrm{id}(A) = P$. One gets:

$$\psi \otimes \mathrm{id}(e^{2\mathbf{i}\pi A}) = e^{2\mathbf{i}\pi P} = \mathbf{1} \ .$$

So that $B = e^{2\mathbf{i}\pi A} \in (\mathcal{J} \otimes \mathcal{K})^\sim$ and is unitary in $(\mathcal{J} \otimes \mathcal{K})^\sim$. The class of B gives an element of $\mathrm{K}_1(\mathcal{J})$ which is by definition $\mathrm{Exp}([P])$. One can check that this definition makes sense. In much the same way, let now U be an element of $(\mathcal{B} \otimes \mathcal{K}^\sim)$ which we may assume without loss of generality to be the image by $\psi \otimes \mathrm{id}$ of a partial isometry W in $(\mathcal{A} \otimes \mathcal{K})^\sim$. Then $\mathrm{Ind}([U])$ is the class of $[WW^*] - [W^*W]$ in $\mathrm{K}_0(\mathcal{J})$. One can also check that this definition makes sense.

In order to compute these groups in practice we will also need two other kinds of results. The first one is a theorem by Pimsner and Voiculescu [PV80a], and concerns $\mathbb{Z}$-actions on a C^*-algebra.

Theorem 13. *Let $\mathcal{A}$ be a separable C^*-algebra, and α be a *-automorphism of $\mathcal{A}$. The crossed product of $\mathcal{A}$ by $\mathbb{Z}$ via α is the C^*-algebra generated by $\mathcal{A}$ and a unitary U such that $UAU^{-1} = \alpha(A)$ for all A's in $\mathcal{A}$. Then there exists a six-terms exact sequence of the form:*

$$\begin{tikzcd}
\mathrm{K}_0(\mathcal{A}) \arrow[r, "\mathrm{id}-\alpha_*"] & \mathrm{K}_0(\mathcal{A}) \arrow[r, "j_*"] & \mathrm{K}_0(\mathcal{A} \rtimes_\alpha \mathbb{Z}) \arrow[d, "\mathrm{Exp}"] \\
\mathrm{K}_1(\mathcal{A} \rtimes_\alpha \mathbb{Z}) \arrow[u, "\mathrm{Ind}"] & \mathrm{K}_1(\mathcal{A}) \arrow[l, "j_*"'] & \mathrm{K}_1(\mathcal{A}) \arrow[l, "\mathrm{id}-\alpha_*"']
\end{tikzcd}$$

where j is the canonical injection of $\mathcal{A}$ into the crossed product $\mathcal{B} = \mathcal{A} \rtimes_\alpha \mathbb{Z}$.

The connection homomorphisms in Theorem 13, are defined from the following exact sequence (called the Toeplitz extension of $\mathcal{A}$): let S be a non unitary isometry such that $S^*S = 1$, $SS^* = \mathbf{1} - P$, with $P \neq 0$. The C^*-algebra $C^*(S)$ defined by S does not depend upon the choice of S [CO67]. Choosing U in $\mathcal{B}$ such that $\alpha(A) = U^*AU$, one gets a homomorphism ψ from $\mathcal{A} \otimes \mathcal{K}$ into the subalgebra $\mathcal{T}$ of $\mathcal{B} \otimes C^*(S)$ generated by $\mathcal{U} \otimes \mathcal{S}$ and $\mathcal{A} \otimes \mathbf{1}$ by the formula:

$$\psi(A \otimes e_{i,j}) = (U \otimes S)^i A \otimes P (U \otimes S)^{*j} ,$$

where $e_{i,j}$ is the matrix with all its elements equal to zero but for the element (i,j), equal to one. Then the image of ψ is the ideal $\mathcal{J}$ generated by $\mathbf{1} \otimes P$ in $\mathcal{T}$ and the quotient algebra $\mathcal{T}/\mathcal{J}$ is isomorphic to $\mathcal{B}$. In other words the sequence $0 \to \mathcal{A} \otimes \mathcal{K} \to \mathcal{T} \to \mathcal{B} \to 0$ is exact.

We will use this exact sequence to compute the gap labelling of the 1D Schrödinger operator with a potential given by an automatic sequence.

At last, let us indicate the Connes analog of Thom's isomorphism [CO81]

Theorem 14. *Let $\mathcal{A}$ be a separable C^*-algebra, and α be a one parameter group of *-automorphisms of $\mathcal{A}$. Then there is a natural isomorphism between $\mathrm{K}_i(\mathcal{A} \rtimes_\alpha \mathbb{R})$ and $\mathrm{K}_i(\mathcal{A} \otimes \mathcal{C}_0(\mathbb{R})) \approx \mathrm{K}_{i+1}(\mathcal{A})$, for $i \in \mathbb{Z}/2$.*

These rules allow us to compute the K-groups for inductive limits, ideals, quotients, extensions, tensor product by $M_n, \mathcal{K}, \mathcal{C}_0(\mathbb{R})$, and by crossed products by $\mathbb{Z}$ or $\mathbb{R}$. It is more than enough for the class of C^*-algebras we have developed previously in this paper to compute it.

5. Gap Labelling Theorems for 1D Discrete Hamiltonians

5.1 Completely Disconnected Hull

Let Ω be a compact metrizable space. In this Section we will assume that Ω is completely disconnected. Let T be a homeomorphism of Ω. We will assume that T is 'topologically transitive' namely that it admits at least one dense orbit in Ω. At last μ will denote a T-invariant ergodic probability measure on Ω.

We will consider 1D discrete Schrödinger operators of the form:

$$H_\omega \psi(n) = \psi(n+1) + \psi(n-1) + v(T^{-n}\omega)\psi(n) \,, \qquad \omega \in \Omega, \psi \in l^2(\mathbb{Z}) \,, \tag{5.1.1}$$

where v is a continuous function on Ω. More generally, we will consider bounded self-adjoint operators acting on $l^2(\mathbb{Z})$ as follows:

$$H_\omega \psi(n) = \sum_{m \in \mathbb{Z}} v(T^{-n}\omega; m-n)\psi(m) \,, \quad \omega \in \Omega, \psi \in l^2(\mathbb{Z}) \,, \tag{5.1.2}$$

where each map $v_j : \omega \in \Omega \to v(\omega; j) \in C$ is continuous on Ω.

The oldest examples investigated in the literature are the case of a 1D Schrödinger operator with random potentials of Bernouilli type [CF87] or with a limit periodic potential [MO81, AS81]. On the lattice they give rise to a completely disconnected hull indeed. The case of 1D quasicrystals falls into this class (see Section 3.2). More recently potentials with a hierarchical structure had been investigated [KC88, LM88, KL89]. They also belong to such a class, together with potentials given by a substitution (Section 3.3). More generally every operator like the one given by (5.1.1) in which the coefficients take finitely many values, is of this type.

Recall that the C^*-algebra generated by the family of translated of H is the crossed product $C^*(\Omega, T) = \mathcal{C}(\Omega) \rtimes_T \mathbb{Z}$ of the algebra $\mathcal{C}(\Omega)$ of continuous functions on Ω, by the $\mathbb{Z}$-action defined by T. The probability μ defines a trace τ on it, which coincides with a trace per unit length.

Thanks to results of Section 4, the gap labelling will be given by the K_0-group of $C^*(\Omega, T)$, whereas the values of the IDS on the gaps will belong to the image of this group by the trace τ. The main tool to compute the K_0-group is provided by the Pimsner-Voiculescu exact sequence (cf. Section 4.3) where $\mathcal{A} = \mathcal{C}(\Omega)$ and $\alpha = T_*$ is the automorphism induced by T. We need first the following lemmas proved in [see BB91]:

Lemma 5.1.1. *Let Ω be a completely disconnected compact metrizable space. Then* $\mathrm{K}_1(\mathcal{C}(\Omega)) = \{0\}$.

Lemma 5.1.2. *Let Ω be a completely disconnected compact metrizable space. Then $\mathrm{K}_0(\mathcal{C}(\Omega))$ is isomorphic to the group $\mathcal{C}(\Omega, \mathbb{Z})$ of integer valued continuous functions on Ω.*

The main result of this Section will be the following

Theorem 15. *Let Ω be a completely disconnected compact metrizable space, and T a homeomorphism on Ω. Then:*

(i) If T is topologically transitive $\mathrm{K}_1(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

(ii) $\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z})$ is isomorphic to the quotient $\mathcal{C}(\Omega, \mathbb{Z})/\mathcal{E}_D$, where $\mathcal{E}_D$ is the subgroup $\mathcal{E}_D = \{f \in \mathcal{C}(\Omega, \mathbb{Z});\ \exists g \in \mathcal{C}(\Omega, \mathbb{Z}),\ f = g - g \circ T^{-1}\}$.

(iii) Let μ be a T-invariant ergodic probability measure on Ω, and τ be the corresponding trace on $\mathcal{C}(\Omega) \rtimes_T \mathbb{Z}$. Then, the image of $\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z})$ by τ is equal to the countable subgroup $\mu(\mathcal{C}(\Omega, \mathbb{Z}))$ of $\mathbb{R}$.

Proof. Thank to Lemmas 5.1.1 and 5.1.2, and using the Pimsner-Voiculescu exact sequence for $\mathcal{A} = \mathcal{C}(\Omega), \alpha = T_*$ the automorphism defined by T, we get the following exact sequence

$$(5.1.3)\quad 0 \to \mathrm{K}_1(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z}) \to \mathcal{C}(\Omega, \mathbb{Z}) \xrightarrow{\mathrm{id}-T_*} \mathcal{C}(\Omega, \mathbb{Z}) \to \mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z}) \to 0\ ,$$

where the middle arrow is given by $\mathrm{id} - T_*$. Since the sequence is exact, it follows that $\mathrm{K}_1(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z})$ is isomorphic to the kernel of $\mathrm{id} - T_*$. If T is topologically transitive, the kernel of $\mathrm{id} - T_*$ is the set of constant functions in $\mathcal{C}(\Omega, \mathbb{Z})$ namely $\mathbb{Z}$ and (i) is proved.

For the same reason, since the sequence is exact, the fourth arrow i_* is surjective. Thus $\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{Z}) \approx \mathcal{C}(\Omega, \mathbb{Z})/\mathrm{Ker}(i_*)$, which proves (ii) because $\mathrm{Ker}(i_*) = \mathrm{Im}\{\mathrm{id} - T_*\} = \mathcal{E}_D$.

If i is the canonical injection from $\mathcal{C}(\Omega)$ into $\mathcal{C}(\Omega) \rtimes_T \mathbb{Z}$, it follows from the definition of the trace given by μ that $\tau \circ i = \mu$. By functoriality $\tau_* \circ i_* = \mu_*$ on the corresponding K_0-groups. One can show that μ_* is nothing but μ acting on $\mathcal{C}(\Omega, \mathbb{Z})$ (see the proof of the Lemma 5.1.2 in [BB91]), and i_* is surjective, (iii) holds. □

5.2 Potentials Taking Finitely Many Values

Let us now consider a Hamiltonian on $l^2(\mathbb{Z})$ of the form given by:

$$(5.2.1)\qquad H\psi(n) = \sum_{|m| \leq N} t_m(n)\psi(n+m)\ , \psi \in l^2(\mathbb{Z})\ ,$$

where the coefficients take finitely many values. Then the hull Ω can be constructed as to be the closure of an orbit of T in $A^{\mathbb{Z}}$ where A is a finite set (the possible values of the coefficients), and T the two-sided shift on $A^{\mathbb{Z}}$. It is therefore completely disconnected. By a 'letter' we mean a point in the finite set A, while a 'word' will be a finite sequence $w = (a_0 a_1 \cdots a_n)$ of letters; $n = |w|$ is the length of w. Let μ be a T-invariant ergodic probability measure on Ω.

