DYNAMICAL LOCALIZATION : MATHEMATICAL FRAMEWORK[1]

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Abstract

We reconsider the various definitions of dynamical localization in view of Solid State Physics analog. To treat the semiclassical limit on a rigorous background we introduce an algebraic formalism already used for disordered crystals. We get a complete analogy between classical motion and quantum motion in term of the language of Non Commutative Geometry. Various rigorous results on the semiclassical limit are obtained and we discuss the main difficulties in getting a rigorous approach of the Chirikov- Izrailev-Shepelyansky theory.

Keywords : dynamical localization / non commutative geometry / semiclassical limit

1 Introduction

One of the important results obtained during the last ten years in the field of quantum chaos is the analogy between Anderson's localization and the suppression of classical diffusion by quantum interferences. This is now called "dynamical localization". This analogy was described in the famous paper by Fishman, Grempel and Prange [1] interpreting the eigenvalues equation for the quasienergy in the kicked rotor (KR) problem in terms of a one dimensional chain with random potential. In the KR the sites of the chain correspond to quantized values of the angular momentum, giving localization in momentum space, instead, compared with disordered systems where the localization takes place in the real space.
The relation between quantum localization and classical dynamics in the quasi integrable regime is pretty well understood mathematically by mean of a quantum version of the KAM theorem [2] involving the control of tunneling effect. This regime corresponds, for disordered systems, to the case of strong disorder, whenever the model is diagonal dominant. In the strongly chaotic regime, however the situation is more involved no rigorous argument is known yet. This latter case has been emphasized by Shepelyansky [3] on the basis of a previous argument of Chirikov, Izrailev and Shepelyansky [4]. He proposed a formula relating the quantal localization length ℓ to the classical diffusion constant D. For The KR problem this formula reads :

$$\ell = \frac{1}{2}\frac{D}{\hbar^2} , \tag{1}$$

where $\hbar$ is Planck's constant. The constant 1/2 in front of (1) was discussed by Shepelyansky [3] on the basis of a numerical study.
Actually a very similar argument already existed in the Solid State Physics literature [5] which led Fishman, Prange and Griniasty [6] to rederive (1) through the finite size scaling theory of 1D disordered chains and Einstein's formula for the conductivity. However no self-contained satisfactory argument has

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been given yet to justify (1). In particular no-one knows whether higher order corrections are present in (1) either if they can be computed. The very same difficulty arises in justifying Kubo's formulae for transport coefficients and Einstein's relation still requires ergodic assumptions to be proved. Due to the relative simplicity of the KR model we may expect to find eventually a way of proving these formulae from first principles without using any kind of additionnal assumptions.

Mathematically Anderson's localization has been expressed in terms of the spectral type of the quantum Hamiltonian. In particular it has been proved rigorously that for one dimensional nearest neighbour interaction and random potential the spectrum is pure point and all eigenstates are exponentially localized [7, 8, 9]. In higher dimensions the same result holds at large disorder or on the band edges [10]. Dynamical localization was investigated for the first time on the pulsed rotor (PR) model (namely the kicked rotor with smooth kicks) at small coupling. It has been proved that the quasienergy spectrum is pure point with exponentially localized eigenstates in momentum space [11]. Later on using a global argument Howland [12] proved for a large class of Hamiltonians periodically perturbed in time that the quasienergy spectrum is pure point.

On the other hand the localization length of 1D chains is oftenly expressed through the Lyapunov exponent.

It can be computed by mean of the Thouless-Herbert-Jones formula [13, 14]. The Pastur-Ishii-Kotani theorem [15] insures that the Lyapunov exponent is positive on an open interval of energy if and only if the energy spectrum in this interval is purely singular. However examples such as the Lloyd's model [16] which gives localization and the Maryland's model [1] for which no state is localized have the same Lyapunov exponent showing that this expression for the localization length is not suitable for proving Anderson's localization. On the other hand it has been proved by Pastur [7] that the finiteness of the inverse participation ratio is a necessary and sufficient condition for the energy spectrum to have a pure point component.

Our aim in this lecture is to report on a recent work to be published elsewhere [17] in which we try to get a semiclassical formula for the localization length and relate it to the spectral type of the Floquet operator.

Physicists usually prefer to use the Schrödinger's point of view in its most elaborate version, namely the Feynman path integral [18]. Using stationnary phase method it leads to the so-called Gutzwiller formula [19]. We will show that for the kicked rotor problem, we get a finite dimensional Feynman path integral at finite times leading rigorously to a Gutzwiller-type formula. However we have not been able so far to draw any sound conclusion from it at least in the strongly chaotic regime.

We have preferred an algebraic approach in Heisenberg's spirit based upon Connes' non commutative geometry [20]. We will present several rigorous results which are relevant in our case. The Spectral Duality Theorem proposed by Bellissard and Testard [21] in 1981 and recently proved by Chojnacki [22] gives results on absence of localization, for instance in the kicked Harper model [23, 24, 25]. On the other hand we give a calculation of the inverse participation ratio and we recover Pastur's result. We also propose a definition for the localization length which will be related to the quantum diffusion constant along the corresponding chain; a finite localization length implies existence of a pure point spectrum. We also relate the localization length to the time average of the kinetic energy.

Beyond its capability in performing calculations this algebraic formalism also permits to get a rigorous control of the semiclassical limit of various quantities of interest such as correlation function. This last point will be discussed at the end of this lecture. The main result in our discussion is that the $\hbar \to 0$ limit is not reached uniformly in time unless the corresponding classical system is completely integrable. In a classically chaotic phase space region, there is a breaking time presumably of order $O(\hbar^{-2})$ beyond which the quantal correlation functions do not converge to the classical ones as $\hbar \to 0$. This breaking time should also depend upon the size of the considered phase space region. So that at small coupling where many resonances occur, there is presumably an infinite family of time scales appearing one after the others in the semiclassical limit.

2 Systems Periodically Driven in Time, Mathematical Framework

2.1 Models

We will consider time dependent quantal Hamiltonians $H(t)$, periodic of period T in time, namely $H(t+T) = H(t)$. The evolution is then described by the time dependent Schrödinger equation :

$$i\hbar \frac{\partial \Psi}{\partial t} = H(t)\Psi \ . \tag{2}$$

The solutions of (2) define an evolution operator, namely the evolution during one period of time defined by :

$$F_t = U(t, t+T) \Rightarrow \Psi_{t+T} = F_t^{-1}\Psi(t) \ . \tag{3}$$

Because of the periodicity, the long time evolution is descibed by the spectral properties of F_t, for $\Psi_{t+nT} = F_t^{-n}\Psi_t$. On the other hand, if $t \neq t'$ F_t and $F_{t'}$ are unitarily equivalent.
Among the simplest examples investigated so far in the literature is the pulsed rotor (PR) model describing a rigid rotor spinning around an axis, with a time dependent potential. If θ denotes the angle of rotation, L its angular momentum, the classical Hamiltonian takes the form :

$$\mathcal{H}_{PR}(t) = \frac{L^2}{2I} + V\left(\theta, \frac{2\pi t}{T}\right) \ , \tag{4}$$

where I is the moment of inertia and $V(\theta, \tau)$ is 2π-periodic and smooth in both variables θ, τ. The quantal analog is defined as the operator on $\mathcal{L}^2(\mathbf{T}, d\theta/2\pi)$ by (4) where L is replaced by the operator $\hat{L} = \frac{\hbar}{i}\frac{\partial}{\partial\theta}$ and $V(\theta, \tau)$ by the operator $V(\hat{\theta}, \tau)$ of multiplication by $V(\theta, \tau)$.

Kicked models are given by a Hamiltonian of the form :

$$H(t) = H_0 + V\sum_{n\in\mathbf{Z}} \delta(t - nT) \ , \tag{5}$$

where H_0 and V are suitable self adjoint operators. The δ-function may produce some technical difficulty, which can be controlled [26, 27]. However the Floquet's operator takes on an especially simple form namely :

$$F_0 = e^{iV/\hbar}e^{iH_0T/\hbar} \ . \tag{6}$$

The examples we are interested in have classically the following form :

$$\mathcal{H}(L, \theta, t) = \mathcal{H}_0(L) + V(\theta)\sum_{n\in\mathbf{Z}} \delta(t - nT) \ , \tag{7}$$

where θ is one angle and L the corresponding angular momentum. Here V is a 2π-periodic smooth function of θ, whereas $\mathcal{H}_0$ is smooth in L. The various explicit examples so far investigated in the literature are :

(i) the kicked rotor (KR) [28]

$$\mathcal{H}_0(L) = \frac{L^2}{2I} + \mu BL \ , \ V(\theta) = k\cos(\theta) \ , \tag{8}$$

where I represents the moment of inertia, B a uniform magnetic field parallel to the axis of rotation, μ the magnetic moment and k a measure of the potential strength.

(ii) The kicked Harper model (KH) [23, 25] for which

$$\mathcal{H}_0(L) = k_1\cos\left(\frac{L}{L_0}\right) \ , \ V(\theta) = k_2\cos(\theta) \ , \tag{9}$$

where k_1, k_2 are some coupling constants and L_0 is some unit of angular momentum.

