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Resistance and fluctuation of a fractal network of random resistors: a non-linear law of large numbers

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Abstract. We study rigorously the resistance and fluctuation of resistance of a large deterministic fractal lattice in the limit of an infinite number of resistors. We give estimates on corrections to the effective medium approximation of the total resistance. We prove scaling laws for the relative fluctuation, and prove that the normalised relative fluctuation converges in distribution to the standard normal variable. This is a kind of non-linear law of large numbers.

1. Introduction

In this paper we investigate rigorously an example of a fractal network of random resistors. The motivation can be found in recent work by Giraud *et al* [1, 2], measuring flicker noise of the determinstic fractal lattice (DFL), a model proposed by Kirkpatrick to mimic some properties of percolation clusters in random media and disordered systems [3, 4]. Our goal is to study theoretically the influence of noise on the same lattice.

We will restrict ourselves to the case for which the resistances of each branch of the network are independent identically distributed positive random variables and we would like to compute the behaviour of the total resistance as the size of the network goes to infinity. We will give exact corrections to an effective medium approach [5-8] produced by fluctuations of the average resistance. We will also study the variance of the fluctuation which is related to the magnitude power spectrum of flicker noise. We prove rigorously that scaling laws obtained from a first-order calculation hold [9-12]; however we produce exact correction to the leading terms.

On the other hand, in the limit of infinitely many resistors the fluctuation actually decreases to zero fast enough to allow a linear theory to hold. As a result the total normalised fluctuation will converge in distribution to the standard normal variable, even though the total resistance is a non-linear function of the individual ones.

From some recent experimental results it seems that the $1/f$ law dependence of flicker noise may be due to fluctuation of microscopic local resistance [13-16]. This explains the recent interest of random resistor network [9, 12, 13, 17, 18], where most studies also concern the effect of geometrical self-similarity on electrical noise in a macroscopic resistor network.

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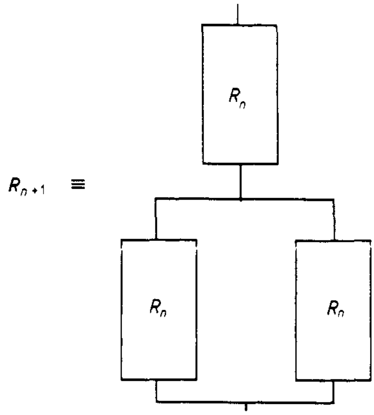

Figure 1. The recurrence definition of the equivalent circuit of the direct current deterministic fractal lattice.

The deterministic fractal lattice itself which is well defined in [3, 4, 19] is built out of the equivalent circuit defined recursively by figure 1.

Let R_n be the random resistance at each step n of the lattice, and $\langle R_n\rangle$ and σ_n be, respectively, the mean value and the variance of R_n. Finally let ρ_n be the normalised fluctuation given by

$$\rho_n = \frac{R_n - \langle R_n\rangle}{\sigma_n}.$$

The main result of this paper is the following.

Theorem. If R_0 is a positive random variable such that $\langle R_0^2\rangle < \infty$ then:

(i) effective medium estimates:

(*a*) $\lim_{n\to\infty} (\frac{2}{3})^n\langle R_n\rangle = R_\infty$ exists and $(\frac{2}{3})^n R_n$ converges almost surely to R_∞;

(*b*) $\langle R_0\rangle - \frac{3}{2}\sigma_0 \leqslant R_\infty \leqslant \langle R_0\rangle$.

(ii) If, in addition, $\langle R_0^4\rangle < \infty$, $R_\infty > 0$ and $\sigma_0 > 0$ then

(*a*) $\lim_{n\to\infty} (\frac{2}{3})^n 2^{n/2}\sigma_n = \sigma_\infty$ exists;

(*b*) $|\sigma_\infty/\sigma_0 - 1| \leqslant O(\sigma_0)$ as $\sigma_0 \to 0$;

(*c*) the sequence ρ_n of the normalised random variable converges in distribution to the standard normal variable.