Given a sequence $\omega = (\omega(n))_{n \in \mathbb{Z}} \in A^{\mathbb{Z}}$, the occurrence number $L_\omega(w)$ of a word w in ω is defined (whenever it exists) by:

(5.2.2)
$$L_\omega(w) = \lim_{L\to\infty} 1/(2L+1)\#\{n \in [-L, L];\ (\omega(n+1), \cdots, \omega(n+N)) = w\}\ .$$

By Birkhoff's ergodic theorem, this limit exists for μ-almost all $\omega \in \Omega$; it is independent of ω and coincides with $\mu(\chi_w)$ where χ_w is the characteristic function of the cylinder set $\Omega_w = \{\omega \in \Omega; (\omega(0), \cdots, \omega(N-1)) = w\}$. A direct consequence of Theorem 15 is given by:

Proposition 5.2.1. *The values of the IDS of H on the spectral gaps are linear combinations with integer coefficients of the occurrence numbers of any possible word of A.*

Ex 1: Bernouilli Process

Corollary 5.2.2. *If $\Omega = \{-1, +1\}^{\mathbb{Z}}$, T is the two-sided shift, and $\mu = \otimes_{n\in\mathbb{Z}}\mu_p$ the Bernouilli measure where $\mu_p\{-1\} = p, \mu_p\{+1\} = 1-p$, the possible values of the IDS on gaps of H given by (5.1.2), are linear combinations with integer coefficients of the numbers $p^m(1-p)^n$.*

Remark. 1) In this last corollary, T is not topologically transitive on Ω. Nevertheless (ii) and (iii) of Theorem 15 still hold.

2) If H is given by (5.1.1), where $V(n) = v(T^{-n}\omega)$ takes values $\pm V$ with probabilities $p, (1-p)$ respectively, we cannot expect more than one gap. However for models with longer range interaction of the type (5.1.2), or another function v in $\mathcal{C}(\Omega)$, one may expect more than one gap, even though their number may be finite.

Ex 2: Induced Process

Another example of systems described by such a construction is given as follows: let $(M, S, \mathbf{P})$ be a dynamical system, namely M is a compact metrizable space, S is a homeomorphism of M, and $\mathbf{P}$ is a S-invariant ergodic probability measure on M. Let now Λ_i $(1 \le i \le N)$ be a finite family of $\mathbf{P}$-measurable subsets of M. Let $\mathcal{A}_0$ be the *-algebra generated by the functions $\{\chi_i \circ S^n;\ 1 \le i \le N,\ n \in \mathbb{N}\}$, where χ_i is the characteristic function of Λ_i. Without loss of generality we can assume that the Λ_i's form a partition of M. Let $\mathcal{A}$ be the norm closure of $\mathcal{A}_0$ in $L^\infty(M, \mathbf{P})$. $\mathcal{A}$ is an Abelian C^*-algebra, so that there is a compact space Ω, (its spectrum namely the set of characters of $\mathcal{A}$), such that $\mathcal{A}$ is isomorphic to $\mathcal{C}(\Omega)$. The map $g \in \mathcal{A}_0 \to g \circ S \in \mathcal{A}_0$ induces a *-automorphism of $\mathcal{A}$, which in turn gives rise to a homeomorphism also denoted by S, of Ω. The map $g \in \mathcal{A}_0 \to \int d\mathbf{P}(\omega)g(\omega) \in \mathbb{C}$ extends to a trace on $\mathcal{A}$ which in turn defines a S invariant ergodic probability measure on Ω. Thus we get a new dynamical system $(\Omega, S, \mathbf{P})$, induced by the family $\{\Lambda_i; 1 \le i \le N\}$.

The partition $\mathcal{P} = \{\Lambda_i; 1 \leq i \leq N\}$ is called 'generating' if there is a compatible metric on M for which, for any $\varepsilon > 0$, there are $m \leq n$ such that the partition $\mathcal{P}_{m,n} = S^m\mathcal{P} \wedge S^{m+1}\mathcal{P} \wedge \cdots \wedge S^n\mathcal{P}$ contains only atoms of diameter less than or equal to ε.

Application of the Theorem 15 gives [BB91]:

Proposition 5.2.3 . *(i) The compact space Ω induced by the partition $\mathcal{P} = \{\Lambda_i; 1 \leq i \leq N\}$ is completely disconnected.*

(ii) If $\mathcal{P}$ is generating, $\mathcal{C}(M)$ is a closed subalgebra of $\mathcal{C}(\Omega)$. If in addition (M, S) is uniquely ergodic, then (Ω, S) is also uniquely ergodic, and the unique ergodic invariant measures on M and Ω agree.

(iii) The IDS of a self adjoint element of $\mathcal{C}(\Omega) \rtimes_S \mathbb{Z}$ takes values in the $\mathbb{Z}$-module generated by the numbers $\mathbf{P}(\Lambda_{i(0)} \cap S\Lambda_{i(1)} \cap \cdots S^k\Lambda_{i(k)})$ where $1 \leq i(n) \leq N$ for all n's.

Among the examples which have been investigated in the literature, let us mention the following:

Ex 3: The Kohmoto model [KK83, OP83, KO84, OK85, KL86, LP86, CA87, SU87, WI89, BI89, LE89, SU89, BI90]

The algebra is generated by the characteristic function $\chi_{[1-\alpha,1)}$ of the interval $\Lambda_0 = [1-\alpha, 1)$ on the torus $M = \mathbb{T}$, with $S = R_\alpha$ is the rotation by $\alpha \in (0,1)\backslash\mathbb{Q}$ and $\mathbf{P} = \lambda$ is the normalized Lebesgue measure. Then $(\mathbb{T}, R_\alpha)$ is uniquely ergodic. The Kohmoto model is given by:

(5.2.3)

$$H_x\psi(n) = \psi(n+1) + \psi(n-1) + V\chi_{[1-\alpha,1)}(n\alpha - x)\psi(n)\ , x \in \mathbb{T}\ , \psi \in l^2(\mathbb{Z})\ .$$

Hamiltonians on a 1D quasicrystal also belong to the same algebra. Applying the Proposition 5.2.3 we get:

Proposition 5.2.4. *The IDS of the Kohmoto model, or of Hamiltonians in the same algebra, takes values in the set $(\mathbb{Z} + \mathbb{Z}\alpha) \cap [0,1]$. If we denote by Ω the corresponding hull, the* K_0*-group of the algebra $\mathcal{C}(\Omega) \rtimes_\alpha \mathbb{Z}$ defined above, is isomorphic to $\mathbb{Z}^2$. Its image under the trace defined by the Lebesgue measure on Ω is the dense subgroup $\mathbb{Z} + \mathbb{Z}\alpha$ of $\mathbb{R}$.*

Ex 4: The B-S Model [BS82a]:

It is given by the same dynamical system $(\mathbb{T}, R_\alpha, \lambda)$, but now $\Lambda_0 = [1-\beta, 1)$, where β is an irrational number rationally independent of α. One possible example of Hamiltonian is given by:

(5.2.4)

$$H_x\psi(n) = \psi(n+1) + \psi(n-1) + V\chi_{[1-\beta,1)}(n\alpha - x)\psi(n)\ , x \in \mathbb{T}\ , \psi \in l^2(\mathbb{Z})\ .$$

More generally, one can introduce the sets $\Lambda_i = [1-\beta_i, 1]$ $(1 \leq i \leq N)$, where the numbers $1, \alpha, \beta_i$ are all rationally independent. Then the same argument leads to a gap labelling given by the $\mathbb{Z}$-module generated by $1, \alpha$ and the β_i 's.

5.3 Some Almost Periodic Hamiltonians

1) Limit-Periodic Case. Let us consider the Hamiltonian H given by (5.1.2), where the coefficients are limit periodic sequences on $\mathbb{Z}$. We recall that a sequence $\mathbf{V} = (V(n))_{n\in\mathbb{Z}}$ is limit periodic [BO47] if it is the uniform limit of a family of periodic sequences. It implies that there is a sequence $(q_i)_{i\in\mathbb{N}}$ of integers, such that $q_0 = 1$, $q_{i+1}/q_i = a_i \in \mathbb{N}\backslash\{0,1\}$, and for each $i \geq 0$, a q_i-periodic sequence $\mathbf{V}_i$ on $\mathbb{Z}$, such that $\sup_{n\in\mathbb{Z}}|\mathbf{V}(n) - \mathbf{V}_i(n)| \to 0$ as $i \to \infty$.

Such models have been investigated in [MO81, AS81, BB82, KC88, LM88, KL89].

The hull of such a sequence can then be constructed as follows: as a set $\Omega = \Pi_{i\in\mathbb{N}}\{0,1,\cdots,a_i-1\}$. If ω', $\omega'' \in \Omega$, the sum $\omega' + \omega'' = \omega$ is defined as follows: ω_0 is the unique integer in $\{0,1,\cdots,a_0-1\}$ such that $\omega'_0 + \omega''_0 = \omega_0 + a_0 r_1$, where r_1 takes values 0 or 1. By recursion if $r_i \in \{0,1\}$ is defined, then ω_i is the unique integer in $\{0,1,\cdots,a_i-1\}$ such that $\omega'_i + \omega''_i + r_i = \omega_i + a_i r_{i+1}$, where now r_{i+1} takes values 0 or 1. One can check that this sum is associative and commutative, that the sequence 0 with all coordinates equal to 0 is a neutral element, and that every element $\omega \in \Omega$ has an opposite ω' defined by $\omega'_0 = a_0 - \omega_0$, $w'_i = a_i - 1 - \omega_i$ $(i \geq 1)$. Thus Ω is a compact Abelian group.

We then denote by ε the element $(\delta_{i,0})_{i\in\mathbb{N}}$ and we check that $T\omega = \omega + \varepsilon$ is a homeomorphism of Ω, whereas the orbit of 0 is dense. Actually, if $n \in \mathbb{N}$, one can decompose n in a unique way in the form $n = \nu_0 + \nu_1 q_1 + \cdots + \nu_L q_L$ with $0 \leq \nu_i < a_i$ and we check that $T^n 0 = n\varepsilon = (\nu_0, \nu_1 \cdots, \nu_L, 0, \cdots, 0, \cdots)$, namely $n \in \mathbb{N} \to T^n 0 \in \Omega$ extends in a unique way to a group homomorphism with a dense image.

Any character χ of Ω is associated to a unique rational number (mod 1) of the form k/q_L by $\chi(\omega) = \Pi_{i\leq L} e^{2\mathrm{i}\pi k\omega_i q_i/q_L}$. Thus the dual group of Ω can be identified with the subgroup of the torus $\mathbb{T}$ given by numbers of the form k/q_L; it is isomorphic to the inductive limit $\Omega^* = \mathrm{Lim}_{L\in N}\mathbb{Z}/q_L\mathbb{Z}$, where the injection of $\mathbb{Z}/q_L\mathbb{Z}$ into $\mathbb{Z}/q_{L+1}\mathbb{Z}$ is given by the multiplication by a_{L+1}. The 'frequency module' is the $\mathbb{Z}$-module generated by the k/q_L's in $\mathbb{Q}$. In the sequel we will denote by $\mathbb{Z}[a_i; i \in \mathbb{N}]$ the group Ω so constructed. Clearly it is completely disconnected. So we can summarize it as follows:

Proposition 5.3.1. *A bounded sequence $(V(n))_{n\in\mathbb{Z}}$ is limit periodic if and only if there is a sequence $(a_i)_{i\in\mathbb{N}}$ of integers bigger than 1, and a continuous function v on the compact group $\mathbb{Z}[a_i; i \in \mathbb{N}]$ such that $V(n) = v(n\varepsilon)$.*

T is uniquely ergodic, since any T-invariant measure must coincide with the Haar measure μ. Since Ω is a product space, and one can show that the Haar measure is the product measure $\mu = \otimes_{i\in\mathbb{N}}\mu_i$ where μ_i is the uniform measure on $\{0, 1, \cdots, a_i - 1\}$, namely the measure which affects the weight $1/a_i$ to each point. So as a corollary of Theorem 15 we get:

Theorem 16. *Let H be given by (5.1.2) where the coefficients are limit periodic sequences. Then the IDS on the gaps of H takes values in the frequency module of the hull.*