(iii) The Lima-Shepelyansky (LS) modified kicked Harper model [24] :

$$\mathcal{H}_0(L) = k_1\left[\cos\left(\frac{\hat{L}}{L_0}\right) - \frac{k_1'}{2}\sin\left(\nu\frac{\hat{L}}{L_0}\right)\right], V(\theta) = k_2\left[\cos(\theta) - \frac{k_2'}{2}\sin(2\theta)\right] \ , \tag{10}$$

where k_1, k_1', k_2, k_2' are coupling constants and ν is some real number.
Let us notice that other models have also been studied : the kicked rotor with spin [29, 30], two or three coupled kicked rotors [31] where now the system admits two or three angles of rotation, Hamiltonians $H(t)$ with two or more independent frequencies in time, namely $H(t) = H'(2\pi t/T_1, 2\pi t/T_2, \ldots, 2\pi t/T_n)$ where $H'(\tau_1, \tau_2, \ldots, \tau_n)$ is 2π-periodic in $\tau_1, \tau_2, \ldots, \tau_n$ [32, 33].
All these models have their classical counterpart. For a kicked model given by (7), the classical equations of motion reduce to :

$$L_{n+1} = L_n - V'(\theta_n) \ , \ \theta_{n+1} = \theta_n + T\omega(L_{n+1}) \ , \tag{11}$$

where $\omega(L) = \partial H_0/\partial L$, $\theta_n = \theta(nT-0)$, $L_n = L(nT-0)$. For example the KR model is described by :

$$L_{n+1} = L_n + k\sin(\theta_n) \ , \ \theta_{n+1} = \theta_n + T\left(\frac{L_{n+1}}{I} + \mu B\right) \ , \tag{12}$$

so that if we set $A = T\left(\frac{L}{I} + \mu B\right)$, $K = kT/I$ we get :

$$A_{n+1} = A_n + K\sin(\theta_n) \ , \ \theta_{n+1} = \theta_n + A_{n+1} \ , \tag{13}$$

which is called the Standard Map [34]. The classical phase space for these models is a cylinder $\mathcal{C} = \mathbf{T}\times\mathbf{R}$ namely $\theta \in \mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ whereas $L \in \mathbf{R}$. However in many cases like for the KR or the KH, LS models with $\nu \in \mathbf{Z}$, the equations of motion are invariant by some translations in L. For example changing L into $L + 2\pi I/T$ in (12) does not change the equations of motion. Therefore one can see this motion as leaving on the corresponding quotient space, namely a 2-torus $\mathbf{T}^2$.
In all these models one must first determine the minimum number of dimensionless parameters. For instance, in the KR model, only one parameter survives in classical mechanics, namely the dimensionless coupling strength $K = kT/I$. The magnetic field does not play any role in the motion for it describes a global translation of the phase space in the L-direction. On the other hand whereas θ is a dimensionless variable (being an angle), L is not. In all these models, there is a natural constant L_0 such that L/L_0 be dimensionless. If the phase space is a torus we will then choose the new variable $A = \frac{L}{L_0} +$ const. and call it "action". The classical observables will then be given by 2π-periodic functions of (A, θ) in the torus case, whereas in the cylinder case, they will be periodic in θ and vanishing at infinity with respect to A. In the LS model with $\nu \notin \mathbf{Q}$, it is convenient to introduce an extra independent action variable $A_2 = \frac{\nu L}{L_0} +$ const. so that the phase space becomes a 3-torus foliated by the lines $L \in \mathbf{R} \mapsto \left(\frac{L}{L_0} + A_1(0), \frac{\nu L}{L_0} + A_2(0)\right)$. The classical observables will be 2π-periodic in (A_1, A_2, θ).
In the corresponding quantal models there is always one more parameter that will be denoted by γ called the dimensionless Planck's constant such that :

$$\gamma = \hbar \text{ const.} \tag{14}$$

For instance in the KR model,

$$\gamma = \frac{\hbar T}{I} = 4\pi\frac{\nu_{\text{QM}}}{\nu_{\text{CL}}} \ , \tag{15}$$

which represents the ratio between the frequency ν_{QM} of the free rotor and ν_{CL} the frequency of the external kicks. For the KH model, one finds $\gamma = \hbar/L_0$. The semiclassical limit will be reached whenever $\gamma \to 0$.

2.2 Feynman Path Integral for The KR Model

Let us consider the expression of the evolution operator given by (6) applied to the kicked rotor problem with magnetic field (8) :

$$F_{\text{KR}} = e^{iK\cos(\hat{\theta})/\gamma} e^{i\hat{A}^2/2\gamma} \ . \tag{16}$$

Usually we compute the matrix element $\langle\theta|F_{\text{KR}}^{-1}|\theta'\rangle = \langle\theta|e^{-i\hat{A}^2/2\gamma}|\theta'\rangle e^{-iK\cos(\theta')/\gamma}$.
Using Poisson's formula, if $x = -\mu BT$ we obtain

$$\langle\theta|F_{\text{KR}}^{-1}|\theta'\rangle = \sqrt{\frac{2\pi}{\gamma}} e^{-i\pi/4} \sum_l e^{i(2\pi l + \theta' - \theta - x)^2/2\gamma} e^{-iK\cos(\theta')/\gamma} \ . \tag{17}$$

If we iterate t-times we get after a suitable change of variables

$$\left(F_{\text{KR}}^{-t}\Psi\right)(u) = \frac{e^{-it\pi/4}}{(2\pi\gamma)^{t/2}} \int_{\mathbf{R}^t} du_1 \cdots du_t e^{i\mathcal{L}_t(u)/\gamma} \Psi(u_t) \ , \tag{18}$$

where $\mathcal{L}_t(u)$ is the Percival Lagrangean [35] namely

$$\mathcal{L}_t(u) = \sum_{s=1}^{t} \left(\frac{1}{2}(u_s + u_{s-1} - x)^2 - K\cos(u_s)\right) \ . \tag{19}$$

The usual way of treating such an integral consists in finding the stationnary phase points satisfying here $2u_s - u_{s+1} - u_{s-1} + K\sin(u_s) = 0$ for $1 \le s \le t-1$ and $u_t - u_{t-1} - x + K\sin(u_t) = 0$.

Therefore the trajectories of interest are determinated by $(\theta_0, A_0)\dots(\theta_t, A_t)$ such that $\theta_0 = u_0 \bmod 2\pi$ for the initial position and $A_{t+1} = x \bmod 2\pi$ for the final action.
Let us simply remark that the Percival Lagrangean works here because of the specific form of the kinetic energy $\hat{A}^2/2\gamma$ term. It would not work say for the kicked Harper problem, eventhough a Feynman path integral does exist. We will discuss the Gutzwiller formula later on.

2.3 The Non Commutative Phase Space

In the previous section we have described the classical frame for the models of interest, and also the quantum frame in the Schrödinger picture. However, as explained in the introduction, we will prefer to describe the quantum motion in the Heisenberg picture. For indeed, we believe that the semiclassical limit is easier to control in this framework than in Schrödinger's case [38].
As explained in the previous paragraph the classical phase space for the KR and for the KH models can be seen as a 2-torus generated by (θ, A). The corresponding classical observables will be 2π-periodic functions of (θ, A). Due to Fourier's expansions they will be algebraically generated by the two coordinate functions :

$$U : (\theta, A) \mapsto e^{i\theta} \in \mathbf{C} \ , \ V : (\theta, A) \mapsto e^{-iA} \ . \tag{20}$$

Quantum mechanically one also obtains the observables through the unitary operators U and V but they no longer commute. Actually their commutation relation becomes :

$$UV = e^{i\gamma} VU \ , \tag{21}$$

where γ is the dimensionless Planck's constant previously defined. In other words, everything looks like if the phase space would become "non commutative". By this we mean that the observables algebra becomes non commutative, but most of the geometrical properties of the phase space should survive for $\gamma \neq 0$. This is the key observation in the philosophy of "Non Commutative Geometry". It will be convenient to consider γ as a variable giving rise to a continuous "deformation" of our algebra or of our phase space.

For the LS model it is convenient to introduce two functions $V_1 = e^{iA_1}$, $V_2 = e^{iA_2}$ and the quantal commutation rules will be

$$V_1 V_2 = V_2 V_1 \ , \ UV_1 = e^{i\gamma} V_1 U \ , \ UV_2 = e^{i\nu\gamma} V_2 U \ . \tag{22}$$

We can therefore investigate also this situation easily in the same spirit. To make the exposition simpler we will ignore this latter case and focus on (13).
For I a closed interval of $\mathbf{R}$ let us denote by $\mathcal{A}_I^{(0)}$ the set of polynomials in U and V given by (20) with coefficients taken in the space $\mathcal{C}(I)$ of continuous functions on I. Introducing the following notation

$$W(m_1, m_2) = U^{m_1} V^{m_2} e^{-i\gamma m_1 m_2 / 2} \ , \tag{23}$$

any element a of $\mathcal{A}_I^{(0)}$ can be expanded in a non commutative Fourier expansion

$$a = \sum_{\mathbf{m} \in \mathbf{Z}^2} a(\mathbf{m}, \gamma) W(\mathbf{m}) \ , \tag{24}$$

where $\gamma \mapsto a(\mathbf{m}, \gamma) \in \mathcal{C}(I)$ and $a(\mathbf{m}, \gamma) = 0$ for $\mathbf{m}$ large enough. The Fourier coefficients of the product ab will be given by :

$$(ab)(\mathbf{m}, \gamma) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} a(\mathbf{m}', \gamma) b(\mathbf{m} - \mathbf{m}') e^{i\gamma(\mathbf{m}' \wedge (\mathbf{m} - \mathbf{m}'))/2} \ , \tag{25}$$

where $\mathbf{m}' \wedge \mathbf{m} = m'_1 m_2 - m'_2 m_1$. The adjoint will be given by

$$(a^*)(\mathbf{m}, \gamma) = \overline{a(-\mathbf{m}, \gamma)} \ . \tag{26}$$

The identity corresponds to $\mathbf{1}(\mathbf{m}, \gamma) = \delta_{m_1,0}\delta_{m_2,0}$ whereas

$$U(\mathbf{m}, \gamma) = \delta_{m_1,1}\delta_{m_2,0} \ , \qquad V(\mathbf{m}, \gamma) = \delta_{m_1,0}\delta_{m_2,0} \ .$$

We will denote by $\mathcal{A}_\gamma^{(0)}$ the algebra $\mathcal{A}_I^{(0)}$ whenever $I = \{\gamma\}$. The evaluation homomorphism η_γ is a map from $\mathcal{A}_I^{(0)}$ into $\mathcal{A}_\gamma^{(0)}$ (for $\gamma \in I$) defined by

$$\eta_\gamma(a)(m) = a(m, \gamma) \ . \tag{27}$$

We then remark that $\eta_\gamma(a^*) = \eta_\gamma(a)^*$, $\eta_\gamma(ab) = \eta_\gamma(a)\eta_\gamma(b)$ and $\eta_\gamma(\mathbf{I}) = \mathbf{I}$.
For $\gamma = 0$, $\mathcal{A}_0^{(0)}$ is isomorphic to the algebra of trigonometric polynomials, through the Fourier transform. Namely if $a \in \mathcal{A}_0^{(0)}$ we set

$$\tilde{a}(\theta, A) = \sum a(m)e^{i(m_1\theta - m_2 A)} \ , \tag{28}$$

and we get $(\widetilde{ab})(\theta, A) = \tilde{a}(\theta, A)\tilde{b}(\theta, A)$ and $\widetilde{(a^*)}(\theta, A) = \overline{\tilde{a}(\theta, A)}$.
Seeing a as a function on a "non commutative phase space", one can define various rules for calculus generalizing the rules valid on the classical phase space.