We organise the paper as follows: in § 2 we compute fluctuation laws and some moment inequalities; in § 3 we give a strong law of large numbers for R_n and we study the behaviour of the variance; in § 4 we study the normalised variable ρ_n.

2. Fluctuation law and moment inequalities

2.1. Fluctuation law

For deterministic resistors (cf figure 1) Ohm's law gives immediately

$$R_n = (\tfrac{3}{2})^n R_0.$$

Writing the resistance at each step of the recursion as $R_n = \lambda_n r_n$ with $\lambda_n = (\frac{3}{2})^n \langle R_0 \rangle$ and r_n a positive random variable, we obtain recursively

$$\begin{aligned} r_{n+1} &= \frac{2}{3}\left(r_n^{(0)} + \frac{r_n^{(1)} r_n^{(2)}}{r_n^{(1)} + r_n^{(2)}} \right) \\ &= \frac{2}{3}\left(r_n^{(0)} + \tfrac{1}{4}(r_n^{(1)} + r_n^{(2)}) - \frac{1}{4}\,\frac{(r_n^{(1)} - r_n^{(2)})^2}{r_n^{(1)} + r_n^{(2)}} \right) \end{aligned} \tag{2.1}$$

where $r_n^{(0)}$, $r_n^{(1)}$, $r_n^{(2)}$ are independent identically distributed positive random variables.

We denote by s_n the variance relative to r_n defined by

$$s_n^2 = (\langle r_n^2 \rangle - \langle r_n \rangle^2).$$

We have therefore $\sigma_n = \lambda_n s_n$.

We define A_n, Z_n and $\bar{\rho}_n$ as follows:

$$A_n = \frac{1}{6}\,\frac{(r_n^{(1)} - r_n^{(2)})^2}{r_n^{(1)} + r_n^{(2)}}$$

$$Z_n = A_n - \langle A_n \rangle$$

$$\bar{\rho}_n = \tfrac{2}{3} s_n [\rho_n^{(0)} + \tfrac{1}{4}(\rho_n^{(1)} + \rho_n^{(2)})].$$

For any n we get

$$s_{n+1}\rho_{n+1} = \bar{\rho}_n - Z_n. \tag{2.2}$$

2.2. *Moment inequalities*

We want to compute some inequalities between moments of r_n.

Proposition 1. Let α be real.

(i) If $\max r_0 \leqslant \alpha$ then $r_n \leqslant \alpha$ for all n.

(ii) If $\min r_0 \geqslant \alpha$ then $r_n \geqslant \alpha$ for all n.

Proof. By induction it suffices to prove that if $r_n^{(i)} \leqslant$ (or $\geqslant$) α ($i = 0$–2) then $r_{n+1} \leqslant (\geqslant)$ α. We remark that

$$r_{n+1} = f(r_n^{(0)}, r_n^{(1)}, r_n^{(2)})$$

where

$$f(z, y, x) = \frac{2}{3}\left(z + \frac{xy}{x+y} \right)$$

is a non-decreasing function of each variable. Therefore

$$x \leqslant \alpha,\ y \leqslant \alpha,\ z \leqslant \alpha \Rightarrow f(z, y, x) \leqslant f(\alpha, \alpha, \alpha) = \alpha$$

and the same is true for the lower bound.

This result may hold for a general network, as one can easily verify.

Lemma 2. If r_0 is positive then $0 \leqslant A_n \leqslant \frac{1}{6}|(r_n^{(1)} - r_n^{(2)})|$.