2) Harper's Model and related ones [HA55, HO76, RI81, BE88b]. Let us now consider the models described in Section 3.1. The algebra is now $\mathcal{A}_\alpha = C^*(U, V)$ where U and V are two unitary operators such that $UV = e^{2\mathrm{i}\pi\alpha} VU$. Let us remark that this algebra is isomorphic to the crossed product $\mathcal{C}(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ where $\mathbb{Z}$ acts on $\mathbb{T}$ through the rotation R_α by α. For indeed, $\mathcal{C}(\mathbb{T})$ is the C^*-algebra generated by one unitary operator V, namely the function $V : x \in \mathbb{T} \to e^{2\mathrm{i}\pi x} \in \mathbb{C}$. Thus $V \circ R_\alpha = UVU^{-1}$, showing that U is the generator of the rotation in the crossed product. We get the following gap labelling in that case [RI81, PV80b]:

Theorem 17. *Let α be an irrational number in $(0, 1)$. The K_0-group of the algebra $\mathcal{A}_\alpha = C^*(U, V) \approx \mathcal{C}(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ defined above, is isomorphic to $\mathbb{Z}^2$. Its image under the trace defined by the Lebesgue measure on $\mathbb{T}$ is the dense subgroup $\mathbb{Z} + \mathbb{Z}\alpha$ of $\mathbb{R}$.*

Proof. Let α be irrational now, and let $\mathcal{B}_\alpha = \mathcal{C}(\Omega) \rtimes_\alpha \mathbb{Z}$ be the algebra corresponding to the Kohmoto model (see Section 5.2). Since the characteristic functions of the intervals $[n\alpha, m\alpha]$ generate $\mathcal{C}(\Omega)$, and since these intervals can be as small as we want, it follows that $\mathcal{C}(\Omega)$ contains $\mathcal{C}(\mathbb{T})$ as a closed subalgebra. Thus $\mathcal{A}_\alpha$ is contained as a closed subalgebra of $\mathcal{B}_\alpha$. In particular, by functoriality, $\mathrm{K}_0(\mathcal{A}_\alpha)$ is a subgroup of $\mathrm{K}_0(\mathcal{B}_\alpha)$. Since $\mathrm{K}_0(\mathcal{B}_\alpha) \approx \mathbb{Z}^2$, it is enough to show that the generators of $\mathrm{K}_0(\mathcal{B}_\alpha)$ can be taken in $\mathrm{K}_0(\mathcal{A}_\alpha)$. M. Rieffel [RI81] found one projection given by:

$$P_R = fU + g + U^{-1}f \,. \tag{5.3.1}$$

Here f and g are continuous functions on $\mathbb{T}$ constructed as follows: given $0 < \varepsilon < \alpha, 1 - \alpha$, $g(x) = 0$ if $\alpha \le x \le 1 - \varepsilon$, $g(x) = 1$ if $0 \le x \le \alpha - \varepsilon$, $0 \le g(x + \alpha) = 1 - g(x) \le 1$ if $1 - \varepsilon \le x \le 1$; then $f(x) = \{g(x) - g(x)^2\}^{1/2}$ if $1 - \varepsilon \le x \le 1$, and $f(x) = 0$ otherwise on $[0, 1]$. We continue them by periodicity on $\mathbb{R}$. We can even choose f and g in $\mathcal{C}^\infty(\mathbb{T})$. Then P_R is a projection in $\mathcal{A}_\alpha$. Now, let S be the element of $\mathcal{B}_\alpha$ given by $S = aU + b$, where $a(x) = g(x + \alpha)^{1/2}\chi_{[1-a,1)}$, $b(x) = g(x)^{1/2}\chi_{[1-\alpha,1)}$. We check that both $a, b \in \mathcal{C}(\Omega)$, and that $SS^* = \chi_{[1-\alpha,1)}$ whereas $S^*S = P_R$. Thus $\chi_{[1-\alpha,1)} \approx P_R$ in $\mathcal{B}_\alpha$, which shows that the two generators of $\mathrm{K}_0(\mathcal{B}_\alpha)$, namely $[\mathbf{1}]$ and $[\chi_{[1-\alpha,1)}]$ can be taken in $\mathcal{A}_\alpha$. Thus $\mathrm{K}_0(\mathcal{B}_\alpha) = \mathrm{K}_0(\mathcal{A}_\alpha)$. □

3) Denjoy's Diffeomorphism of the Circle [HE79]. We consider now the dynamical system $(\mathbb{T}, S)$, where S is an orientation preserving homeomorphism. One can lift S as an increasing function from $\mathbb{R}$ to $\mathbb{R}$, also denoted by S, such that $S(x+1) = S(x)+1$. By Poincaré's theorem [HE79], the rotation number $\alpha = \lim_{n\to\infty}\{S^n(x)-x\}/n$ is well defined modulo 1. We will assume that it is irrational. Then, S is uniquely ergodic, and has no periodic orbits. If μ is the corresponding invariant measure, we set $h(x) = \int_{[0,x]} d\mu$ to get $h(x+1) = h(x)+1$ and $h \circ S = R_\alpha \circ h$, showing that S and the rotation R_α are semi-conjugate. The support M of μ is always the unique minimal S-invariant closed subset of $\mathbb{T}$. Denjoy's theorem [HE79] asserts that if S has a bounded-variation first derivative, then $M = \mathbb{T}$ and h is a homeomorphism. In this latter case, h induces an isomorphism between the C^*-algebras $\mathcal{C}(\mathbb{T}) \rtimes_\alpha \mathbb{Z}$ and $\mathcal{C}(\mathbb{T}) \rtimes_S \mathbb{Z}$.

Examples with $M \neq \mathbb{T}$ have been constructed by Denjoy [HE79]. Then, M is a nowhere dense set without isolated points, so it is completely disconnected. Its complement is a countable union of intervals $(a_{j,n}, b_{j,n})$ (the 'gaps' of M) where $n \in \mathbb{N}$, $j \in J$ (J is a countable set), $a_{j,n} < b_{j,n}$, and $S^n(a_{j,0}) = a_{j,n}, S^n(b_{j,0}) = b_{j,n}$. h is constant on each gap, and we denote by $\theta_{j,n}$ its value on $(a_{j,n}, b_{j,n})$. It follows that $h(S(x)) = h(x)+\alpha \pmod 1$, namely $\theta_{j,n} = \{\theta_{j,0} + n\alpha\}$ ($\{x\}$ denotes the fractional part of x). From Theorem 15, we get [PS86]:

Proposition 5.3.2. *If S is a homeomorphism of $\mathbb{T}$, with irrational rotation number and minimal set $M \neq \mathbb{T}$, the values of the IDS of a self adjoint element of $\mathcal{C}(\mathbb{T}) \rtimes_S \mathbb{Z}$ or $\mathcal{C}(M) \rtimes_S \mathbb{Z}$ on spectral gaps belong to the $\mathbb{Z}$-module generated by the numbers $\{\theta_{j,0} - \theta_{j',0} + n\alpha\}$, where $j, j' \in J$.*

Remark. Every point x outside M is non-wandering, and $\lim_{n\to\pm\infty} \mathrm{dist}(S^n x, M) = 0$. Let a be the closest point of M from x. If v is a continuous function on $\mathbb{T}$, the potential $V_x(n) = v(S^{-n}x)$ differs from V_a by a sequence converging to zero as $|n| \to \infty$. In this way, the corresponding Schrödinger operator $H_x = \Delta + V_x$ differs from $H_a = \Delta + V_a$ by a potential converging to zero at infinity, namely a *localized impurity*. It will give rise to isolated eigenvalues of multiplicity one which cannot be seen in the IDS, since their multiplicity per unit length is zero. For example if v is supported by the gap $(a_{j,0}, b_{j,0})$, V_x is a continuous family of rank one operators vanishing on gaps $(a_{j',n}, b_{j',n})$ with $j' \neq j$. However, as x varies in $\mathbb{T}$, this eigenvalue also varies continuously, giving rise to a band in $\mathrm{Sp}(H) = \cup_{x\in\mathbb{T}} \mathrm{Sp}(H_x)$.

5.4 Automatic Sequences: Potentials Given by a Substitution

Let us consider now the situation where Ω is generated by a substitution (see Section 3.3). Namely let A be a finite set (the 'alphabet') with L elements, and let $A^* = \cup_{k\leq 1} A^k$ be the set of words. A substitution is a map ζ from A to A^*, which can be extended to A^* by concatenation. We denote by $M_{b,a}(\zeta)$ the occurrence number of the letter b in the word $\zeta(a)$: it gives an $L \times L$ matrix $M(\zeta)$ with integer coefficients and called the 'occurrence matrix'. Then if ζ and η are two substitutions we get

$$M(\zeta)M(\eta) = M(\zeta\eta) \ . \tag{5.4.1}$$

By Perron-Frobenius's theorem, $M(\zeta)$ has an eigenvalue θ of highest module which is positive with corresponding eigenvector V having non negative coordinates. If in addition $M(\zeta)$ is primitive, namely if there is an integer n such that $M(\zeta)^n$ has positive coefficients (a condition fulfilled whenever S3 is satisfied), this eigenvalue is simple and V has positive coordinates. We will normalize V by the condition $\sum_a V_a = 1$.

We now assume that the substitution ζ satisfies the hypotheses S1-S3 of Section 3.3. In particular there is a letter 0 such that the word $\zeta(0)$ begins with 0. So that there is an infinite word, or an infinite sequence of letters $u = \lim_{n\to\infty}\zeta^n(0)$. The occurrence number of a letter a in this sequence is then given by:

$$L(a) = \lim_{n\to\infty} M(\zeta^n)_{a,0}\left\{\sum_b M(\zeta^n)_{b,0}\right\}^{-1} = V_a \ . \tag{5.4.2}$$

The limit is reached as a consequence of the spectral theorem. In particular, if Ω is the two-sided hull of u, and if T is the two-sided shift, any T-invariant ergodic measure μ on Ω satisfies $\mu(\chi_a) = V_a$, where χ_a is the characteristic function of the set of doubly infinite sequences w of letters such that $w(0) = a$.

More generally, let A_N be the set of all words of length N in the sequence u. The substitution ζ induces on A_N a substitution ζ_N for any $N \geq 1$ as follows: A_N is now considered as the set of letters of a new alphabet. If $w \in A_N$ begins with a assume that $\zeta(w) = a_0a_1\cdots a_n$ while the length of $\zeta(a)$ is m. Then we set:

$$\zeta_N(w) = (a_0a_1\cdots a_{N-1})(a_1a_2\cdots a_N)\cdots(a_{m-1}a_m\cdots a_{m+N-2}) \ . \tag{5.4.3}$$

So defined, $|\zeta_N(w)| = |\zeta(a)|$ and we will extend it to A_N^* by concatenation. In much the same way, we get an occurrence matrix denoted by $M_N(\zeta)$, again satisfying (5.4.1), with integer coefficients. In particular, if the axioms S1–S3 are satisfied, $M_N(\zeta)$ is primitive and admits the same highest eigenvalues θ as $M(\zeta)$ [QU87, Proposition V.15], with a corresponding eigenvector $V^{(N)}$ having positive coordinates and normalized according to $\sum_w V_w^{(N)} = 1$. Now we remark that if $u = 0u_1\cdots u_{N-1}\cdots$ one has $\zeta_N^n(0u_1\cdots u_{N-1}) = (0u_1\cdots u_{N-1})(u_1\cdots u_N)\cdots$ in such a way that the occurrence number of a word w in u is nothing but the limit:

$$L(w) = \lim_{n\to\infty} M_N(\zeta^n)_{w,0^{(N)}}\left\{\sum_v M_N(\zeta^n)_{v,0^{(N)}}\right\}^{-1} = V_w^{(N)} \ , \tag{5.4.4}$$

where $0^{(N)} = 0u_1 \cdots u_{N-1}$. In particular any T-invariant ergodic measure μ on Ω satisfies $\mu(\chi_w) = V_w^{(N)}$, where χ_w is the characteristic function of the set of doubly infinite sequences of letters such that $v(0)v(1)\cdots v(N) = w$. This implies that Ω is uniquely ergodic [QU87 Theorem V.13].