(a) Integration : integration over the torus is then given by picking up the 0^{th} order Fourier coefficient namely

$$\tau_\gamma(a) = a(\mathbf{0}, \gamma) \ . \tag{29}$$

Then τ is linear in a and satisfy for $a \neq 0$

$$\begin{array}{l} \tau(a^*a) > 0 \text{ positivity} \\ \tau(ab) = \tau(ba) \text{ trace} \\ \tau(\mathbf{I}) = 1 \text{ normalization} \ . \end{array} \tag{30}$$

(b) Angle average

$$\langle a \rangle = \sum_{m_2} a(0, m_2, \gamma)V^{m_2} \ , \tag{31}$$

consists in picking up all Fourier coefficients with $m_1 = 0$. It corresponds classically to $\int \frac{d\theta}{2\pi}\tilde{a}$.

(c) Derivation

$$\begin{array}{l} (\partial_\theta a)(m_1, m_2, \gamma) = im_1 a(m_1, m_2, \gamma) \\ (\partial_A a)(m_1, m_2, \gamma) = im_2 a(m_1, m_2, \gamma) \ . \end{array} \tag{32}$$

Then $\partial_i (i = \theta, A)$ satisfies the axioms for $*$-derivations namely

$$\begin{array}{l} \partial_i \text{ are linear} \\ \partial_i(ab) = (\partial_i a)b + a(\partial_i b) \\ \partial_i(a^*) = (\partial_i a)^* \ . \end{array} \tag{33}$$

(d) Poisson-Moyal brackets
For $\gamma \neq 0$ we set

$$\{a, b\} = \frac{1}{i\gamma}(ab - ba) \ . \tag{34}$$

More precisely :

$$\{a, b\}(\mathbf{m}, \gamma) = \sum_{\mathbf{m}' \in \mathbf{Z}^2} a(\mathbf{m}', \gamma)b(\mathbf{m} - \mathbf{m}', \gamma)\frac{2}{\gamma}\sin\left(\frac{\gamma}{2}\mathbf{m}' \wedge (\mathbf{m} - \mathbf{m}')\right) \ . \tag{35}$$

This expression shows that, at $\gamma \to 0$, $\{a, b\}(\mathbf{m}, \gamma)$ converges to

$$\{a, b\}(\mathbf{m}, 0) = \sum a(\mathbf{m}', 0)b(\mathbf{m} - \mathbf{m}', 0)\mathbf{m} \wedge (\mathbf{m} - \mathbf{m}') = (\partial_\theta a \partial_A b - \partial_A a \partial_\theta b)(\mathbf{m}, 0) \ . \tag{36}$$

(e) At last we define the derivative with respect to Planck's constant as follows :

$$(\partial_\gamma a)(\mathbf{m}, \gamma) = \frac{\partial a}{\partial \gamma}(\mathbf{m}, \gamma) \ . \tag{37}$$

Then ∂_γ is linear and satisfies properties of Ito's derivative (see in [39]) namely :

$$\begin{array}{l} \partial_\gamma(a^*) = (\partial_\gamma a)^* \\ \partial_\gamma(ab) = (\partial_\gamma a)b + a(\partial_\gamma b) + \frac{i}{2}(\partial_\theta a \partial_A b - \partial_A a \partial_\theta b) \\ \tau(\partial_\gamma a) = d\tau(a)/d\gamma \ . \end{array} \tag{38}$$

The last step in building up the observables algebra consists in defining a topology on $\mathcal{A}_I^{(0)}$ which will permit to describe all continuous functions on the "non commutative phase space". The way to do this is to construct a norm of C^*-algebra.

A norm in $\mathcal{A}_I^{(0)}$ is a mapping $a \in \mathcal{A}_I^{(0)} \mapsto \|a\| \in \mathbf{R}_+$ satisfying :

$$\begin{array}{ll} \text{(i)} & \text{triangle inequality } \|a+b\| \leq \|a\| + \|b\| \\ \text{(ii)} & \|\lambda a\| = |\lambda| \|a\| \lambda \in \mathbf{C} \\ \text{(iii)} & \|a\| = 0 \Leftrightarrow a = 0 \\ \text{(iv)} & \text{a norm is algebraic if } \|ab\| \leq \|a\| \|b\| \\ \text{(v)} & \text{it is a } *\text{-norm if } \|a^*\| = \|a\| \\ \text{(vi)} & \text{it is a } C^*\text{-norm if } \|aa^*\| = \|a\|^2 \ . \end{array} \tag{39}$$

Such norms are constructed by using the theory of representations; a representation of $\mathcal{A}_I^{(0)}$ is given by a Hilbert space $\mathcal{H}$ and a mapping π which assigns to each element $a \in \mathcal{A}_I^{(0)}$ an operator $\pi(a)$ on $\mathcal{H}$ such that the algebraic structure is respected namely :

$$\begin{array}{ll} (1) & \pi(ab) = \pi(a)\pi(b) \\ (2) & \pi(a)^* = \pi(a^*) \\ (3) & \pi(\mathbf{I}) = \mathbf{I} \ . \end{array} \tag{40}$$

In particular $\pi(U)$ and $\pi(V)$ are unitary operators.
Two representations $(\mathcal{H}, \pi)$ and $(\mathcal{H}', \pi')$ are equivalent if there is a unitary operator $S : \mathcal{H} \mapsto \mathcal{H}'$ such that :

$$S\pi(a)S^{-1} = \pi' \ , \forall a \in \mathcal{A}_I^{(0)} \ . \tag{41}$$

A norm is defined as follows :

$$\|a\|_I = \sup_{\pi \in Rep(\mathcal{A}_I^{(0)})} \|\pi(a)\| \ , \tag{42}$$

where the supremum runs over the family of all representations of $\mathcal{A}_I^{(0)}$ and $\|\pi(a)\|$ is the usual operator norm in Hilbert spaces. One can show that this is a C^*-norm [37, 36].
The algebra $\mathcal{A}_I$ will be obtained by taking the completion of $\mathcal{A}_I^{(0)}$ under this norm, namely all the limits of Cauchy's sequences. For $I = \{\gamma\}$ we will denote it by $\mathcal{A}_\gamma$. For $\gamma = 0$ we then get $\mathcal{A}_0 \approx \mathcal{C}(\mathbf{T}^2)$ by Fourier transform.
Various dense subalgebras will be technically useful. Among them consider the subalgebra $\mathcal{A}_I(r)$ (for $r > 0$), made of elements in $\mathcal{A}_I$ whose Fourier coefficients satisfy :

$$\|a\|_{r,I} = \sup_{\gamma \in I} \sum_{\mathbf{m} \in \mathbf{Z}^2} |a(\mathbf{m}, \gamma)| e^{r|\mathbf{m}|} < +\infty \ , \tag{43}$$

where $|\mathbf{m}| = |m_1| + |m_2|$ if $\mathbf{m} = (m_1, m_2) \in \mathbf{Z}^2$. Elements of $\mathcal{A}_I(r)$ are said to be "analytic", by analogy with the classical case for which analytic doubly periodic functions of (A, θ) have exponentially decreasing Fourier series.

We remark that for the KH or the LS model (with $\nu \in \mathbf{Z}$), the Floquet operator itself belongs to the algebra $\mathcal{A}_I$ for any I not containing $\gamma = 0$. For indeed one can write $L/L_0 = A$ so that $2\cos(L/L_0) = 2\cos(A) = U + U^*$ and therefore :

$$F_{\text{KH}} = e^{ik_1(U+U^*)/2\gamma} e^{ik_2(V+V^*)/2\gamma} \ , \tag{44}$$

whereas (if $\nu \in \mathbf{Z}$)

$$F_{\text{LS}} = e^{ik_1\left[(U+U^*) - k_1'(U^2 - U^{*2})/4i\right]/2\gamma} e^{ik_2\left[(V+V^*) - k_2'(V^\nu - V^{*\nu})/4i\right]/2\gamma} \ , \tag{45}$$

The evolution of observables is described by the following map :

$$\beta(a) = FaF^{-1} \ , a \in \mathcal{A}_I \ . \tag{46}$$

This map satisfies :

$$\begin{array}{ll} \text{(a)} & \beta : \mathcal{A}_I \mapsto \mathcal{A}_I \text{ is linear} \\ \text{(b)} & \beta(a^*) = \beta(a)^* \\ \text{(c)} & \beta(ab) = \beta(a)\beta(b) \\ \text{(d)} & \beta(\mathbf{I}) = \mathbf{I} \\ \text{(e)} & \|\beta(a)\| = \|a\| \ . \end{array} \tag{47}$$

Such a map is called a $*$-automorphism of the algebra $\mathcal{A}_I$.