Proof. This is simple due to the fact that, for two non-negative real x, y,

$$|x - y| \leqslant x + y.$$

Proposition 3. Let p be a number such that $\langle r_0^p\rangle$ exists. Then for all n and for all $j \leqslant p$ $\langle r_n^j\rangle$ exists. Moreover

(i) if $\langle r_0\rangle$ exists then $\langle r_n\rangle$ is a decreasing sequence;

(ii) if $\langle r_0^2\rangle$ exists then

$$\langle r_{n+1}^2\rangle \leqslant \min(\tfrac{1}{2}(\langle r_n^2\rangle + \langle r_n\rangle^2), \tfrac{1}{9}(4\langle r_n^2\rangle + 5\langle r_n\rangle^2))$$

and $\langle r_n^2\rangle$ is also a decreasing sequence.

(iii) For any $p \geqslant 2$ such that $\langle r_0^p\rangle$ exists. There are two positive real numbers $\alpha \leqslant \frac{1}{2}$ and $\beta(p)$ such that $\langle r_{n+1}^p\rangle \leqslant \alpha\langle r_n^p\rangle + \beta(p)$.

Proof. From (2.1) one deduces easily that

$$\langle r_{n+1}^p\rangle \leqslant (\tfrac{2}{3})^p\langle [r_n^{(0)} + \tfrac{1}{4}(r_n^{(1)} + r_n^{(2)})]^p\rangle$$

where one obtains (i) and a part of (ii) and (iii) for $p = 1$ and $p = 2$, respectively. Now using proposition 1 and lemma 2 one gets

$$\langle r_{n+1}^2\rangle = \left\langle\left[\frac{2}{3}\left(r_n^{(0)} + \tfrac{1}{4}(r_n^{(1)} + r_n^{(2)}) - \frac{1}{4}\,\frac{(r_n^{(1)} - r_n^{(2)})^2}{r_n^{(1)} + r_n^{(2)}}\right)\right]^2\right\rangle$$

$$\leqslant \left(\frac{2}{3}\right)^2 \langle [r_n^{(0)} + \tfrac{1}{4}(r_n^{(1)} + r_n^{(2)})]^2\rangle - \tfrac{1}{36}\langle (r_n^{(1)} - r_n^{(2)})^2\rangle$$

where one obtains immediately the desired result of (ii) of proposition 3. From the binomial expansion and some elementary inequalities one obtains

$$\langle r_{n+1}^p\rangle \leqslant (\tfrac{2}{3})^p\{[1 + 2(\tfrac{1}{4})^p]\langle r_n^p\rangle + [(\tfrac{3}{2})^p - (\tfrac{1}{2})^p - 1](1 + \langle r_n^{p-1}\rangle)^3\}.$$

Assuming by induction on p that $\sup_n \langle r_n^{p-1}\rangle = \mu < \infty$ we get (iii) by identifying α with $(\frac{2}{3})^p[1 + 2(\frac{1}{4})^p]$ and $\beta(p)$ to $[1 - (\frac{1}{3})^p - (\frac{2}{3})^p](1 + \mu)^3$. Then it follows that $\sup_n \langle r_n^p\rangle \leqslant 2\beta + \alpha\langle r_0^p\rangle$ which is finite if $\langle r_0^p\rangle < \infty$, leading to the conclusions. Let $\lim_{n\to\infty}\langle r_n\rangle = r_\infty$. This obviously means also that $R_\infty = \langle R_0\rangle r_\infty$. From the previous lemma one obtains corollary 4.

Corollary 4. If $\langle r_0\rangle$ exists then

$$r_\infty = \langle r_0\rangle - \sum_{n=0}^{\infty} \langle A_n\rangle$$

where the series converges.

Remark 1. It is important to note that, even if we consider $\langle r_0\rangle = 1$, we cannot have, for all n, $\langle r_n\rangle = 1$. But by proposition 3 clearly we will have $\langle r_n\rangle \leqslant 1$, where the equality holds only for a Dirac probability density. The term $\Sigma_{n=0}^{\infty}\langle a_n\rangle$, due to the non-linearity of the network, brings a correction to an effective medium approach which gives only the leading term of r_∞: $\langle r_0\rangle$. One can verify that in the simple case where r_0 takes two values with probability 0.5, $\langle A_0\rangle$ and $\langle A_1\rangle$ are different to zero, so that $\Sigma_{n=0}^{\infty}\langle a_n\rangle \neq 0$.