Our main result for substitutions is the following:

Theorem 18. *Let H be given by Eq. (5.1.2) where the coefficients are determined by a substitution ζ on a finite alphabet A, which satisfies the hypothesis S1–S3. Then the values of the IDS of H on the spectral gaps (contained in $[0,1]$) belong to the $\mathbb{Z}[\theta^{-1}]$-module generated by the coordinates of the normalized positive eigenvectors V and $V^{(2)}$ of the occurrence matrices $M(\zeta)$ and $M_2(\zeta)$, where θ is the common highest eigenvalue of each of them.*

Proof. For p large enough, namely if $\theta^p >$ const. N, ζ_{N^p} is entirely determined on $w \in A_N$ by the knowledge of the first two letters of w. In the sequel, let p assume such a condition. To compute μ it is thus sufficient to compute the positive eigenvectors of all the $M_N(\zeta)$'s. Actually there is a remarkable property of these matrices namely $M_2(\zeta)$ will suffice to get it. To see this one defines $\pi_{N,2}$ as the map from A_N into A_2 which gives the restriction to the first two letters extended to the set of corresponding words by concatenation. One also defines $\tau_{2,N,p}$ as the map from A_2 into A_N^* given by:

(5.4.5)

$$\tau_{2,N,p}(w) = (a_0a_1\ldots a_{N-1})(a_1a_2\ldots a_N)\ldots(a_{|\zeta^p(a)|-1}a_{|\zeta^p(a)|}\cdots a_{|\zeta^p(a)|+N-2}) \ ,$$

where $\zeta^p(w) = a_0a_1\ldots a_n$ and w begins by a. Then one immediately obtains:

$$\tau_{2,N,p}\circ\pi_{N,2} = \zeta_N^p, \quad \pi_{N,2}\circ\tau_{2,N,p} = \zeta_2^p, \quad \zeta_N \circ_{2,N,p} = \tau_{2,N,p}\circ\zeta_2 \ . \tag{5.4.6}$$

If we denote by $M_{2,N,p}$ the occurrence matrix associated to $\tau_{2,N,p}$, we get :

$$M_{2,N,p}M_2 = M_N M_{2,N,p} \ . \tag{5.4.7}$$

In particular M_2 and M_N have the same non zero eigenvalues and $V_N = M_{2,N,p}(V^{(2)})$ is a positive eigenvector of M_N associated to the highest eigenvalue θ. We just need to normalize V_N to get the occurrence number of any word in u. This is done by remarking that $\sum_w \{M_{2,N,p}\}_{w,v} = |\tau_{2,N,p}(v)| = |\zeta^p(v_0)| = |\zeta_2^p(v)| = \sum_{v'}\{M_2^p\}_{v',v}$ if v begins by the letter v_0. Thus, $\sum_v \{M_{2,N,p}V^{(2)}\}_v = \sum_v\{M_2^pV^{(2)}\}_v = \theta^p \sum_v V_v^{(2)} = \theta^p$.

Consequently, the occurrence number of a word w in u is therefore given by the coordinates of $V^{(N)} = V_N/\sum_w\{V_N\}_w = V_N\theta^{-p}$, which, since $M_{2,N,p}$ has integer coefficients, belongs to the set of linear combinations with integer coefficients of the coordinates of $V^{(2)}$ divided by some power of θ. Since $M_{2,N,p}$ has integer coefficients, this set is the $\mathbb{Z}[\theta^{-1}]$-module generated by the coordinates of $V^{(2)}$, where $\mathbb{Z}[X]$ is the set of polynomials in X with integer coefficients. □

To illustrate this result, let us treat a few examples.

1) The Fibonacci Sequence is given by an alphabet with two letters $A = \{0,1\}$, and the substitution is $\zeta(0) = 01$, $\zeta(1) = 0$, which obeys S1-S3. The alphabet A_2 contains only the words $\{00, 01, 10\}$, giving $\zeta_2(00) = (01)(10)$, $\zeta_2(01) = (01)(10)$, $\zeta_2(10) = (00)$, and:

$$M(\zeta) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad M_2(\zeta) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \tag{5.4.8}$$

and the highest eigenvalue is given by $\theta = (\sqrt{5}+1)/2$, namely the inverse of the golden mean $\sigma = (\sqrt{5}-1)/2$. The corresponding eigenvectors V and V_2 are given by:

$$V = \begin{bmatrix} \sigma \\ 1-\sigma \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 2\sigma - 1 \\ 1-\sigma \\ 1-\sigma \end{bmatrix}. \tag{5.4.9}$$

Thus, since $\sigma^2 = 1 - \sigma$, the IDS on the gaps takes values of the form $m + n\sigma$ where $m, n \in \mathbb{Z}$.

2) The Thue-Morse sequence is again made with two letters with $\zeta(0) = 01$, $\zeta(1) = 10$. This sequence also obeys S1–S3. The alphabet A_2 contains the four words $\{00, 01, 10, 11\}$ and we get $\zeta_2(00) = (01)(10)$, $\zeta_2(01) = (01)(11)$, $\zeta_2(10) = (10)(00)$, $\zeta_2(11) = (10)(01)$. The two matrices $M(\zeta)$ and $M_2(\zeta)$ are given by:

$$M(\zeta) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad M_2(\zeta) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{5.4.10}$$

and the highest eigenvalue is given by $\theta = 2$. The corresponding eigenvectors are:

$$V = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \qquad V_2 = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/3 \\ 1/6 \end{bmatrix}. \tag{5.4.11}$$

Thus the IDS in gaps will take values in the set $1/3\mathbb{Z}[1/2] \cap [0,1]$, namely the set of numbers of the form $k/(3 \cdot 2^N)$ where $k \in N$ and $N \in \mathbb{N}$. We then remark that for the Hamiltonian $H = -\Delta + V$, where Δ is the discrete Laplacian on $\mathbb{Z}$, whereas $V(n)$ takes values V_0 or V_1 according to whether $u_n = 0$ or 1, it has been shown [BE90a] that all gaps corresponding to $k = 3j+1$ or $3j+2$ ($j \in N$) are indeed open, whereas the others are closed due to a special symmetry of the Thue-Morse potential. However a generic perturbation of V in $\mathcal{C}(\Omega)$ will open these gaps too.

3) The period-doubling sequence [BB90] is also defined with two letters by $\zeta(0) = 01, \zeta(1) = 00$. This sequence also obeys S1–S3. We get $A_2 = \{00, 01, 10\}$

and $\zeta_2(00) = (01)(10), \zeta_2(01) = (01)(10), \zeta_2(10) = (00)(00)$. The two matrices $M(\zeta)$ and $M_2(\zeta)$ are given by:

$$M(\zeta) = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad M_2(\zeta) \begin{bmatrix} 0 & 0 & 2 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \tag{5.4.12}$$

with the highest eigenvalue $\theta = 2$, and eigenvectors $V = (2/3, 1/3)$, whereas $V_2 = (1/3, 1/3, 1/3)$. So again the IDS takes values on gaps in the set of $k/(3 \cdot 2^N)$ where $k \in \mathbb{N}$ and $N \in \mathbb{N}$. If $H = -\Delta + V$, where Δ, is the discrete Laplacian on $\mathbb{Z}$, whereas $V(n)$ takes values V_0 or V_1 according to whether $u_n = 0$ or 1, it has been shown [BB90] that indeed all gaps are open.

4) The Rudin-Shapiro sequence $(r_n)_{n\geq 0}$ [RU59, SH51, QU87] is defined recursively by $r_0 = 1, r_{2n} = r_n, r_{2n+1} = (-1)^n r_n$. It is actually given by $r_n = (-1)^{f(n)}$ where $f(n)$ is the number of 11 in the dyadic representation of n. It can also be defined through the substitution involving 4 letters [CK80] given by $\zeta(0) = 02, \zeta(1) = 32, \zeta(2) = 01, \zeta(3) = 31$. If $u = \lim_{n\to\infty} \zeta^n(0)$, and if τ is the map from A to $\{-1, +1\}$ given by $\tau(0) = \tau(2) = 1, \tau(1) = \tau(3) = -1$, then $\tau(u_n) = r_n$. The alphabet A_2 contains the eight words $\{01, 02, 10, 13, 20, 23, 31, 32\}$ and we get $\zeta_2(01) = (02)(23), \zeta_2(02) = (02)(20), \zeta_2(10) = (32)(20), \zeta_2(13) = (32)(23), \zeta_2(20) = (01)(10), \zeta_2(23) = (01)(13), \zeta_2(31) = (31)(13), \zeta_2(32) = (31)(10)$. The two matrices $M(\zeta)$ and $M_2(\zeta)$ are given by Eq. (5.4.13a and b) below. We get $\theta = 2$, and $V = (1/4, 1/4, 1/4, 1/4)$, whereas $V_2 = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$. Thus the values of the IDS on gaps are of the form $k2^{-N}$ where $k \in \mathbb{N}$ and $N \in \mathbb{N}$. The structure of the gaps in that case is quite involved [LU90] and no rigorous result has been proved yet in this case.

$$M(\zeta) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \tag{5.4.13a}$$

$$M_2(\zeta) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.4.13b}$$

6. Gap Labelling Theorems

Whereas in Section 5 we investigated 1D discrete Schrödinger operators it remains to compute the gap labels in a more general situation, namely either for discrete Hamiltonians in higher dimension, or for Hamiltonians given by homogeneous pseudodifferential operators on $\mathbb{R}^n$.

It turns out that this latter case can be solved using geometrical techniques coming from the study of smooth foliations on manifolds and developed by A. Connes [CO82]. On the other hand, the discrete case which corresponds in the physicist's language to the so-called "tight-binding representation" [BE86], can be solved by associating a continuous system, its suspension, the C^*-algebra of which (i.e. the Non-Commutative Brillouin zone) being 'Morita equivalent' to the algebra of the discrete case. As a result, their K-groups are the same and the gap labelling of the discrete version can be computed through Connes formulæ from the continuous one.

Specializing to one dimensional situations, we get a much more precise result in the case of ODE's thanks to the approach proposed by R. Johnson [JM82, JO83, JO86]. It gives a generalization of the Sturm-Liouville gap labelling theorem (see Section 1.5) for homogeneous 1D systems. Very recently [JO90] he proposed an extension of this theory to odd dimensions for Schrödinger operators.

6.1 Connes Formulæ for Group Actions

In Section 2 we have shown that the Non-Commutative Brillouin zone for a homogeneous Hamiltonian H acting on $L^2(\mathbb{R}^n)$, is given by the C^*-algebra $\mathcal{A} = C(\Omega) \rtimes_T \mathbb{R}^n$, where Ω is a compact metrizable space and T is a continuous action of the translation group $\mathbb{R}^n$ on Ω. Let $\mathbf{P}$ be a T-invariant ergodic probability measure on Ω. Then we get a trace $\tau_{\mathbf{P}}$ on $\mathcal{C}(\Omega) \rtimes_T \mathbb{R}^n$ equal for $\mathbf{P}$-almost all $\omega \in \Omega$ to the trace per unit volume of the representative H_ω of H. If a uniform magnetic field is present, the Non-Commutative Brillouin zone is $\mathcal{A} = C^*(\Omega \times \mathbb{R}^n, B)$, described in Section 2.5.

If H is bounded below, Shubin's formula asserts that the IDS on spectral gaps of H is equal to $\tau(\chi_{\leq E}(H))$ where $\chi_{\leq E}(H)$ is the eigenprojection of H corresponding to energies smaller than E. Since E is in a spectral gap, the continuous functional calculus implies that $\chi_{\leq E}(H)$ is a projection in the C^*-algebra $\mathcal{A}$ and therefore the IDS takes values in the countable subgroup $\tau_*(\mathrm{K}_0(\mathcal{A}))$. Thus as in Section 5 we want to get rules for calculating this group. The first important tool is Theorem 14 of A. Connes [CO81] on crossed products, which relates the K-groups of the Non-Commutative Brillouin zone to the topology of the space Ω. This space represents the lack of translation invariance of H, namely the amount of disorder in the system described by H. An important consequence is the following

Proposition 6.1.1. *We get:*

(i) $\mathrm{K}_0(C^*(\Omega \times \mathbb{R}^n, B)) \approx \mathrm{K}_0(\mathcal{C}(\Omega))$ whenever n is even,

(ii) $\mathrm{K}_0(C^*(\Omega \times \mathbb{R}^n, B)) \approx \mathrm{K}_1(\mathcal{C}(\Omega))$ whenever n is odd.