The kicked rotor Floquet operator

$$F_{\mathrm{KR}} = e^{iK\cos(\hat{\theta})/\gamma} e^{i\hat{A}^2/2\gamma} \tag{48}$$

is a product of two unitaries. The first one (potential energy) $e^{iK\cos(\hat{\theta})/\gamma} = e^{iK(U+U^*)/2\gamma}$ is an element of $\mathcal{A}_I$ as long as $0 \notin I$. But the second one $F_0 = e^{i\hat{A}^2/2\gamma}$ cannot be written in any way as an element of $\mathcal{A}_I$. However, we remark that

$$F_0 V F_0^{-1} = V \ , \ F_0 U F_0^{-1} = U V^{-1} e^{i\gamma/2} \ . \tag{49}$$

Therefore the map $\beta_0 : a \in \mathcal{A}_I \mapsto F_0 a F_0^{-1}$ is also a $*$-automorphism of the algebra eventhough F_0 does not belong to $\mathcal{A}_I$. Moreover, if $w = K\cos(\hat{\theta}) = K(U+U^*)/2$, let us set :

$$\mathcal{L}_w(a) = \{w, a\} \text{ for } a \in \mathcal{A}_I \ . \tag{50}$$

This is defined for any $\gamma \in I$. Then if $\gamma \neq 0$ one has :

$$e^{\mathcal{L}_w}(a) = e^{iK\cos(\hat{\theta})/\gamma} a e^{-iK\cos(\hat{\theta})/\gamma} \ , \tag{51}$$

showing that $e^{\mathcal{L}_w}$ also defines a $*$-automorphism of $\mathcal{A}_I$ even if $0 \in I$. Thus the evolution of the observables for the KR is given by the $*$-automorphism :

$$\beta_{\mathrm{KR}} = e^{\mathcal{L}_w} \circ \beta_0 \text{ with } w = K(U+U^*)/2 \text{ in } \mathcal{A}_I \ . \tag{52}$$

It will be convenient however to consider a bigger algebra $\mathcal{B}_I$, generated by the three unitaries (U, V, F_0) satisfying (21) and (49). In much the same way one can define on $\mathcal{B}_I$ a non commutative calculus namely, the evaluation homomorphism, the trace ($\tau(aF_0^n) = 0$ unless $n = 0$), the angle average, the derivative ∂_θ (but not ∂_A!), the Poisson brackets. The derivative with respect to γ can also be defined but it does not satisfy (38). $\mathcal{A}_I$ can be seen as a subalgebra of $\mathcal{B}_I$. We also remark that while $\mathcal{A}_{\gamma=0}$ is commutative (classical observables algebra) $\mathcal{B}_{\gamma=0}$ is not. This is due to the fact that the free evolution is non trivial even in the classical case. In practical calculations we will use the physical representation defined on the space $\ell^2(\mathbf{Z})$ that will be seen as the momentum representation. Such representations $(\pi_{\gamma,x,y})$ of $\mathcal{B}_I$ are labelled by three variables $\gamma \in I$, $(x, y) \in \mathbf{T}^2$ and are defined as follows, if $\Psi \in \ell^2(\mathbf{Z})$:

$$\begin{array}{ll} \text{(a)} & [\pi_{\gamma,x,y}(U)\Psi](n) = [T\Psi](n) = \Psi(n-1) \\ \text{(b)} & [\pi_{\gamma,x,y}(V)\Psi](n) = e^{i(x-n\gamma)}\Psi(n) \\ \text{(c)} & [\pi_{\gamma,x,y}(F_0)\Psi](n) = e^{i(y-nx+\frac{n^2\gamma}{2})}\Psi(n) \ . \end{array} \tag{53}$$

We remark that if we set

$$\gamma = \frac{\hbar T}{I} \ , \ x = -\mu B T \ , \ y = \frac{\mu^2 B^2 T I}{2\hbar} \ , \tag{54}$$

then (53c) gives the expression of the Floquet operator $e^{i\hat{A}^2/2\gamma}$ for the free KR model in the magnetic field B proportional to x. So that x is a dimensionless magnetic field. Then we get the following results

(i) if $a \in \mathcal{A}_I$, $\pi_{\gamma,x,y}(a)$ does not depend on y.
(ii) $a \in \mathcal{B}_I$,

$$\tau_\gamma(a) = \int\int_{\mathbf{T}^2} \langle 0|\pi_{\gamma,x,y}(a)|0\rangle \frac{dxdy}{4\pi^2} \ , \tag{55}$$

if γ is irrational one gets :

$$\tau_\gamma(a) = \lim_{L\mapsto\infty} \frac{1}{2L+1} \sum_{n=-L}^{+L} \langle n|\pi_{\gamma,x,y}(a)|n\rangle \ . \tag{56}$$

Thus, τ_γ appears as a trace per unit length in the momentum space.
(iii)

$$T\pi_{\gamma,x,y}(a)T^{-1} = \pi_{\gamma,x+\gamma,y+x+\frac{\gamma}{2}}(a) \ . \tag{57}$$

(iv) On the other hand if $a = \sum a(m_1, m_2, m_0, \gamma) U^{m_1} V^{m_2} e^{i\gamma m_1 m_2/2} F_0^{m_0}$ the angle average is :

$$\langle a \rangle = \sum a(0, m_2, m_0, \gamma) V^{m_2} F_0^{m_0} \ . \tag{58}$$

We associate to $\langle a\rangle$ the function :

$$\langle a\rangle(\gamma,x,y)=\sum a(0,m_2,m_0,\gamma)e^{im_2x}e^{im_0y}\ , \tag{59}$$

and we remark that $[\langle a\rangle\langle b\rangle](\gamma,x,y)=\langle a\rangle(\gamma,x,y)\langle b\rangle(\gamma,x,y)$. Moreover

$$\langle a\rangle(\gamma,x,y)=\langle 0|\pi_{\gamma,x,y}(a)|0\rangle\ , \tag{60}$$

so that using the translation operator T :

$$\langle a\rangle(\gamma,x-n\gamma,y-nx+\frac{n^2\gamma}{2})=\langle n|\pi_{\gamma,x,y}(a)|n\rangle\ . \tag{61}$$

(v) We also get for $a\in\mathcal{B}_I$:

$$\pi_{\gamma,x,y}(\partial_\theta a)=i\,[N,\pi_{\gamma,x,y}(a)]\ , \tag{62}$$

where N is the momentum operator defined by :

$$(N\Psi)(n)=n\Psi(n)\ ,\ \Psi\in\ell^2(\mathbf{Z})\ . \tag{63}$$

If $a\in\mathcal{A}_I$ only, we have (recall that ∂_A is not well defined on $\mathcal{B}_I$ and that if $a\in\mathcal{A}_I$, $\pi_{\gamma,x,y}(a)$ does not depend on y) :

$$\pi_{\gamma,x}(\partial_A a)=\partial\pi_{\gamma,x}(a)/\partial x\ . \tag{64}$$

(vi) At last

$$\|a\|_I=\sup_{\gamma\in I}\ \sup_{(x,y)\in\mathbf{T}^2}\|\pi_{\gamma,x,y}(a)\|_{op}\ , \tag{65}$$

where $\|\cdot\|_{op}$ denotes the operator norm on $\ell^2(\mathbf{Z})$.

2.4 Spectral Theorems

We recall that a is an element of a C^*-algebra $\mathcal{A}$ with unit $\mathbf{I}$, its spectrum denoted by $\mathrm{Sp}(a)$ is the set of complex numbers $z\in\mathbf{C}$ such that $z\mathbf{I}-a$ has no inverse in $\mathcal{A}$. We also recall that a is normal if $aa^*=a^*a$. Selfadjoint elements (i.e. $a=a^*$) as well as unitaries (i.e. $aa^*=a^*a=\mathbf{I}$) are the main examples of normal elements. Selfadjoint elements have their spectrum contained in $\mathbf{R}$, unitaries in the unit circle.

We also recall (see for instance [27]) that if now A is a normal operator acting on a Hilbert space $\mathcal{H}$, with spectrum $\mathrm{Sp}(A)\subset\mathbf{C}$, its spectral measure relative to a vector $\psi\in\mathcal{H}$ is the unique probability measure on $\mathrm{Sp}(A)$ such that

$$\frac{\langle\psi|f(A)|\psi\rangle}{\langle\psi|\psi\rangle}=\int_{\omega\in\mathrm{Sp}(A)}d\mu_\psi(\omega)f(\omega)\ , \tag{66}$$

for any continuous function f on $\mathrm{Sp}(A)$. Then there exists a probability measure μ (which is not unique) on $\mathrm{Sp}(A)$ such that $d\mu_\psi=F_\psi(\omega)d\mu$, $\forall\psi\in\mathcal{H}$ where $F_\psi(\omega)\in L^1(d\mu)$ and such that a measurable subset Σ of $\mathrm{Sp}(A)$ has zero μ-measure if and only if $\mu_\psi(\Sigma)=0$, $\forall\psi\in\mathcal{H}$. μ is unique modulo equivalence of measure (namely modulo a measure having same sets of zero measure). Each such measure μ can be decomposed in a unique way as $\mu_{ac}+\mu_{sc}+\mu_{pp}$ where μ_{pp} is a countable set of Dirac measure (pure point spectrum), μ_{ac} is equivalent to the Lebesgue's measure (the absolutely continuous one) and μ_{sc} (the singular continuous part) is the remainder. Correspondingly $\mathcal{H}$ can be decomposed in a direct sum of three orthogonal subspaces $\mathcal{H}=\mathcal{H}_{ac}\bigoplus\mathcal{H}_{sc}\bigoplus\mathcal{H}_{pp}$ where $\psi\in\mathcal{H}_i$ $(i=ac,sc,pp)$ if and only if $d\mu_\psi=F_\psi d\mu_i$ where $F_\psi\in L^1(d\mu_i)$. $\mathcal{H}_{ac}$ (resp. $\mathcal{H}_{sc}$, $\mathcal{H}_{pp}$) is called the absolutely continuous (resp. the singular continuous, the pure point) component. We also call $\mathcal{H}_c=\mathcal{H}_{ac}\bigoplus\mathcal{H}_{sc}$ and $\mathcal{H}_s=\mathcal{H}_{sc}\bigoplus\mathcal{H}_{pp}$ where c stands for continuous and s for singular.

Let us consider now the special case for which I is reduced to the point γ. Then one gets the following results.

Theorem 1 *If $\gamma=2\pi p/q$ where p and q are integers, prime to each other, let "a" be a normal analytic element of $\mathcal{A}_\gamma$. Then for any $x\in\mathbf{T}$ the spectrum of $\pi_{\gamma,x}(a)$ contains at most q eigenvalues of infinite multiplicity, the rest of the spectrum being absolutely continuous.*

Theorem 2 *If $\gamma=2\pi p/q$ where p and q are integers, prime to each other, let "a" be a normal analytic element of $\mathcal{B}_\gamma$, then for any $x\in\mathbf{T}$ such that $x/2\pi$ be rational, $\pi_{\gamma,x}(a)$ has at most a finite number of eigenvalues of infinite multiplicity and the rest of the spectrum is absolutely continuous.*

Applied to the KR model, the Theorem 2 was proved in 1980 by Izrailev and Shepelyansky [40]. It is remarkable that the magnetic field, in Theorem 2, may break the result if it is not rational!