Remark 2. From corollary 4, we do not know whether $r_\infty > 0$ or if r_∞ vanishes unless σ_0 is small enough. We have not been able to eliminate the possibility that $r_\infty = 0$ for high values of σ_0. If $r_\infty = 0$ the dominant contribution to R_n as n tends to infinity is not given by $(\frac{3}{2})^n$ constant, but is corrected by the effect of large fluctuation. We will assume that $r_\infty > 0$ in the following; a sufficient condition for it is that σ_0 be small enough as we will see in the next section.

3. Convergence of R_n

3.1. Upper and lower bound of the first and second moment of A_n

Proposition 5. If $\langle r_0^2 \rangle$ exists then

$$\text{(i)} \qquad \langle A_n \rangle \leqslant \frac{\sqrt{2}}{6} s_n. \tag{3.1}$$

$$\text{(ii)} \qquad \langle A_n^2 \rangle \leqslant \tfrac{1}{18} s_n^2. \tag{3.2}$$

In addition, if $r_\infty \neq 0$ then

$$\text{(iii)} \qquad \langle A_n \rangle \leqslant \frac{2}{3} \frac{s_n^2}{r_\infty}. \tag{3.3}$$

$$\text{(iv)} \qquad \langle A_n^2 \rangle \leqslant \frac{1}{9} \left(\frac{s_n^2}{r_\infty} \right)^2 (\langle \rho_n^4 \rangle + 1). \tag{3.4}$$

Proof. From lemma 2 one gets for all p

$$A_n^p \leqslant [\tfrac{1}{4} |(r_n^{(1)} - r_n^{(2)})|]^p$$

leading to (2.2). Using the Cauchy–Schwartz inequality (3.1) follows for $p = 1$.

On the other hand, let a be a positive number such that, for all n, $\langle r_n \rangle - a > 0$. Then

$$\langle A_n^p \rangle \leqslant \langle A_n^p \chi[s_n(\rho_n^{(1)} + \rho_n^{(2)}) \leqslant -2a] \rangle + \langle A_n^p \chi[s_n(\rho_n^{(1)} + \rho_n^{(2)}) \geqslant -2a) \rangle$$

where $\chi(A)$ is the characteristic function of the set A. Thus

$$\langle A_n^p \chi[s_n(\rho_n^{(1)} + \rho_n^{(2)}) \geqslant -2a] \rangle \leqslant (\tfrac{1}{12})^p s_n^{2p} \left(\frac{1}{\langle r_n \rangle - a} \right)^p \langle (\rho_n^{(1)} - \rho_n^{(2)})^{2p} \rangle.$$

Moreover

$$\chi[s_n(\rho_n^{(1)} + \rho_n^{(2)}) \leqslant -2a] \leqslant \left(\frac{s_n}{2a} |\rho_n^{(1)} + \rho_n^{(2)}| \right)^p.$$

Using lemma 2 one has

$$\langle A_n^p \chi[s_n(\rho_n^{(1)} + \rho_n^{(2)}) \leqslant -2a] \rangle \leqslant \left\langle (\tfrac{1}{6} |r_n^{(1)} - r_n^{(2)}|)^p \left(\frac{s_n}{2a} |\rho_n^{(1)} + \rho_n^{(2)}| \right)^p \right\rangle.$$

Applying the Cauchy–Schwartz inequality and using proposition 3, one obtains for $p = 1$

$$\begin{aligned} \langle A_n \rangle &\leqslant \tfrac{1}{6} s_n^2 \left(\frac{1}{\langle r_n \rangle - a} + \frac{1}{a} \right) \\ &\leqslant \tfrac{1}{6} s_n^2 \left(\frac{1}{r_\infty - a} + \frac{1}{a} \right) \end{aligned}$$

and for $p = 2$

$$\begin{aligned} \langle A_n^2 \rangle &\leqslant s_n^4 \left(\frac{\langle \rho_n^4 \rangle + 3}{(\langle r_n \rangle - a)^2} + \frac{\langle \rho_n^4 \rangle - 1}{a^2} \right) \\ &\leqslant \tfrac{1}{72} s_n^4 \left(\frac{\langle \rho_n^4 \rangle + 3}{(r_\infty - a)^2} + \frac{\langle \rho_n^4 \rangle - 1}{a^2} \right). \end{aligned}$$