Sketch of the proof. The main remark about $C^*(\Omega \times \mathbb{R}^n, B)$ is that it can be written as a double crossed product [XI88] as follows $C^*(\Omega \times \mathbb{R}^n, B) \approx ((\mathcal{C}(\Omega) \rtimes_{T'} \mathbb{R}^{l+s}) \rtimes_\beta \mathbb{R}^l)$, that is to be described below. Theorem 14 used twice, we then get the result.

Now the double crossed product structure comes from the way the magnetic field acts (see Section 2.5, Eq. (2.5.6)): there is a decomposition of $x \in \mathbb{R}^n$ into $x = (x_+ \oplus x_- \oplus x_0) \in \mathcal{E}_+ \oplus \mathcal{E}_- \oplus \mathcal{E}_0$ such that $\dim(E_\pm) = l$, $\dim(\mathcal{E}_0) = s = n - 2l$, and $(Bx)_+ = \mathbf{b}x_-$, $(Bx)_- = -\mathbf{b}x_0$, $(Bx)_0 = 0$, where $\mathbf{b}$ is an $l \times l$ real symmetric matrix. Then if f is a continuous function with compact support on $\mathbb{R}^n$, we define $\underline{f}$ by $\underline{f}(\omega, x) = f(\omega, x)e^{\mathbf{i}\pi(e/h)\langle x_+ | \mathbf{b}x_- \rangle}$. The map $f \to \underline{f}$ defines a *-isomorphism between $C^*(\Omega \times \mathbb{R}^n, B)$ and an algebra $\mathcal{A}'$ on which the structure is

$$
(6.1.1) \qquad
\begin{aligned}
\underline{f}_1 \underline{f}_2(\omega, x) &= \int_{\mathbb{R}^n} d^n y \underline{f}_1(\omega, x) \underline{f}_2(T^{-y}\omega, x - y) e^{\mathbf{i}\pi(e/h)\langle x_+ - y_+ | \mathbf{b}y \rangle} , \\
\underline{f}^*(\omega, x) &= \underline{f}(T^{-x}\omega, -x) e^{\mathbf{i}2\pi(e/h)\langle x_+ | \mathbf{b}x_- \rangle} .
\end{aligned}
$$

If we define on the crossed product $\mathcal{C}(\Omega) \rtimes_{T'} \mathcal{E}_+ \oplus \mathcal{E}_0$ (where T' is the restriction of T to $\mathcal{E}_+ \oplus \mathcal{E}_0$), a group of *-automorphisms by $\beta_{x''}(g)(\omega, x') = g(T'^{-x''}\omega, x')e^{\mathbf{i}\pi(e/h)\langle x' | \mathbf{b}x'' \rangle}$ for $x' \in \mathcal{E}_+ \oplus \mathcal{E}_0, x'' \in \mathcal{E}_-$, it is then tedious but elementary to check that $(\mathcal{C}(\Omega) \rtimes_{T'} \mathcal{E}_+ \oplus \mathcal{E}_0) \rtimes_\beta \mathcal{E}_-$ is isomorphic to $\mathcal{A}'$. □

This result shows that there is certainly a difference in the treatment of odd or even dimensions. In the odd case, we associate to the projection $\chi_{\leq E}(H)$ an invertible matrix valued map on Ω, whereas in the even case, one has to find a fiber bundle over Ω, or equivalently a projection valued map on Ω. As we will see this problem is not yet solved. A solution has been found by R. Johnson in 1D for Schrödinger operators, and he has recently found an important step to treat the odd dimensional case.

However, there is another general result by A. Connes which permits to compute the group $\tau_*(\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{R}^n))$ in many relevant situations. Let us assume that Ω is a smooth manifold, and that the action of $\mathbb{R}^n$ is smooth namely that T^a is a diffeomorphism for $a \in \mathbb{R}^n$. Therefore, one gets smooth vector fields $X_1, X_2, \cdots, X_n$ on Ω formally defined by

$$
(6.1.2) \qquad X_\mu(\omega) = \left. \frac{\partial T^a \omega}{\partial a_\mu} \right|_{a=0} , \qquad 1 \leq \mu \leq n .
$$

Now a smooth differential form η of degree n on Ω, will give a volume element on each n-dimensional subspace of the tangent space $T_\omega \Omega$ at $\omega \in \Omega$. Evaluating it on the subspace tangent to the orbits of $\mathbb{R}^n$ gives rise to the smooth function $\langle \eta | X_1 \wedge X_2 \wedge \cdots \wedge X_n \rangle$ on Ω. The averaged value of this function

$$(6.1.3) \qquad \langle C|\eta\rangle = \int_{\Omega} \mathbf{P}(d\omega)\langle \eta|X_1 \wedge X_2 \wedge \cdots \wedge X_n\rangle$$

defines a de Rham current of degree n called the 'Ruelle-Sullivan current' of the smooth dynamical system $(\Omega, T, \mathbf{P})$. Its main properties are the following:

(i) C is *closed* namely $\langle C|d\theta\rangle = 0$ for any $(n-1)$-form θ

(ii) C is *positive* namely for any n-form η positive along the orbits of $\mathbb{R}^n$, $\langle C|\eta\rangle \geq 0$ (η is positive along the orbits of $\mathbb{R}^n$ whenever $\langle \eta|X_1 \wedge X_2 \wedge \cdots \wedge X_n\rangle \geq 0$).

One can show [CO82] that any such current is automatically of the form given by Eq. (6.1.3). Since it is closed, it defines a class [C] in the de Rham homology group $\mathrm{H}_n(\Omega, \mathbb{R})$, and therefore, its evaluation $\langle C|\eta\rangle$ on a *closed* form η can also be written in terms of the duality between the homology and the cohomology as $\langle [C]|[\eta]\rangle$ where $[\eta]$ is the cohomology class of η. Recall that a closed n-form η has integer coefficients whenever its evaluation on any n-cycle is an integer. The set of such forms defines a discrete countable subgroup $\mathrm{H}^n(\Omega, \mathbb{Z})$ of the n-th cohomology group.

The next theorem by A. Connes is the main result concerning the gap labelling in the continuous case [CO82]:

Theorem 19. *If $\mathbb{R}^n$ acts freely on Ω by means of diffeomorphisms, the countable subgroup $\tau_*(\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{R}^n))$ of $\mathbb{R}$ coincides with the group $\langle [C]|\mathrm{H}^n(\Omega, \mathbb{Z})\rangle$ obtained by evaluating the Ruelle-Sullivan current C on the n-th cohomology group with integer coefficients.*

In practice, it will require the calculation of a set of closed n-forms with integer coefficients generating $\mathrm{H}^n(\Omega, \mathbb{Z})$. The evaluation of C on it is then purely computational.

Let us now relate the Proposition 6.1.1 to Theorem 19 by showing how the trace acts on $\mathrm{K}_{i+n}(\mathcal{C}(\Omega))$ through the isomorphism with $\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{R}^n)$. In the case $n = 1$, $\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_T \mathbb{R})$ is isomorphic to $\mathrm{K}_1(\mathcal{C}(\Omega))$. A typical element of the latter may be generated by a smooth map $\omega \in \Omega \to U(\omega) \in \mathrm{GL}_N(\mathbb{C})$. Then we get a closed 1-form by considering $\eta_U = (1/2\mathbf{i}\pi)\mathrm{Tr}(U^{-1}dU)$. It represents the differential of the logarithm of $\mathrm{Det}(U)$ (up to the normalization factor). It is well known that the variation of $\mathrm{LogDet}(U)$ on any closed path in Ω is an integer multiple of $2\mathbf{i}\pi$, which implies that η has integer coefficients. Actually any element in $\mathrm{H}^1(\Omega, \mathbb{Z})$ can be obtained in this way and therefore it is enough to consider numbers of the form $\langle C|\eta_U\rangle$. It is simple to check that if U is homotopic to U', η_U and $\eta_{U'}$ are also homotopic, and therefore they admit the same equivalence class in $\mathrm{H}_1(\Omega, \mathbb{Z})$. Thus $U \to \eta_U$ defines a map η_* from $\mathrm{K}_1(\mathcal{C}(\Omega))$ into $\mathrm{H}^1(\Omega, \mathbb{Z})$ Moreover, it is elementary to check that $\eta_{UV} = \eta_U + \eta_V$, showing that η_* is actually a group homomorphism. Using Connes's Theorem 19 and Birkhoff's ergodic theorem, all possible gap labels are given by numbers (where $U_\omega(a) = U(T^{-a}\omega)$)

$$
(6.1.4)\quad
\begin{aligned}
\langle [C]|\eta_*[U]\rangle &= \int_\Omega \mathbf{P}(d\omega)\langle (1/2\mathbf{i}\pi)\mathrm{Tr}(U^{-1}dU)|X\rangle = \\
&= \lim_{L\to\infty}(1/2\mathbf{i}\pi)(1/2L)\int_{-L}^{+L} da\,\mathrm{Tr}(U_\omega(a)^{-1}dU_\omega(a)/da)\,,
\end{aligned}
$$

which is nothing but the average of the variation of the phase of $\mathrm{Det}(U)$ along $\mathbf{P}$-almost every orbit. We will see later on how to associate canonically to each gap of a 1D Schrödinger equation a map $\omega\in\Omega\to U(\omega)\in\mathbb{C}\backslash\{0\}$.

We can generalize this construction to the odd dimensional case by considering the n-form $\eta_U^{(n)} = c_n\mathrm{Tr}((U^{-1}dU)^n)$ instead, where c_n is a suitable normalization factor insuring that $\eta_U^{(n)}$ has integer coefficients.

If now $n=2$, a typical element of $\mathrm{K}_0(\mathcal{C}(\Omega))$ is given by a smooth map $\omega\in\Omega\to P(\omega)\in M_N(\mathbb{C})$ such that $P(\omega)^2=P(\omega)$. Such a map defines a fiber bundle over Ω by taking the image of the projection $P(\omega)$ as the fiber above ω. A closed differential 2-form is then given by the trace of the curvature of this bundle, namely the second Chern class $\theta_P=(1/2\mathbf{i}\pi)\mathrm{Tr}(PdPdP)$. That θ_P has integer coefficients is a classical result about Chern classes. The map $P\to\theta_P$ defines also a group homomorphism θ_* between $\mathrm{K}_0(\mathcal{C}(\Omega))$ and $\mathrm{H}^2(\Omega,\mathbb{Z})$. It is surjective also (but not injective!), so that gap labels are provided by numbers $\langle [C]|\theta_*[P]\rangle$ for all possible P's. Remark that using again Birkhoff's ergodic theorem and setting $P_\omega(a)=P(T^{-a}\omega)$,

(6.1.5)

$$
\langle [C]|\theta_*[P]\rangle = \lim_{L\to\infty}(1/2\mathbf{i}\pi)(1/2L)\int_{[-L,L]} da\mathrm{Tr}(P_\omega[\partial P_\omega/\partial a_1,\partial P_\omega/\partial a_2])\,,
$$

which is the averaged Chern class of the bundle over Ω defined by P along $\mathbf{P}$-almost every orbit. The main problem is that nobody yet has been able to associate explicitly such a bundle to a given spectral gap of the original Hamiltonian H.

The generalization for $n=2p$ consists in replacing θ_P by the higher Chern classes, namely $\theta_P^{(n)}=c_n\mathrm{Tr}(P(dPdP)^{n/2})$ where c_n is a suitable normalization constant to make sure that we have an n-form with integer coefficients.

Let us finish by remarking that formulae (6.1.4) and (6.1.5) do not require the smoothness of Ω. We conjecture that they are still true whenever Ω is a compact space, owing to the fact that it is always possible to regularize any continuous function over Ω along the orbits of $\mathbb{R}^n$ by means of a convolution with a smooth function on $\mathbb{R}^n$.

6.2 Gap Labelling Theorems for Quasiperiodic Hamiltonians on $\mathbb{R}^n$

Thanks to Theorem 19, we are able to compute possible gap labels in full generality, at least if the disorder space Ω is a manifold on which $\mathbb{R}^n$ acts freely by diffeomorphisms. By a free action, we mean that if there are $\omega\in\Omega\,,\ a,b\in\mathbb{R}^n$ such that $T^a\omega=T^b\omega$ then $a=b$.