The third result is a generalization of the so-called Aubry duality [41]. There is a unique $*$-automorphism $\mathcal{F}$ of $\mathcal{A}_\gamma$ such that

$$\mathcal{F}(U) = V^{-1} \ , \mathcal{F}(V) = U^{-1} \ . \tag{67}$$

Then one gets $\tau(\mathcal{F}(a)) = \tau_\gamma(a)$, $\forall a \in \mathcal{A}_\gamma$. If $a \in \mathcal{A}_\gamma$ is analytic the spectral properties of $\pi_{\gamma,x}(a)$ and of $\pi_{\gamma,x}(\mathcal{F}a)$ can be related as follows :

Theorem 3 *[21, 42, 22] [Chojnacki] Let $\gamma/2\pi$ being irrational and let "a" be a normal analytic element of $\mathcal{A}_\gamma$. Let also B be a Borel subset of $\mathbf{R}$. If for almost every $x \in \mathbf{T}$, $\pi_{\gamma,x}(a)$ has pure point spectrum in B, then for almost all x, $\pi_{\gamma,x}(\mathcal{F}a)$ has purely continuous spectrum in B.*

Corollary 1 *Under the previous hypothesis if "a" is quasi self dual, namely if there is a unitary element S in $\mathcal{A}_\gamma$ and $\lambda \in \mathbf{C}$ such that $a = \lambda S\mathcal{F}(a)S^{-1}$, then for almost all $x \in \mathbf{T}$, $\pi_{\gamma,x}(a)$ has a purely continuous spectrum.*

This result applies in particular to the self dual generalized kicked Harper model for which

$$F = e^{ig(V)/\gamma} e^{ig(U)/\gamma} \ , \tag{68}$$

where g is an analytic function on $\mathbf{T}$. Then

$$\mathcal{F}(F) = e^{ig(U)/\gamma} F e^{-ig(U)/\gamma} \ , \tag{69}$$

namely $\mathcal{F}(F)$ is unitarily equivalent to F.
It also applies to the Harper model namely

$$H = U + U^* + (V + V^*) \ , \tag{70}$$

which is self dual (namely $\mathcal{F}(H) = H$, analytic and self adjoint). In this latter case Delyon [43] proved that if ψ is an eigenvector of $\pi_x(H)$ then it is in $\ell^2(\mathbf{Z})$ but not in $\ell^1(\mathbf{Z})$. Actually Chojnacki's result shows that there is no such vector [22].
The next kind of result, still valid only in the case of $\mathcal{A}_\gamma$ is due to Chulaevski and Dynaburg [44] :

Theorem 4 *Let us consider an element $H = H_0 + \epsilon H_I \in \mathcal{A}_\gamma$, $\epsilon \in \mathbf{C}$ where :*
(i) γ satisfies a diophantine condition
$\exists \sigma > 0$, $C > 0$ such that $|\gamma - 2\pi p/q| \geq C/q^\sigma$, $\forall p/q \in \mathbf{Q}$.
(ii) $H_0 = h(V)$ where "h" is a real function in $\mathcal{C}^2(\mathbf{T})$ and has only one regular maximum and one regular minimum (namely the second derivative does not vanish at the extrema).
(iii) H_I is an analytic element of $\mathcal{A}_\gamma$.
Then there is $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$, $\pi_x(H)$ has pure point spectrum of finite multiplicity for almost every $x \in \mathbf{T}$ and the corresponding eigenstates are exponentially localized.

Comments
(i) For the almost Mathieu Hamiltonian $H = (V + V^*) + \epsilon(U + U^*)$, $\pi_x(H)$ has a pure point spectrum at small ϵ's for almost all x's [45, 46, 47, 48].
(ii) In this result H_I needs not be self adjoint.
(iii) Presumably the same result should apply to unitaries of the form

$$F = e^{ih(V)} e^{i\epsilon H_I} \ , \tag{71}$$

with the same hypothesis on γ, h, H_I and ϵ.
In particular we do expect that the KH model described by

$$F_{\mathrm{KH}} = e^{ik_1(V+V^*)} e^{ik_2(U+U^*)} \ , \tag{72}$$

for $k_2 \ll 1$ and k_1 not too big, the same is true.

The kinetic energy is defined as follows : let F be a unitary element of $\mathcal{B}_\gamma$ and let N be the momentum operator in $\ell^2(\mathbf{Z})$ namely :

$$[N\Psi](n) = n\Psi(n) \ , \text{ if } \psi \in \ell^2(\mathbf{Z}) \ . \tag{73}$$

We set $t \in \mathbf{Z}$:

$$\mathcal{E}_{\gamma,x,y}(t) = \gamma^2 \langle 0|\pi_{\gamma,x,y}(F)^t N^2 \pi_{\gamma,x,y}(F)^{-t}|0\rangle \ . \tag{74}$$

We remark that :

$$\mathcal{E}_{\gamma,x-n\gamma,y-n+n^2\frac{\gamma}{2}}(t) = \gamma^2 \langle n|\pi_{\gamma,x,y}(F)^t (N-n)^2 \pi_{\gamma,x,y}(F)^{-t}|n\rangle = \gamma^2 \|N_t - N_0\|^2 \ , \tag{75}$$

where $N_t = \pi_{\gamma,x,y}(F)^t N \pi_{\gamma,x,y}(F)^{-t}$ and $N_0 = N$. This quantity has been introduced by Casati et al. [28] as a tool to measure the diffusion in the phase space.
We will rather consider now the average of this quantity on (x, y) namely :

$$\mathcal{E}_\gamma(t) = \gamma^2 \tau_\gamma(|\partial F^t|^2) \ . \tag{76}$$

2.5 The Semiclassical Limit

In this section we will summarize a certain number of rigorous results. The first one is the following [27] :

Proposition 1 *For any $a \in \mathcal{B}_I$, the map $\gamma \in I \mapsto \tau_\gamma(a) \in \mathbf{C}$ is continuous.*

Let now a be a normal element of $\mathcal{B}_I$, namely $aa^* = a^*a$. Then, if f is any continuous function of the spectrum of a, we define the Density Of States (DOS) of a as the unique family $(dN_\gamma)_{\gamma \in I}$ of probability measures such that

$$\tau_\gamma(f(a)) = \int_{\mathrm{Sp}(a)} dN_\gamma(\omega) f(\omega) \ . \tag{77}$$

If $a = a^*$, $\mathrm{Sp}(a) \subset \mathbf{R}$ and then we define the Integrated Density Of States (IDOS) as :

$$N_\gamma(\omega) = \int_{-\infty}^{\omega} dN_\gamma(\omega') \ . \tag{78}$$

N_γ is the non decreasing function on $\mathbf{R}$ with $0 \le N_\gamma(\omega) \le 1$. If a is unitary $\mathrm{Sp}(a)$ is included in the unit circle and if $|z| = 1$, $|z| = e^{i\omega}$, $\omega \in [0, 2\pi]$ we then define the IDOS as

$$N_\gamma(\omega) = \int_{\omega' \in [0,\omega]} dN_\gamma(\omega') \ . \tag{79}$$

In this latter case N_γ can be continued as a non decreasing function on $\mathbf{R}$ such that $N_\gamma(\omega + 2\pi) = N_\gamma(\omega) + 1$.
If $\gamma = 0$, $\mathcal{A}_0$ is isomorphic to $\mathcal{C}(\mathbf{T}^2)$ (the phase space is now commutative and is the torus). Then for $H = H^* \in \mathcal{A}_0$, if $\tilde{H}$ is its Fourier transform (see (28)) we get

$$\tau_0(f(H)) = \int \frac{d\theta dA}{4\pi^2} f(\tilde{H}(\theta, A)) \ . \tag{80}$$

Therefore

$$N_0(\omega) = \int_{\tilde{H}(\theta,A) \le \omega} \frac{d\theta dA}{4\pi^2} \ . \tag{81}$$

For $U \in \mathcal{A}_0$ unitary, there is a similar formula.
Then we get the following

Proposition 2 *Let H be self adjoint (resp. unitary) in $\mathcal{B}_I$. If $\gamma \in I$ and ω is a point of continuity of N_γ then*

$$\lim_{\gamma' \in I, \gamma' \mapsto \gamma} N_{\gamma'}(\omega) = N_\gamma(\omega) \ . \tag{82}$$

Corollary 2 *Let H be self adjoint in $\mathcal{A}_I$ and I contains $\gamma = 0$. Then if $\{(\theta, A) \in \mathbf{T}, \tilde{H}(\theta, A) = \omega\}$ has a zero Lebesgue measure, we get :*

$$\lim_{\gamma \mapsto 0} N_\gamma(\omega) = \int_{\tilde{H}(\theta,A) \le \omega} \frac{d\theta dA}{4\pi^2} \quad \textit{(Weyl's formula)}, \tag{83}$$

where $\tilde{H}(\theta, A)$ is the Fourier transform of $\eta_\gamma(H)$ for $\gamma = 0$. If U is unitary a similar formula holds.

We remark that actually [37].

Proposition 3 *If H is a self adjoint or a unitary in $\mathcal{B}_\gamma$ which is a polynomial in U, V and F_0 then its IDOS is continuous.*

The next type of result concerns the time evolution in $\mathcal{A}_\gamma$. Then one gets the following result [27, 38]

Theorem 5 *Let "w" be a self adjoint element of $\mathcal{A}_I(r)$ where I is an interval containing $\gamma = 0$. For any ρ such that $0 < \rho < r$:*
(i) The Liouville operator $\mathcal{L}_w$ associated to "w" is linear and bounded from $\mathcal{A}_I(r)$ to $\mathcal{A}_I(r-\rho)$.
(ii) For t small enough, depending on ρ, $e^{t\mathcal{L}_w}$ is also linear and bounded from $\mathcal{A}_I(r)$ to $\mathcal{A}_I(r-\rho)$.
(iii) $e^{t\mathcal{L}_w}$ can be extended as a $*$-automorphism of $\mathcal{A}_I$ for any $t \in \mathbf{R}$ such that if $a \in \mathcal{A}_I(r)$

$$\frac{d}{dt}\left(e^{t\mathcal{L}_w}\right)(a) = e^{t\mathcal{L}_w}[\mathcal{L}_w(a)] \ . \tag{84}$$

Using Proposition 1 and Theorem 5, it immediately follows that :