Our first important result concerns the quasi periodic potentials. Recall that V is a quasi periodic function on $\mathbb{R}^n$ whenever there exists $\nu > n$, and a continuous function $\mathcal{V}$ on $\mathbb{R}^\nu$ periodic of period 1 in each variable, such that if [BO47]

$$V_\omega(x_1, \cdots, x_n) = \mathcal{V}(\omega_1 - \sum_j \alpha_{1j} x_j, \cdots, \omega_\nu - \sum_j \alpha_{\nu j} x_j) , \tag{6.2.1}$$

then $V_{\omega=0} = V$. Moreover the rectangular $\nu \times n$ matrix $\alpha = ((\alpha_{\mu j}))$ is called the 'frequency matrix'. It is 'irrational' if the subspace $\Delta_\alpha = \{\alpha x ; x \in \mathbb{R}^n\}$ intersects the lattice $\mathbb{Z}^\nu$ at $\{0\}$ only. The hull of such a function is then the torus $\Omega = \mathbb{T}^\nu$ and $\mathbb{R}^n$ acts on it through the translation $T^x \omega = \omega + \alpha x$. This action is always smooth (it is actually analytic), and it is free if and only if α is irrational.

There is a unique T-invariant ergodic probability measure on $\mathbb{T}^\nu$ namely the Lebesgue measure $\mathbf{P}(d\omega) = d^\nu \omega$. The action of $\mathbb{R}^n$ is defined by the n constant vector fields $X_i = (\alpha_{\mu j})_{\mu \in [1,\nu]}$. The calculation of the n-th cohomology group of $\mathbb{T}^\nu$ is actually quite easy. The generators are the n-forms $d\omega_{i(1)} \wedge \cdots \wedge d\omega_{i(n)}$ where $1 \leq i(1) < i(2) < \cdots < i(n) \leq \nu$. By definition, they have integer coefficients so that $\mathrm{H}_n(\mathbb{T}^\nu, \mathbb{Z}) \approx \mathbb{Z}^N$ with $N = \nu!/(\nu - n)!n!$. Therefore we immediately get [BL85]

Proposition 6.2.1. *Let H be a pseudo-differential operator on $\mathbb{R}^n$ bounded from below with quasiperiodic coefficients, the frequency matrix α of which being irrational. Then its* IDS *on spectral gaps takes values in the dense subgroup $\sum_\beta n_\beta \det(\beta)$ where the sum runs over the set of square submatrices β of maximal rank of α.*

6.3 Johnson's Approach for Schrödinger Operators

Let us consider now the one-dimensional Schrödinger operator acting on $L^2(\mathbb{R})$ by

$$H_\omega \psi(x) = -d^2\psi/dx^2 + v(T^{-x}\omega)\psi(x) = E\psi(x) , \tag{6.3.1}$$

where Ω is a compact metrizable space, $\omega \in \Omega$, T is a group action of $\mathbb{R}$ by homeomorphisms and v is a continuous real function on Ω. This equation can be written as a first order differential system:

$$d\Psi/dx = M(T^{-x}\omega)\Psi(x) , \tag{6.3.2}$$

with

$$\Psi(x) = \begin{bmatrix} \psi(x) \\ \frac{\psi'(x)}{\sqrt{E}} \end{bmatrix} , \qquad M(\omega) = \begin{bmatrix} 0 & \sqrt{E} \\ -\sqrt{E} + \frac{v(\omega)}{\sqrt{E}} & 0 \end{bmatrix} , \tag{6.3.3}$$

The full spectrum of H is the union $\mathrm{sp}(H) = \cup_{\omega\in\Omega}\mathrm{Sp}(H_\omega)$. Remark that if $\omega \in \Omega$ has a dense orbit, then $\mathrm{sp}(H) = \mathrm{Sp}(H_\omega)$. We will also denote by $\mathbf{P}$ a T-invariant ergodic probability measure on Ω.

Proposition 6.3.1. *If E belongs to a spectral gap of the operator $H_\omega = -d^2/dx^2 + v(T^{-x}\omega)$ defined by Eq. (6.3.1), up to normalization, there is a unique real solution Ψ_+ (resp. Ψ_-) of Eq. (6.3.2) converging to zero at $+\infty$ (resp. $-\infty$). This solution belongs to $\mathcal{C}^2(\mathbb{R})$, depends continuously on $\omega \in \Omega$ and decays exponentially fast at infinity. Moreover these two solutions give linearly independent vectors of $\mathbb{R}^2$ at every $x \in \mathbb{R}$.*

Sketch of a proof. Uniqueness: given two solutions Ψ_1 and Ψ_2 of (6.3.2) their Wronskian is constant for M is traceless. Thus if they both converge to zero at $+\infty$ their Wronskian vanishes identically and they must be equal up to a constant.

Existence: let us consider the Green function $G_E(\omega; x, y)$ defined by

$$G_E(\omega; x, y) = \langle \delta_x | (H_\omega - E)^{-1} \delta_y \rangle \,, \tag{6.3.4}$$

where δ_x is the Dirac measure at x. It is easy to check that $u_x = (\mathbf{1}+p^2)^{-1/2}\delta_x$ belongs to $L^2(\mathbb{R})$ if $p = -\mathbf{i}d/dx$. Then u_x is Hölder continuous of exponent $\alpha < 1/2$ and bounded with respect to x. Moreover, $E_0 \leq -(1 + \|v\|) \Rightarrow H_\omega - E_0 \geq \mathbf{1}+p^2$. Using the resolvent equation, it follows that $(\mathbf{1}+p^2)^{1/2}(H_\omega - z)^{-1}(\mathbf{1} + p^2)^{1/2}$ is a family of bounded operators, strongly continuous with respect to $\omega \in \Omega$, and norm-analytic with respect to z in the resolvent set of H_ω. Thus Eq. (6.3.4) defines for every $R > 0$, a bounded continuous function of the variables $(E, \omega, x, y) \in \mathbb{C}\backslash\mathrm{sp}(H) \times \Omega \times \mathbb{R}^2$, analytic in E, continuous in ω, Hölder continuous of exponent $\alpha < 1/2$ in (x, y).

On the other hand, in the sense of distributions, one gets

$$(\partial^2/\partial x^2 + v(T^{-x}\omega) - E)\, G_E(\omega; x, y) = \delta(x - y) \,. \tag{6.3.5}$$

In particular, if $x > x_0$, the map $\psi : x \in \mathbb{R} \to G_E(\omega; x, x_0)$ is a solution of (6.3.1), which, by uniqueness, can be continued in a unique way as a solution of (6.3.1) on the full line. Standard results on ODE's show that this solution is automatically in $\mathcal{C}^2(\mathbb{R})$.

To show that ψ decays exponentially fast at $+\infty$, let $W(a)$ be the unitary operator on $L^2(\mathbb{R})$ of multiplication by $e^{\mathbf{i}ax}$. Then $H_\omega(a) = W(a)H_\omega W(a)^* = H_\omega + 2ap + a^2$ and

$$\langle \delta_x | (H_\omega(a) - E)^{-1} \delta_y \rangle = e^{\mathbf{i}a(x-y)} G_E(\omega; x, y) \,. \tag{6.3.6}$$

Since p is H_ω-bounded, it follows that $H_\omega(a)$ is analytic with respect to a in the norm-resolvent sense. In particular, if $E \notin \mathrm{Sp}(H_\omega)$ there is $\rho_E > 0$ such that if $a \in \mathbb{C}$ and $|a| \leq \rho_E$, one has $\|(\mathbf{1}+p^2)^{1/2}(H_\omega(a) - E)^{-1}(\mathbf{1}+p^2)^{1/2}\| \leq C_E < \infty$. Thus, using (6.3.6) we get

$$(6.3.7) \qquad |G_E(\omega; x, y)| \leq C_E e^{-\rho_E |x-y|} \, .$$

Hence ψ is exponentially decaying at $+\infty$ and defines the solution Ψ_+ of the proposition. A similar argument holds for Ψ_-. □

Let us now consider the trivial bundle $\mathcal{E}_0 = \Omega \times \mathbb{R}^2$. Using the previous proposition, one can find for each $\omega \in \Omega$ a vector $\Phi_\pm(\omega)$, unique up to a normalization factor, such that the unique solution of (6.3.2) with initial condition $\Phi_\pm(\omega)$ at $x = 0$, decays exponentially fast at $\pm\infty$. Moreover, we know that these vectors vary continuously with respect to ω, and are linearly independent. This gives a splitting $\mathcal{E}_0 = \mathcal{E}_+ \oplus \mathcal{E}_-$ into the Whitney sum of two line bundles. Denoting by $\Phi_\pm(\omega, x)$ the solution of (6.3.2) with initial condition $\Phi_\pm(\omega)$ at $x = 0$, we remark that $\Phi_\pm(T^{-a}\omega, x - a)$ satisfies also the equation (6.3.2) and also converges to zero at $\pm\infty$. By the uniqueness theorem, it follows that $\Phi_\pm(\omega, x) \in \mathcal{E}_+(T^{-x}\omega)$. On the other hand, as x varies, $\Phi_\pm(\omega, x)$ never vanishes otherwise it would be identically zero, and therefore it rotates around the origin in $\mathbb{R}^2$. Let us parametrize $\Phi_\pm(\omega, x)$ by means of the angle $\theta_\pm(T^{-x}\omega)$ with the first axis. If x varies between $-L$ and $+L$, the total variation of this angle can be written as $\Delta_L\theta_\pm = \int_{-L}^{+L} d/dx(\theta_\pm(T^{-x}(\omega))$ and $\Delta_L\theta_{\pm/\pi}$ differs from the number of zeroes of $\psi_\pm$ in the interval $[-L, +L]$ by at most 2. Sturm-Liouville's theory (see Section 1.5) shows that this number of zeroes is equal to the number of eigenvalues smaller than E of the Hamiltonian H_L given by the restriction of H_ω to $[-L, +L]$ with suitable boundary conditions. Thus as $L \to \infty$, $\Delta_L\theta_\pm/2L\pi$ converges $\mathbf{P}$-almost surely to the IDS $\mathcal{N}(E)$. Using Birkhoff's ergodic theorem again, one gets

$$(6.3.8) \qquad \mathcal{N}(E) = 1/\pi \int_\Omega d\mathbf{P}(\omega) d/dx(\theta_\pm(T^{-x}\omega))|_{x=0} \, .$$

If Ω were a manifold, and if the action of $\mathbb{R}$ were smooth, denoting by X the vector field defined by the flow, we could write

$$(6.3.9) \qquad d/dx(\theta_\pm(T^{-x}\omega))|_{x=0} = \langle d\theta_\pm | X \rangle_\omega \, .$$

Moreover if we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$, $d\theta_\pm$ is homologous to $-\mathbf{i}\Phi_\pm^{-1} d\Phi_\pm$. Adopting these notations even if Ω is not a manifold, we get for the value of the IDS on the previous gap, a Connes formula with an explicit 1-form, namely the rotation angle of a solution vanishing at infinity [JM82, JO83, JO86, BE86]:

Theorem 20. *If $H_\omega = -d^2/dx^2 + v(T^{-x}\omega)$ acts on $L^2(\mathbb{R})$, where $v \in \mathcal{C}(\Omega)$, and $(\Omega, \mathbb{R}, \mathbf{P})$ is a topological dynamical system with $\mathbf{P}$ an invariant ergodic probability, the* IDS *on gaps is given by*

$$\mathcal{N}(E) = \tau(\chi_{\leq E}(H)) = 1/\mathbf{i}\pi \int_\Omega d\mathbf{P} \langle \Phi_\pm^{-1} d\Phi_\pm | X \rangle$$

where $\Phi_{\pm} = \Psi_{\pm} + \mathbf{i}\Psi'_{\pm}/\sqrt{E}$ *and* $\Psi_{\pm}$ *is the solution of the Schrödinger equation* $H_{\omega}\psi = E\psi$ *vanishing at* $\pm\infty$*.*

Remark. We get therefore a result compatible with the form found in Section 6.1. For indeed here the maps $\omega \in \Omega \to \Phi_{\pm}(\omega) \in \mathbb{C}$ have inverses and they define elements in $\mathrm{GL}(\mathcal{C}(\Omega))$.

The previous construction has been extended recently by R. Johnson [JO90] to the case of a Schrödinger operator of the type given by Eq. (6.3.1) on $\mathbb{R}^n$, for n odd. Let D be a bounded open domain with $\mathcal{C}^{\infty}$ boundary ∂D given as the zero set of a $\mathcal{C}^{\infty}$ function from $\mathbb{R}^n$ into $\mathbb{R}$, having no critical point on ∂D. Let $V(x)$ be a bounded continuous function on the closure of D. We consider the Schrödinger operator $H_D = -\Delta + V$ with Dirichlet boundary conditions on ∂D, and we assume that the potential V is such that H_D has a simple spectrum given by the eigenvalues $E_1 < E_2 < \cdots < E_i < \cdots$. If $(\psi_i)_{i\geq 1}$ are the corresponding eigenfunctions, there is $y \in D$ such that $\psi_i(y) \neq 0$ for every $i \geq 1$. Without loss of generality, one can choose the origin of the coordinates in such a way that $y = 0$.

As in (6.3.4) let $G_D(E; x, y)$ be the Green function of H_D at the energy E, and we set $g(x, E) = G_D(E; x, 0)$. Let ζ be the map from $D \times \mathbb{R}$ to the projective space $\mathbb{RP}(n)$ (the manifold of lines in $\mathbb{R}^{n+1}$) given by

$$\zeta(x, E) = [g(x, E), \partial_1 g(x, E), \cdots, \partial_n g(x, E)] \,, \tag{6.3.10}$$

where $\partial_i = \partial/\partial x_i$, and $[\mathbf{u}]$ denotes the line through the vector $\mathbf{u}$. This map is well defined at every 'non singular point', namely points where the vector in brackets does not vanish. Whenever g does not vanish, this line is nothing but the line $[1, \nabla_x \mathrm{Log}(g)]$.

Since n is odd, the space $\mathbb{RP}(n)$ is an orientable manifold. So let 'vol' be its volume form normalized in such a way that

$$\int_{\mathbb{RP}(n)} \mathrm{vol} \;=\; \Omega_n/2 \,, \qquad \Omega_n = \text{volume of the } n\text{-sphere } S^n \,. \tag{6.3.11}$$

The 'oscillation' of g is the integral of the pull-back $\zeta^*\mathrm{vol}$, which can be viewed as a rigorous definition of the differential of $\mathrm{Log}(g)$. More precisely, in order to avoid the singular points, we fix the interval $I = [E_0, E]$ where $E_0 < E$ .Let $\sum$ be the union of the hypersurfaces $\partial D \times I$ and $D \times \{E\}$, and we set:

$$O(D; E) = \int_{\sum} \zeta^*\mathrm{vol} - \sum_{j\in[1,M]} \mathrm{Ind}(s_j) \,, \tag{6.3.12}$$

where $s_1, \cdots, s_M$ are the singular points of g (see [JO90] for a precise definition of them), and $\mathrm{Ind}(s)$ is the index of the lift of the map ζ to S^n at the point s. The main result of Johnson is the following:

Proposition 6.3.2. *With the previous notation, if n is odd, and if E is not an eigenvalue of H_D, the oscillation number satisfies the formula:*

$$O(D;E) = -\Omega_n N_D(E) \tag{6.3.13}$$

where $N_D(E)$ is the number of eigenvalues of H_D in the interval $[E_0, E]$.

Let now assume that Ω is a topological compact metrizable space, endowed with an action of $\mathbb{R}^n$ by a group of homeomorphisms. $\mathbf{P}$ will denote an invariant ergodic probability measure on ω. Let v be a continuous function on Ω, and for $\omega \in \Omega$ let V_ω be the potential on $\mathbb{R}^n$ given by $V_\omega(x) = v(T^{-x}\omega)$. Let $(D_m)_{m\geq 1}$ be a Følner sequence (see Section 4.1) of bounded open domains in $\mathbb{R}^n$ satisfying the same conditions as D above, and covering $\mathbb{R}^n$. Then the mean oscillation number is defined as

$$O^{\sim}(E) = \lim_{n\to\infty} |D_n|^{-1} O(D_n;E) \ , \tag{6.3.14}$$

and by using the Proposition 6.3.2 it follows that the IDS is given by

$$\mathcal{N}(E) = -\Omega_n^{-1} O^{\sim}(E) \ . \tag{6.3.15}$$

We will conclude this Section by addressing the following question:

Problem. Prove that the mean oscillation number can be written in the form $\langle C|\eta\rangle$, where C is the Ruelle-Sullivan current and η a closed n-form related to ζ^*vol above.

6.4 Strong Morita Equivalence and Tight-Binding Approximation

Our last Section will concern the relation between Hamiltonians on the continuum and on a discrete lattice. In 1D this is related to the so-called 'Poincaré' section, and its converse the so-called 'suspension' construction. Let $(\Omega, T, \mathbb{R})$ be a dynamical system, namely Ω is a compact metrizable space endowed with an action of $\mathbb{R}$ by a group of homeomorphisms $(T^s)_{s\in\mathbb{R}}$. By a 'smooth transversal' (or a Poincaré section), we mean a compact subspace N such that for every $\omega \in \Omega$, the orbit of ω meets N and the set $\mathcal{L}(\omega) = \{s \in \mathbb{R}; T^{-s}\omega \in N\}$ is discrete, non empty and depends continuously on $\omega \in \Omega$. We then define on N the 'first return map' ϕ as follows: if $\zeta \in N$, let $t(\zeta)$ be the lowest positive real number t such that $T^t\zeta \in N$, and we set $\phi(\zeta) = T^{t(\zeta)}\zeta$. By hypothesis on N the map $\zeta \in N \to t(\zeta) \in (0,\infty)$ is continuous (in particular there is $t_- > 0$ such that $t(\zeta) \geq t_-$ for all $\zeta \in N$). Then (N, ϕ) gives a $\mathbb{Z}$-action on N.

Conversely, let (N, ϕ) be a $\mathbb{Z}$-action on a compact metrizable space N, and let $\zeta \in N \to t(\zeta) \in (0,\infty)$ be a continuous function. Let us consider on the space $N \times \mathbb{R}$, the map $\Phi : (\zeta, t) \to (\phi(\zeta), t - t(\zeta))$. The suspension of N is then the space $SN = N \times \mathbb{R}/\Phi$, namely the compact topological space obtained by identifying (ζ, t) with $\Phi(\zeta, t)$. We then consider the flow $T^s : (\zeta, t) \to (\zeta, t + s)$ on $N \times \mathbb{R}$. Since T and Φ commute, it follows that T defines an $\mathbb{R}$-action on SN. Then N can be identified with the transversal $N \times \{0\}$ in SN, and one can easily check that the first return map coincides with ϕ. If now N is a smooth transversal of the dynamical system $(\Omega, T, \mathbb{R})$ the map $h : (\zeta, t) \in N \times \mathbb{R} \to T^t\zeta \in \Omega$ defines a homeomorphism of SN onto Ω which intertwins the corresponding $\mathbb{R}$-actions.

The main question is to find the relation between the C^*-algebras $\mathcal{B} = C(\Omega) \rtimes_T \mathbb{R}$ and $\mathcal{C} = C(N) \rtimes_\phi \mathbb{Z}$. These two algebras are actually 'strongly Morita equivalent' [RI82, AMS81]. In particular they are 'stably isomorphic' namely $\mathcal{B} \otimes \mathcal{K}$ and $C \otimes \mathcal{K}$ are isomorphic, implying that they have the same K-groups. In this Section we wish to develop this theory to apply to our problem.

More generally, let $(\Omega, T, \mathbb{R}^n)$ be a dynamical system in n-dimensions. A smooth transversal is defined as before where $\mathbb{R}^n$ replaces $\mathbb{R}$. Then we get immediately:

Lemma 6.4.1. *If N is a smooth transversal of $(\Omega, T, \mathbb{R}^n)$ there is $R > 0$ such that if $\omega \in \Omega$, $s, t \in \mathcal{L}(\omega)$, then $|s - t| \geq R$. Moreover, $s \in \mathcal{L}(\omega)$ if and only if $(s - t) \in \mathcal{L}(T^{-t}\omega)$.*

Let now Γ_N be the set of pairs (ζ, s) where $\zeta \in N$ and $s \in \mathcal{L}(\zeta)$. Since $\mathcal{L}(\zeta)$ is discrete, Γ_N is endowed with an obvious topology which makes it a locally compact space. We define by $\mathcal{C}_0$ the space $\mathcal{C}_c(\Gamma_N)$ of continuous functions on Γ_N with compact support endowed with the structure of *-algebra given by:

$$(6.4.1)\quad c_1c_2(\zeta, s) = \sum_{t \in \mathcal{L}(\zeta)} c_1(\zeta, t)c_2(T^{-t}\zeta, s - t)\ , \qquad c^*(\zeta, t) = c(T^{-t}\zeta, -t)^*\ .$$

For $\zeta \in N$, we then denote by π_ζ the *-representation of $\mathcal{C}_0$ on $l^2(\mathcal{L}(\zeta))$ given by:

$$\pi_\zeta(c)\psi(s) = \sum_{t \in \mathcal{L}(\zeta)} c(T^{-s}\zeta, t - s)\psi(t)\ . \tag{6.4.2}$$

Then a C^*-norm is given by

$$\|c\| = \sup_{\zeta \in N} \|\pi_\zeta(c)\| \tag{6.4.3}$$

We will denote by $\mathcal{C} = C^*(N)$ the completion of $\mathcal{C}_0$ under this norm. We remark that $\mathcal{C}_0$ has a unit namely $\mathbf{1}(\zeta, s) = \delta_{s,0}$. We will denote by $\mathcal{B}_0$ the dense subalgebra $\mathcal{C}_c(\Omega \times \mathbb{R}^n)$ of $\mathcal{B} = \mathcal{C}(\Omega) \times_T \mathbb{R}^n$.

Remark. If $\zeta \in N$, the set $\mathcal{L}(\zeta)$ is a deformed lattice in $\mathbb{R}^n$. The C^*-algebra $C^*(N)$ is then exactly the algebra containing all Hamiltonians on this lattice. It is even possible to define properly the notion of covariance [CO79, CO82, BE86]. In [BE86], a connection between the Schrödinger operator in $\mathbb{R}^n$ with a potential given by a function on Ω, and operators on the lattices $\mathcal{L}(\zeta)$ has been described justifying the so-called 'tight-binding' representation in Solid State Physics.

Now let χ_0 be the space $\mathcal{C}_c(N\times\mathbb{R}^n)$. We define on it a structure of $\mathcal{B}_0-\mathcal{C}_0$-bimodule as follows (where $b\in\mathcal{B}_0, c\in\mathcal{C}_0, k\in\chi_0$):

$$
\begin{aligned}
kc(\zeta,s) &= \sum_{t\in\mathcal{L}(\zeta)} k(T^{-t}\zeta,s-t)c(T^{-t}\zeta,-t)\ ,\\
bk(\zeta,s) &= \int_{\mathbb{R}^n} d^n t b(T^s\zeta,t)k(\zeta,s-t)\ .
\end{aligned}
\tag{6.4.4}
$$

Moreover, we define a 'hermitian' structure [RI82] by means of scalar products with values in the algebras $\mathcal{B}_0$ and $\mathcal{C}_0$, namely if $k,k'\in\chi_0$:

$$
\begin{aligned}
\langle k|k'\rangle_{\mathcal{B}}(\omega,s) &= \sum_{t\in\mathcal{L}(\omega)} k(T^{-t}\omega,t)k'(T^{-t}\omega,t-s)^*\ ,\\
\langle k|k'\rangle_{\mathcal{C}}(\zeta,s) &= \int_{\mathbb{R}^n} d^n t k(\zeta,t)^* k'(T^{-t}\zeta,s-t)\ .
\end{aligned}
\tag{6.4.5}
$$

then we get the following properties which will be left to the reader [RI82]:

Lemma 6.4.2. *The hermitian bimodule structure defined by* (6.4.4) *and* (6.4.5) *satisfies the following identities (here $b,b'\in\mathcal{B}_0, c,c'\in\mathcal{C}_0, k,k',k''\in\chi_0$)*