Theorem 6 *Let $a_1, a_2, \ldots, a_n$ be in $\mathcal{A}_I$ where I contains $\gamma = 0$.*
Let $w_1, w_2, \ldots, w_n$ be analytic selfadjoint elements of $\mathcal{A}_I(r)$ for some $r > 0$. Then

$$\begin{aligned} &\lim_{\gamma\mapsto 0} \tau_\gamma\left(e^{t\mathcal{L}_{w_1}}(a_1)e^{t\mathcal{L}_{w_2}}(a_2)\ldots e^{t\mathcal{L}_{w_n}}(a_n)\right) = \\ &= \int \frac{d\theta dA}{4\pi^2}\tilde{a}_1(t,\theta,A)\tilde{a}_2(t,\theta,A)\ldots\tilde{a}_n(t,\theta,A) \ , \end{aligned} \tag{85}$$

where $\tilde{a}_i(t,\theta,A)$ $(i = 1, 2, \ldots, n)$ solves the Hamilton equation

$$\frac{\partial}{\partial t}\tilde{a}_i = \{w_i, \tilde{a}_i\} \ , \ \tilde{a}_i(t=0,\theta,A) = \tilde{a}_i(\theta,A) \ . \tag{86}$$

The last set of results concerns the continuity of the spectrum with respect to γ. We will say that a family $(\Sigma(\gamma))_{\gamma\in I}$ of compact subsets $\Sigma(\gamma)$ of a topological space X is continuous at $\gamma = \gamma_0$ if :
(i) it is continuous from the outside, namely given any closed set F in X such that $\Sigma(\gamma_0) \cap F = \emptyset$ there is $\delta > 0$ such that if $|\gamma - \gamma_0| \le \delta$ then $\Sigma(\gamma) \cap F = \emptyset$.
(ii) It is continuous from inside, namely for any open set O in X such that $\Sigma(\gamma_0) \cap O \ne \emptyset$ there is $\delta > 0$ such that if $|\gamma - \gamma_0| \le \delta$ then $\Sigma(\gamma) \cap O \ne \emptyset$.
If $X = \mathbf{R}$, a gap of $\Sigma(\gamma)$ is one of the interval corresponding to a connected component of $\mathbf{R} - \Sigma(\gamma)$. To say that $\gamma \mapsto \Sigma(\gamma)$ is continuous, is equivalent to say that the gap edges of $\Sigma(\gamma)$ are continuous functions of γ.
For $a \in \mathcal{B}_I$ we set $\Sigma(\gamma) = \mathrm{Sp}(\eta_\gamma(a))$. Then one gets [37]

Theorem 7 *For any normal element $a \in \mathcal{B}_I$ the spectrum $(\Sigma(\gamma))_{\gamma\in I}$ is continuous at every point of I.*

3 Dynamical Localization

In this section we give several possible definitions of the localization length and discuss the relation between its finiteness and the nature of the spectrum. A detailed part is devoted to the KR problem.
We will either consider a self-adjoint element ($H = H^*$) or a unitary element ($F = (F^*)^{-1}$) of the algebra $\mathcal{B}_I$ previously described. The case of unitary elements reduces to the case of selfadjoint elements provide we identify F with $e^{iTH/\gamma}$ for some $T > 0$ and γ is the effective Planck constant.
In the physical representation we consider the operator $\pi_{\gamma,x,y}(H) = H_{\gamma,x,y}$ instead. Then if Δ is some interval in $\mathbf{R}$ we denote by P_Δ the eigenprojection of H corresponding to energies in Δ namely :

$$P_\Delta = \chi_\Delta(H) \ , \tag{87}$$

where χ_Δ is the characteristic function of the interval Δ. Suppose that $H_{\gamma,x,y}$ has point spectrum in Δ then :

$$\pi_{\gamma,x,y}(P_\Delta) = \sum_{\omega\in\Delta} |\psi_{\omega,\gamma,x,y}\rangle\langle\psi_{\omega,\gamma,x,y}| \ , \tag{88}$$

where the $\psi_{\omega,\gamma,x,y}$'s are the normalized eigenstates of $H_{\gamma,x,y}$ corresponding to the energies ω. All the eigenstates are in $\ell^2(\mathbf{Z})$ namely :

$$\|\psi_\omega\|^2 = \sum_{n\in\mathbf{Z}} |\psi_\omega(n)|^2 = 1 < +\infty \ . \tag{89}$$

In solid state physics, one considers several quantities to measure how ψ_ω is localized. The first among them is the mean inverse participation ratio early introduced by Anderson [49] and studied by Pastur [7] namely

$$A_{n,n'}(\Delta,\gamma,x,y) = \sum_{\omega\in\Delta} |\psi_{\omega,\gamma,x,y}(n)|^2 |\overline{\psi}_{\omega,\gamma,x,y}(n')|^2 \,. \tag{90}$$

The equation (90) is valid only if the spectrum is pure point. Nevertheless $A_{n,n'}(\Delta,\gamma,x,y)$ can be rewritten in a purely algebraic sense as :

$$A_{n,n'}(\Delta,\gamma,x,y) = \lim_{T\mapsto\infty} \frac{1}{T}\int_0^T dt |\langle n|\pi_{\gamma,x,y}(e^{itH_\Delta})|n'\rangle|^2 \tag{91}$$

then we can rephrase Pastur's result as follows :

Theorem 8 *For almost all $(x,y) \in \mathbf{T}^2$, the number of eigenvalues of $H_{\gamma,x,y}$ in Δ is either zero or infinity. The latter is realized if and only if $\xi_\Delta > 0$ where*

$$\xi_\Delta = \int \frac{dxdy}{4\pi^2} A_{0,0}(\Delta,\gamma,x,y) \;, \tag{92}$$

which can be rewritten as

$$\xi_\Delta = \lim_{T\mapsto\infty} \frac{1}{T}\int_0^T dt \int \frac{dxdy}{4\pi^2} |\langle 0|\pi_{\gamma,x,y}(e^{itH_\Delta})|0\rangle|^2 \;, \tag{93}$$

$$\xi_\Delta = \lim_{T\mapsto\infty} \frac{1}{T}\int_0^T dt \int \tau_\gamma\left(e^{itH_\Delta}\rho_\theta(e^{-itH_\Delta})\right) \frac{d\theta}{2\pi} \;, \tag{94}$$

where $H_\Delta = \chi_\Delta(H)H$ is the restriction of H to the spectral interval Δ and ρ_θ is the $*$-automorphism $\rho_\theta = \exp(\theta\partial_\theta)$.

Comments

i) the existence of point spectrum in Δ is therefore equivalent to $\xi_\Delta > 0$.
ii) Anderson introduced the mean inverse participation ratio to serve as a criterion of quantal diffusion. For if $\Delta = \mathbf{R}$, $A_{0,0}(\Delta,\omega,x,y)$ represents the quantal probability for a particle sitting at $n=0$ when $t=0$ to return to $n=0$ after an infinite time. So it gives rise to some weak notion of localization.
iii) This criterion is not sufficient to eliminate continuous spectrum.

Let us introduce a stronger notion giving a measurement of the localization length. Whenever $\pi_{\gamma,x,y}(H)$ has pure point spectrum, the eigenvalues may decrease faster. We are led to introduce quantities like :

$$\ell^{(D)}(\omega,\gamma,x,y) = \left[\sum_{n\in\mathbf{Z}} |\psi_{\omega,\gamma,x,y}(n)|^2 |n|^D\right]^{1/D} \;, \tag{95}$$

for ω an eigenvalue of $H_{\gamma,x,y}$ and $D \geq 1$.
If the eigenstates decrease exponentially fast one can also consider the quantity :

$$\ell(\omega,\gamma,x,y) = \limsup_{n\mapsto\infty} \frac{-\ln|\psi_{\omega,\gamma,x,y}(n)|}{n} \,. \tag{96}$$

However such expressions are very badly behaving with (x,y) in general and they are not suited for comparison with experiments or numerical calculations. The following reasoning is our proposal to give a correct definition of $\ell^{(D)}$ at least for $D=2$ which avoids the assumption that $H_{\gamma,x,y}$ has point spectrum, and gives rise to a quantity independent of γ, x, y.
To do so we will first average over a spectral interval Δ. If $H_{\gamma,x,y}$ has pure point spectrum, this can be done by looking at (for $D=2$) :

$$\xi_{\Delta,n}(\gamma,x,y) = \sum_{\omega\in\Delta}\sum_{n'} |\psi_{\omega,\gamma,x,y}(n)|^2 |\psi_{w,\gamma,x,y}(n')|^2 |n-n'|^2 \,. \tag{97}$$

The weight now is given by $|\psi_{\omega,\gamma,x,y}(n)|^2$ which satisfies

$$\sum_{\omega\in\Delta} |\psi_{\omega,\gamma,x,y}(n)|^2 = \langle n|\pi_{\gamma,x,y}(P_\Delta)|n'\rangle \,. \tag{98}$$

In particular we get

$$\int \frac{dxdy}{4\pi^2}\left[\sum_{\omega\in\Delta}|\psi_{\omega,\gamma,x,y}(n)|^2\right] = \int \frac{dxdy}{4\pi^2}\langle n|\pi_{\gamma,x,y}(P_\Delta)|n\rangle = = \tau_\gamma(P_\Delta) = \int_\Delta dN(\omega)\ , \tag{99}$$

where dN is the density of states (DOS).
The expression (97) is not obviously related to the algebraic one because the energy ω is the same in both wave functions $|\psi_{\omega,\gamma,x,y}(n)|^2$ and $|\psi_{\omega,\gamma,x,y}(n')|^2$. To overcome this difficulty one can use either one of the following tricks :
(i) if $(X_\omega)_{\omega\in\Delta}$ is a sequence then

$$\sum_{\omega\in\Delta}|X_\omega|^2 = \lim_{T\mapsto\infty}\int_0^T\frac{dt}{T}|\sum_{\omega\in\Delta}X_\omega e^{it\omega/\gamma}|^2\ , \tag{100}$$

in our case it gives :

$$\xi_{\Delta,n}(\gamma,x,y) = \lim_{T\mapsto\infty}\int_0^T\frac{dt}{T}\langle n|\pi_{\gamma,x,y}\left|\partial_\theta e^{itH/\gamma}\right|^2|n\rangle\ , \tag{101}$$

where $|A|^2 = AA^*$ if A is an operator.
(ii) One can also consider a partition $\mathcal{P}$ of Δ into a finite set $(\Delta_1,\Delta_2,\cdots,\Delta_N)$ of subintervals and look at :

$$\xi_{\Delta,n}(\gamma,x,y,\mathcal{P}) = \sum_{\Delta_i\in\mathcal{P}}\sum_{\omega\in\Delta_i}\sum_{\omega'\in\Delta_i}\sum_{n'}\psi_{E,\omega}(n)\overline{\psi_{E,\omega}(n')\psi_{E',\omega}(n)}\psi_{E',\omega}(n')|n-n'|^2\ , \tag{102}$$

$$\xi_{\Delta,n}(\gamma,x,y,\mathcal{P}) = \sum_{\Delta_i\in\mathcal{P}}\langle n|\pi_{\gamma,x,y}(|\partial_\theta P_{\Delta_i}|^2)|n\rangle\ . \tag{103}$$