(i) $bk\in\chi_0,\qquad kc\in\chi_0$.
(ii) $b(b'k)=(bb')k,\qquad (kc)c'=k(cc'),\qquad (bk)c=b(kc)$
(iii) $\langle k|k'\rangle_{\mathcal{B}}\in\mathcal{B}_0,\qquad \langle k|k'\rangle_{\mathcal{C}}\in\mathcal{C}_0$.
(iv) $\langle bk|k'\rangle_{\mathcal{B}}=b\langle k|k'\rangle_{\mathcal{B}}\qquad \langle k|k'c\rangle_{\mathcal{C}}=\langle k|k'\rangle_{\mathcal{C}}c$.
(v) $\langle k|k'\rangle_{\mathcal{B}}^*=\langle k'|k\rangle_{\mathcal{B}},\qquad \langle k|k'\rangle_{\mathcal{C}}^*=\langle k'|k\rangle_{\mathcal{C}}$.
(vi) $\langle k|k\rangle_{\mathcal{B}}\geq 0$ *and* $\langle k|k\rangle_{\mathcal{B}}=0\Rightarrow k=0$ *(same property with $\mathcal{C}$).*
(vii) $\langle kc|k'\rangle_{\mathcal{B}}=\langle k|k'c^*\rangle_{\mathcal{B}}\qquad \langle k|bk'\rangle_{\mathcal{C}}=\langle b^*k|k'\rangle_{\mathcal{C}}$
(viii) $k\langle k'|k''\rangle_{\mathcal{C}}=\langle k|k'\rangle_{\mathcal{B}}k''$.
(ix) $\langle kc|kc\rangle_{\mathcal{B}}\leq\|c\|^2\langle k|k\rangle_{\mathcal{B}}, \langle bk|bk\rangle_{\mathcal{C}}\leq\|b\|^2\langle k|k\rangle_{\mathcal{C}}$
(x) $\langle\chi_0|\chi_0\rangle_{\mathcal{B}}$ *is dense in* $\mathcal{B}_0$, $\langle\chi_0|\chi_0\rangle_{\mathcal{C}}=\mathcal{C}_0$.
(xi) There is $u\in\chi_0$ such that $\mathbf{1}_{\mathcal{C}}=\langle u|u\rangle_{\mathcal{C}}$.

We simply want to prove *(xi)* for it will be useful in practice. Let g be a continuous function on $\mathbb{R}^n$ with compact support contained in the open ball of radius $R/2$, and such that $\int_{\mathbb{R}^n} d^n t|g(t)|^2=1$. Then $u(\zeta,s)\equiv g(s)$ satisfies *(xi)*, in particular u is not unique. Then a norm is defined on χ_0 by setting:

$$
\|k\|=\|\langle k|k\rangle_{\mathcal{B}}\|^{1/2}=\|\langle k|k\rangle_{\mathcal{C}}\|^{1/2}\ .
\tag{6.4.6}
$$

We will denote by χ the $\mathcal{C}-\mathcal{B}$-bimodule obtained by completing χ_0 under this norm. It also satisfies all the properties of the Lemma 6.4.2.

Now let $\mathbf{P}$ be an invariant ergodic probability measure on Ω. Then it induces on N a probability measure μ with the property that whenever $s:\zeta\in N\to s(\zeta)\in\mathbb{R}^n$ is a continuous function such that $s(\zeta)\in\mathcal{L}(\zeta)$, then μ is invariant by the map $\phi_s(\zeta)=T^{s(\zeta)}\zeta$. Moreover any Borelian set F in N which is ϕ_s-invariant for every s satisfies $\mu(F)=0$ or 1. Then we get a trace on each of the algebras $\mathcal{B}_0$ and $\mathcal{C}_0$ via:

$$(6.4.7)\qquad \tau_{\mathbf{P}}(b)=\int_{\Omega}\mathbf{P}(d\omega)b(\omega,0)\,,\qquad \tau_{\mu}(c)=\int_{N}\mu(d\zeta)c(\zeta,0)\,.$$

We then get easily:

Lemma 6.4.3. *The traces $\tau_{\mathbf{P}}$ and τ_μ satisfy $\tau_{\mathbf{P}}(\langle k|k'\rangle_{\mathcal{B}})=\tau_\mu(\langle k'|k\rangle_{\mathcal{C}})$, for every k,k' in χ_0.*

Now let $c\in\mathcal{C}_0$, and $u\in\chi_0$ such that $\mathbf{1}_{\mathcal{C}}=\langle u|u\rangle_{\mathcal{C}}$. Then we set $\rho(c)=\langle uc|u\rangle_{\mathcal{B}}$. It is easy to check that $\rho(c)\in\mathcal{B}_0$, and that ρ is a *-endomorphism, namely it satisfies $\rho(cc')=\rho(c)\rho(c')$ and $\rho(c)^*=\rho(c^*)$. Thus ρ extends as a *-endomorphism from $\mathcal{C}$ into $\mathcal{B}$. Moreover, $\rho(\mathbf{1}_{\mathcal{C}})$ is a projection in $\mathcal{B}_0$, and for every projection P in $\mathcal{C}_0$, $\rho(P)$ is a projection in $\mathcal{B}_0$.

Conversely, given any projection Q in $\mathcal{B}_0$, using the property (x), we can find two finite families $\underline{k}=(k_i)_{1\leq i\leq I}$ and $\underline{k}'=(k_i')_{1\leq i\leq I}$ in χ_0 such that $\|Q-\sum_{i\in[1,I]}\langle k_i|k_i'\rangle_{\mathcal{B}}\|\leq\varepsilon<1/2$. It is actually possible to choose $\underline{k'}=\underline{k}$ if we accept to replace χ_0 by χ. Then by an argument similar to the one used in the proof of Lemma 4.2.2, one gets a projection $Q'\approx Q$ in the form $Q'=\sum_{i=1}^{I}\langle k_i|k_i\rangle_{\mathcal{B}}$ for some $\underline{k}\in\chi^I$. If we now set $P'=((\langle k_i|k_j\rangle_{\mathcal{C}}))_{i,j}$, we get a projection in the matrix algebra $M_I(\mathcal{C})$.

More generally through the replacement of $\mathcal{B}_0$ by $M_L(\mathcal{B}_0)$, of $\mathcal{C}_0$ by $M_N(\mathcal{C}_0)$, and of χ_0 by $M_{L\times N}(\chi_0)$, we get in an obvious way a $M_L(\mathcal{B}_0)-M_N(\mathcal{C}_0)$ hermitian bimodule, and any projection in $M_L(\mathcal{B}_0)$ will give rise in the same manner, to a projection in $M_{N'}(C)$, for some N'. This is the basic argument leading to [BR77, RI82]:

Theorem 21. *(i) The C^*-algebras $\mathcal{B}=\mathcal{C}(\Omega)\rtimes_T\mathbb{R}^n$ and $\mathcal{C}=C^*(N)$ are stably isomorphic, namely $\mathcal{B}\otimes\mathcal{K}$ and $\mathcal{C}\otimes\mathcal{K}$ are isomorphic (not in a canonical way).*
(ii) The K_0-groups of $\mathcal{B}$ and $\mathcal{C}$ are isomorphic.
(iii) Their images by the traces coincide namely $\tau_{\mathbf{P}}(\mathrm{K}_0(\mathcal{B}))=\tau_\mu(\mathrm{K}_0(\mathcal{C}))$.

Remark. The property (iii) is a direct consequence of the Lemma 6.4.3.

As a consequence of Theorem 21, the computation of the gap labelling for a discrete system is equivalent to that of its suspension. This gives immediately some results in practice. First of all let us consider a Hamiltonian on $\mathbb{Z}^n$ with quasiperiodic coefficients. It means that it has the form:

$$(6.4.8)\qquad H_\xi\psi(m)=\sum_{m'\in\mathbb{Z}^n}h(\xi-\alpha m,m'-m)\psi(m')\,,\qquad \psi\in l^2(\mathbb{Z}^n)\,,$$

where $h\in\mathcal{C}(\mathbb{T}^\nu)\rtimes_\alpha\mathbb{Z}^n$, and α is a $\nu\times n$ matrix with rationally independent real column vectors, acting on the n-dimensional torus $\mathbb{T}^\nu$ by translation namely $T^m\xi=\xi-\alpha m$.

The suspension of the dynamical system $(\mathbb{T}^\nu, \alpha, \mathbb{Z}^n)$ is actually given by $(\mathbb{T}^{\nu+n}, \beta, \mathbb{R}^n)$, where the matrix β is the $(\nu+n) \times n$ real matrix given by $\beta = [\alpha, \mathbf{1}_n]$ obtained by gluing together the $\nu \times n$ matrix α and the $n \times n$ identity matrix $\mathbf{1}_n$. Using the Proposition 6.2.1, we get the following result [BE81, EL82a, DS83, BL85].

Proposition 6.4.4. *Let H be as in Eq. (6.4.8). Then the* IDS *on spectral gaps takes values in the $\mathbb{Z}$-module in $\mathbb{R}$ generated by* 1 *and all the minors of the matrix α.*

In particular if $n = 1$, this module is generated by 1 and the components of the line α.

Another consequence was given by A. Connes, and concerns the absence of gaps in the spectrum.

Proposition 6.4.5. *Let Ω be a manifold such that the first cohomology group* $\mathrm{H}^1(\Omega, \mathbb{Z}) = 0$. *If ϕ is a minimal diffeomorphism, the algebra $\mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z}$ is simple, has a unit and no non trivial projection.*

In particular, if $(H_\omega)_{\omega \in \Omega}$ is a covariant family of self adjoint bounded operators on $l^2(\mathbb{Z})$ of the form:

$$H_\omega \psi(m) = \sum_{m' \in \mathbb{Z}} h(\phi^{-m}(\omega), m' - m)\psi(m') , \qquad \psi \in l^2(\mathbb{Z}^n) ,$$

with $h \in \mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z}$ then for every $\omega \in \Omega$, the spectrum of H_ω is connected.

Sketch of the proof [CO81]. First of all, the minimality of ϕ implies the simplicity of $\mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z}$ [SA79].

Let μ be a ϕ-invariant probability measure on Ω. Let $S\Omega$ be the suspension corresponding to the constant first return time $t(\omega) = 1$, then the measure $\mathbf{P} = \mu(d\omega)dt$ is invariant and ergodic on $S\Omega$. Using Theorem 21, the image of the K_0-group of $\mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z}$ by the trace τ_μ is identical with the image of the K_0-group of its suspension $\mathcal{C}(S\Omega) \rtimes_T \mathbb{R}$ by the induced trace τ_P. By Theorem 19, this last set is given by $\langle [C] | H^1(S\Omega, \mathbb{Z}) \rangle$, where C is the Ruelle-Sullivan current induced by $\mathbf{P}$. The flow on $S\Omega$ is then generated by the vector field $\partial/\partial t$. Since Ω is connected (ϕ is minimal), and $H^1(\Omega, \mathbb{Z}) = 0$, it follows that $H^1(S\Omega, \mathbb{Z}) = \mathbb{Z}$ with the 1-form dt as generator. Then one clearly has $\langle [C] | [dt] \rangle = 1$, for P is a probability, and we get $\tau_{\mathbf{P}}(\mathrm{K}_0(\mathcal{C}(S\Omega) \rtimes_T \mathbb{R})) = \tau_\mu(\mathrm{K}_0(\mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z})) = \mathbb{Z}$. Now let P be a projection in $\mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z}$, it follows that its trace is an integer, and therefore it must be 0 or 1, because $0 \leq P \leq \mathbf{1} \Rightarrow 0 \leq \tau_\mu(P) \leq 1$. It is elementary to check that since ϕ is minimal, the support of any invariant measure on Ω is Ω itself, implying that the trace τ_μ is faithful, and therefore $P = 0$ or $\mathbf{1}$.

In particular, since ϕ is minimal, and since $TH_\omega T^{-1} = H_{\phi(\omega)}$, the spectrum of H_ω is independent of $\omega \in \Omega$. If it had a gap $G = (a, b)$ then the eigenprojection $\chi_{(-\infty,E]}(H_\omega)$, with $E \in G$, would define a non trivial projection (i.e. different from 0 or 1) in the C^*-algebra $\mathcal{C}(\Omega) \rtimes_\phi \mathbb{Z}$ leading to a contradiction.

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