Then we expect to recover (101) in the limit where the partition $\mathcal{P}$ gets finer and finer while the length of Δ_i tends to zero. Actually it is even better to eliminate the dependence with respect to (x,y) by integrating over them, to get now a purely algebraic object. Then it is no longer necessary to assume that $H_{\gamma,x,y}$ has point spectrum.
Namely if Δ is a Borel subset of the spectrum Σ of H let $\hat{\xi}_\Delta$ be defined as

$$\hat{\xi}_\Delta = \tau(|\partial_\theta P_\Delta|^2)\ . \tag{104}$$

If now $\mathcal{P} = (\Delta_1,\Delta_2,\cdots,\Delta_N)$ is a finite partition of Δ by Borel subsets one sets

$$\xi_\Delta(\mathcal{P}) = \sum_{i=1}^N\hat{\xi}_{\Delta_i}\ . \tag{105}$$

Theorem 9 *As the partition $\mathcal{P}$ gets finer and finer $\xi_\Delta(\mathcal{P})$ increases and converges to its supremum over $\mathcal{P}$. If the limit ξ_Δ is finite then one can find a function $\omega\in\Delta\mapsto\ell(\omega,\gamma)$ depending upon γ, which is measurable with respect to the integrated density of states N and such that :*

$$\xi_\Delta = \lim_{\mathcal{P}}\xi_\Delta(\mathcal{P}) = \sup_{\mathcal{P}}\xi_\Delta(\mathcal{P}) = \int_\Delta dN(\omega)\ell^2(\omega,\gamma)\ . \tag{106}$$

If, in addition, there is a constant $C>0$ such that for any Borel subset Δ' of Δ, $\tau_\gamma(|\partial_\theta P_\Delta|^2)\le C\tau_\gamma(P_{\Delta'})$ then $\ell^2(\omega,\gamma)\le C$ for almost all ω's in Δ with respect to the density of states.

Now this function $\ell(\omega,\gamma)$ will be called the localization length. Roughly speaking $\ell(\omega,\gamma)$ should be equal to the average over (x,y) of $\ell^{(2)}(\omega,\gamma,x,y)$ defined in (95). A similar result is expected for $\ell^{(D)}$ where now $\tau_\gamma(|\partial_\theta P_\Delta|^2)$ must be replaced by $\tau_\gamma(|\partial_\theta^{D/2}P_\Delta|^2)$.
What is nice about these formulae is that localization properties are related to Sobolev's norms over the non commutative phase space, where ∂_θ is the derivative with respect to the angles. This has been remarked for the first time in [39, 50] about the connection between the localization properties and the existence of plateaux in the Quantum Hall Effect.
Thus the faster the average decay, the smoother the corresponding eigenfunction. Note however that P_Δ does not need to be in the C^*-algebra. It is enough that one of its derivative be square integrable. In a non commutative phase space it is not a sufficient condition for P_Δ to be in the C^*-algebra (namely to be continuous).

Now we need to connect this way of defining the localization length to the other trick given by (101). Let us introduce the expression

$$\mathcal{E}_{\gamma,x,y}(t,\Delta) = \gamma^2 \langle 0|\pi_{\gamma,x,y} \left|\partial_\theta e^{itH/\gamma} P_\Delta\right|^2 |0\rangle \ . \tag{107}$$

Remark that by changing (x,y) into $(x-n\gamma, y-nx+n^2\frac{\gamma}{2})$ in (101) one is reduced to $n=0$. We also have

$$\mathcal{E}_{\gamma,x,y}(t,\Delta) = \sum_{n\in\mathbf{Z}} |\langle 0|\pi_{\gamma,x,y}\left(e^{itH/\gamma}\right)|n\rangle|^2 n^2\gamma^2 \ , \tag{108}$$

which reprensents the mean value of the position at time t with respect to the inital state $\langle[\gamma n(t)]^2\rangle$. Actually in the KR problem, the "position" $n\in\mathbf{Z}$ is nothing but the angular momentum in such a way that $\langle[\gamma n(t)]^2\rangle$ is proportional to the mean kinetic energy in time.
To get algebraic quantities let us average over (x,y) to get (dropping the explicit γ-dependence)

$$\mathcal{E}(t,\Delta) = \gamma^2\tau_\gamma\left(|\partial_\theta\left(e^{itH/\gamma}P_\Delta\right)|^2\right) \ . \tag{109}$$

At last we consider the time average of this "kinetic energy" by setting

$$\overline{\mathcal{E}}(\Delta) = \lim_{T\mapsto\infty}\frac{1}{T}\int_0^T dt\mathcal{E}(t,\Delta) \ . \tag{110}$$

Proposition 4 *If $\overline{\mathcal{E}}(\Delta)<\infty$ there exists a function $\omega\in\Delta\mapsto\overline{\mathcal{E}}(\omega)\in\mathbf{R}_+$ which is measurable with respect to the DOS such that $\forall\Delta'\subset\Delta$,*

$$\overline{\mathcal{E}}(\Delta') = \int_{\Delta'}\overline{\mathcal{E}}(\omega)dN(\omega) \ . \tag{111}$$

The argument introduced for (101) and (103) suggests the following result

$$\overline{\mathcal{E}}(\omega) = \gamma^2\ell^2(\omega) \ , \tag{112}$$

for N-almost every ω's.
Eventhough we did not get a complete rigorous proof of this formula in general, it can be proved rigorously in various cases such as the KR model.
The last property concerns the semiclassical limit. If $\Delta=\mathbf{R}$ then one can prove the following :

Proposition 5 *If $H\in\mathcal{B}_I$ and $t\in\mathbf{R}$ is fixed, then the function*

$$\gamma\in I\mapsto\gamma^2\tau_\gamma\left(|\partial_\theta e^{itH/\gamma}|^2\right) = \mathcal{E}_\gamma(t,\mathbf{R}) \tag{113}$$

is continuous even if I contains the origin.

Presumably if Δ is a Borel subset of the spectrum of H, the map $\gamma\mapsto\mathcal{E}_\gamma(t,\Delta)$ is only a Borel one and not continuous.
We also remark that the continuity holds for each given time, but this result does not allow to conclude that the time average $\overline{\mathcal{E}}_\gamma(\mathbf{R})$ be continuous. A sufficient condition for it would be that $\gamma\mapsto\mathcal{E}_\gamma(t,\mathbf{R})$ be continuous uniformly with respect to t; we will see later that it is certainly not true in general.

Let us comment about the connection between the finiteness of $\mathcal{E}(\Delta)$ and the spectral type of H. For physicists the localization occurs provide certain quantities like the localization length or the inverse participation ratio are finite. Mathematicians have proposed to consider the nature of the spectral measure of $\pi_{\gamma,x,y}(H)$ instead. What is the connection between these notions ?
We have given Pastur's criterion for existence of point spectrum. Let us now give another one valid when the mean kinetic energy is finite :

Proposition 6 *Let $H\in\mathcal{B}_\gamma$, be such that $\overline{\mathcal{E}}_\gamma(\Delta)<+\infty$ for some Borel subset Δ of* Sp(H). *Then for almost every $(x,y)\in\mathbf{T}^2$ the spectrum of $\pi_{\gamma,x,y}$ has a pure point component.*

The proof of this proposition consists in showing that $\overline{\mathcal{E}}_\gamma(\Delta)<+\infty\Rightarrow\xi_\Delta>0$ (see (94)) and then to apply Pastur's criterion.
Let us also mention various criterions for characterizing the spectral type of $\pi_{\gamma,x,y}$.

The theorem of Pastur-Ishii-Kotani, valid only for Jacobi matrices as 1D Schrödinger operators, gives a necessary and sufficient condition for absence of absolutely continuous spectrum in term of the positivity of the Lyapunov exponent [15].
All these criterions however are not sufficient to eliminate the possibility of having some singular continuous spectrum. Actually there are examples where it happens. For instance in the KR problem for $\gamma = 0$ and x a Liouville number, one can find examples of C^{∞} potentials giving rise to a singular continuous spectrum [11]. For $\gamma/2\pi$ a Liouville number in the KR model the result of Casati and Guarneri [51] presumably leads to a singular continuous spectrum.
Let us mention however that Howland gave a criterion [12] which permits to characterize the point spectrum. It has been used successfully for a certain class of modified pulsed rotor models. However it is not strong enough to conclude in the KR case.

Application to the KR Model

Let us apply our formalism to the KR model. Now $e^{itH/\gamma}$ is replaced by F given by (16).

Let η_y be the $*$-automorphism of $\mathcal{B}_I$ defined by

$$\eta_y(U) = U \ , \ \eta_y(V) = V \ , \ \eta_y(F_0) = e^{iy} F_0 \ . \tag{114}$$

Then $\eta_y(F) = e^{iy}F$. Therefore η_y translates the spectrum of F by y. Since $\mathrm{Sp}(F) = \mathrm{Sp}(\eta_y(F))$ it follows that

Proposition 7 *[11]* $\mathrm{Sp}(F) = S_1$ *the unit circle.*

Moreover η_y leaves the trace invariant. Therefore for any continuous function f on the circle we get

$$\tau_\gamma(\eta_y(f(F))) = \tau_\gamma(f(F)) = \int_{\mathbf{T}} dN(\omega) f(e^{i\omega}) \ . \tag{115}$$

But

$$\tau_\gamma(\eta_y(f(F))) = \tau_\gamma(f(e^{iy}F)) = \int_{\mathbf{T}} dN(\omega) f\left(e^{i(\omega+y)}\right) \ . \tag{116}$$

Therefore dN is invariant by translation on the torus namely $dN(\omega) = d\omega/2\pi$.

Proposition 8 *The DOS for the KR model is uniform on* $\mathbf{T}$.

The same kind of argument leads to :

Proposition 9 *If the localization length* $\ell_\gamma(\omega)$ *exists for the KR model, it is constant in* ω. *In much the same way the kinetic energy* $\overline{\mathcal{E}}_\gamma(\omega)$ *is independent of* ω.

Proof : Clearly η_y commutes with the derivation ∂_θ (see (33)). Moreover since η_y translates the spectrum by y along the circle we get for any Borel subset Δ of $\mathbf{T}$

$$\eta_y(P_\Delta) = P_{\Delta+y} \ . \tag{117}$$

Thus

$$\tau_\gamma\left(|\partial_\theta P_\Delta|^2\right) = \tau_\gamma\left(\eta_y\left(|\partial_\theta P_\Delta|^2\right)\right) = \tau_\gamma\left(|\partial_\theta(\eta_y P_\Delta)|^2\right) = \tau_\gamma\left(|\partial_\theta P_{\Delta+y}|^2\right) \ , \tag{118}$$

implies

$$\xi_{\Delta+y} = \xi_\Delta \Rightarrow \int_\Delta \ell_\gamma^2(\omega - y) dN = \int_\Delta \ell_\gamma^2(\omega) dN \ , \forall\Delta \ , \tag{119}$$

which proves the result. The same argument leads to $\overline{\mathcal{E}}(\omega) = \overline{\mathcal{E}}, \forall\omega$.

Proposition 10 *For the KR model the following formula holds*

$$\ell^2 = \frac{\overline{\mathcal{E}}}{\gamma^2} = \lim_{T\mapsto\infty} \frac{1}{T} \sum_{t=0}^{T} \tau_\gamma\left(|\partial_\theta(F^t)|^2\right) \tag{120}$$

Proposition 11 $\gamma \mapsto \ell^2(\gamma)$ *is lower semi continuous.*

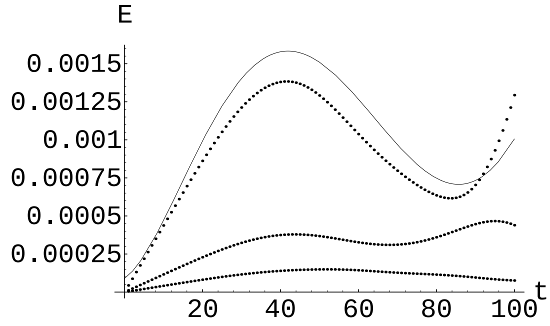

Figure 1: *Comparison of kinetic energies for the classical and the quantum rotator when $K = 0.01$. The full curve is associated to the classical energy. The dashed curves correspond to quantum energies for various values of the effective Planck's constant γ.*

Remark : It is not possible to expect a much better result. The simple examples given by $F = Ve^{if(U)/\gamma}$ show that ℓ^2 is infinite on a dense set of points whenever f is a function with an infinite number of non zero Fourier coefficients. Presumably in the KR model the same is true.

However we may expect $\gamma^2\ell^2(\gamma)$ to converge to some finite quantity as $\gamma \mapsto 0$. This is the contents of the Chirikov-Izrailev-Shepelyansky formula at least in the KR case found on the basis of a numerical work. The well-known observation is that despite the diffusive behavior of the classical model (namely for strong coupling) the quantized version exhibits, up to a certain breaking time τ^*, a diffusion-like motion in phase space and then for $t > \tau^*$ its kinetic energy saturates as a function of time. This numerical result allows us to write

$$\mathcal{E}_\gamma(t) = D\tau^* = \mathcal{E}_{cl} \ , \tag{121}$$

where D is the classical diffusion coefficient. The problem is that D does not exist as an average on the torus. This means in particular that we should replace the trace τ_γ by a quantum state localized in a phase space region tending to zero as $\gamma \mapsto 0$. But for the sawtooth map, for instance, this diffusion coefficient exists [52].
(120) gives another relation $\gamma^2\ell_\gamma^2 = \mathcal{E}_\gamma(t) \Rightarrow$

$$D\tau^* = \gamma^2\ell_\gamma^2 \ . \tag{122}$$

If we then consider that $\mathrm{Sp}(F)$ is continuous for $t < \tau^*$ then $\Delta\omega \sim 2\pi/\ell$ as long as $t\Delta\omega \ll 2\pi$. Therefore $\tau^*\Delta\omega \sim 2\pi$ and then

$$\tau^* \sim \ell \tag{123}$$

For the moment we do not know how to define mathematically the breaking time τ^*.

We would like now to study the behavior of the kinetic energy for the quantized version of the KR model as the effective Planck constant γ tends to zero. For that, we perform a numerical calculation giving the classical and quantum energies of the KR for two different values of K namely $K = 0.01$ for the KAM regime (Fig. 1) , and $K = 4$ for the diffusive regime (Fig. 2).

We computed the quantal energy for different values of Planck's constant γ in both cases; it is easy to see that as γ is decreased the quantal curves tend to the classical one. One could think that this energy converges to its classical limit as $\gamma \mapsto 0$ but a problem arises because of the uniformity of the semiclassical limit with respect to time. In fact one expects the existence of a breaking time destroying this uniformity. Presumably this breaking time τ^* is of order $O(\gamma^{-2})$. This can be shown by heuristic arguments : as pointed out by (123) τ^* is of order $O(\ell)$ which is of order $O(\gamma^{-2})$. One argument is to notice that the quantum effects only appear at order $O(\gamma^{-2})$ [53]. In fact we also remark that this breaking time does not exist when the semiclassical approximation is exact; this can be seen in the hydrogen atom, the harmonic oscillator, the Arnold cat map...

Let us consider now the Feynman path integral in the case of the KR model. Let us recall that the evolution of a solution of Schrödinger's equation is given by

$$(F^{-t}\Psi)(u) = \frac{e^{-it\pi/4}}{(2\pi\gamma)^{t/2}} \int_{\mathbf{R}^t} du_1 \cdots du_t e^{i\mathcal{L}_t(u)/\gamma}\psi(u_t) \ , \tag{124}$$

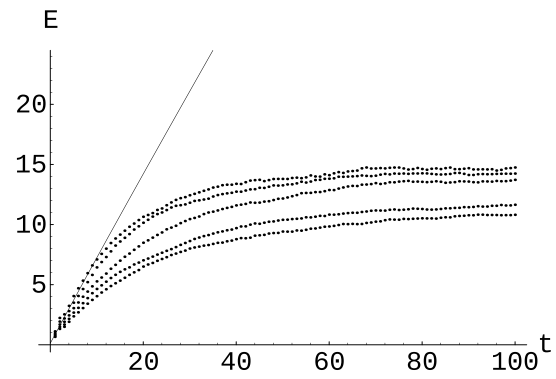

Figure 2: *Comparison of kinetic energies for the classical and the quantum rotator when $K = 4$. The full curve is associated to the classical energy. The dashed curves correspond to quantum energies for various values of the effective Planck's constant γ. When γ is decreasing we reach the classical curve (extracted from [22]).*

where $\mathcal{L}_t(u)$ is defined in (19) and $x = -\mu BT$ is the dimensionless magnetic field.
As explained before the trajectories of interest are determined by $\theta_0 = u_0$; $A_{t+1} = x$ while the stationnary phase points denoted with a hat have to satisfy $2\hat{u}_s - \hat{u}_{s+1} - \hat{u}_{s-1} + K\sin(\hat{u}_s) = 0$ for $1 \leq s \leq t-1$, $\hat{u}_t - \hat{u}_{t+1} + K\sin(\hat{u}_t) = 0$.
Using the expression of the kinetic energy as a function of the evolution operator (120) we are led to :

$$\begin{aligned} \mathcal{E}_\gamma(t) &= \frac{\gamma^2}{(2\pi\gamma)^t}\int \frac{d\theta_0 dx}{4\pi^2} \times \\ &\times \int_{u_0=v_0, u_t=v_t} du_1\cdots du_t dv_1\cdots dv_t e^{\frac{i}{\gamma}(\mathcal{L}_t(u)-\mathcal{L}_t(v))} \times \\ &\times (u_t - u_{t-1} - u_1 + u_0 + K\sin(u_t)) \times \\ &\times ((v_t - v_{t-1} - v_1 + v_0 + K\sin(v_t)) \ , \end{aligned} \tag{125}$$

and considering it at stationnary phase points $(\hat{u}, \hat{v})$:

$$\mathcal{E}_\gamma(t) = \gamma^2 \sum_{\hat{u},\hat{v}} \int \frac{d\theta_0 dx}{4\pi^2} \frac{e^{\frac{i}{\gamma}(\mathcal{L}_t(\hat{u})-\mathcal{L}_t(\hat{v}))}(A_{t+1}-A_1)(\hat{u})(A_{t+1}-A_1)(\hat{v})}{\sqrt{det(M(\hat{u}))det(M(\hat{v}))}} \ , \tag{126}$$

where $M(u)_{ss'} = \partial^2\mathcal{L}/\partial u_s \partial u_{s'}$.
We can separate the previous expression into two parts such that

$$\mathcal{E}_\gamma(t) = \gamma^2 \sum_{\hat{u}=\hat{v}} (\ ) + \gamma^2 \sum_{\hat{u}\neq\hat{v}} (\ ) \ . \tag{127}$$

The first term can be easily identified to the classical energy namely

$$\sum_{\hat{u}=\hat{v}} (\ ) = \int \frac{d\theta_0 dx}{4\pi^2} \frac{(A_{t+1}-A_1)^2(\hat{u})}{det(M(\hat{u}))} = \int_{\mathbf{T}^2} \frac{d\theta_0 dA_0}{4\pi^2}(A_{t+1}-A_1)^2 = \mathcal{E}_{cl}(t) \ . \tag{128}$$

The second term is the interference term. It is expected to be uniformly bounded in the KAM regime namely for $K \ll 1$. We may also expect that it partially compensates the first term at large time for K large, leading to a saturation of the quantum energy. An estimate of this term may probably lead to an estimate of the breaking time and perhaps a calculation of corrections if any.

4 Conclusion

Using the algebraic framework it is possible to define properly the notion of correlation function and of localization length, and to get some properties of these expressions as functions of the effective Planck constant. In particular there is a general connection between this localization length and what is called the "kinetic energy" justifying half of the Chirikov-Izrailev-Shepelyansky formula.
However the results given here are not sufficient to define in a non phenomenological way the notion of "breaking time", as the time scale beyond which the interference effect becomes dominant. Possibly the Feynman path integral may give a hint towards a good mathematical definition of the time scale.

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