Taking $a = \frac{1}{2} r_\infty$ one easily obtains (2.3) and (2.4).

3.2. Weak estimates on the variance

Lemma 6. If $\langle r_0^2\rangle$ exists then for all n

$$\tfrac{7}{18}s_n^2 \leqslant s_{n+1}^2 \leqslant \tfrac{2}{3}s_n^2.$$

Proof. From (2.2) one has

$$\langle (s_{n+1}\rho_{n+1})^2\rangle = \langle \bar\rho_n^2\rangle - 2\langle \bar\rho_n Z_n\rangle + \langle Z_n^2\rangle.$$

By direct computation one finds

$$\langle \bar\rho_n^2\rangle = \tfrac{1}{2}s_n^2$$

and

$$|\langle \bar\rho_n Z_n\rangle| = |\langle \tfrac{1}{6}s_n(\rho_n^{(1)}+\rho_n^{(2)})A_n\rangle|.$$

Applying the Cauchy-Schwartz inequality to the last equality and (3.2) one gets

$$|\langle \bar\rho_n Z_n\rangle| \leqslant \tfrac{1}{18}s_n^2$$

and obviously

$$0 \leqslant \langle Z_n^2\rangle = \langle A_n^2\rangle - \langle A_n\rangle^2 \leqslant \langle A_n^2\rangle \leqslant \tfrac{1}{18}s_n^2.$$

These estimates lead to lemma 6.

3.3. Lower bound of r_∞

Using corollary 4 and lemma 6 one gets the following proposition.

Proposition 7. If $\langle r_0^2\rangle$ exists then

$$r_\infty \geqslant \langle r_0\rangle - \frac{3\sqrt{2}+2\sqrt{3}}{6}\, s_0.$$

By multiplying the relation of proposition 7 by $\langle R_0\rangle$, one obtains obviously

$$R_\infty \geqslant \langle R_0\rangle - \frac{3\sqrt{2}+2\sqrt{3}}{6}\, \sigma_0.$$

From propositions 1 and 7 one obtains two sufficient conditions for $r_\infty \neq 0$.

3.4. Convergence of r_n

Theorem 8. If $\langle r_0^2\rangle$ exists then r_n converges almost surely to r_∞.

Proof. By the Borel-Cantelli lemma and the Tchebyshev inequality [20-23] it suffices to show that

$$\sum \langle |r_n - r_\infty|\rangle < \infty.$$

We have

$$\sum \langle |r_n - r_\infty.|\rangle \leqslant \sum \langle |r_n - \langle r_n\rangle|\rangle + \sum \langle |\langle r_n\rangle - r_\infty.|\rangle$$
$$\leqslant \sum s_n + \frac{\sqrt{2}}{6}\sum_{m=1}^{\infty}\sum_{n=m}^{\infty} s_n$$

where in the last inequality we have used (3.1), so that the conclusion follows from lemma 6.

In other words this theorem means that $(\frac{2}{3})^n R_n$ converges almost surely to R_∞. As a result, when the system is large, fluctuations are small so that the value of the total resistance of the network is almost constant. However it differs from the value obtained by taking the equivalent resistance of the average value of the individual resistance of the network.

3.5. Strong estimates on the variance

Let us introduce the quantities

$$\delta_1(n) = \frac{2\sqrt{2}}{9}\frac{s_n}{r_\infty}(\langle\rho_n^4\rangle)^{1/2} + \frac{2}{9}\left(\frac{s_n}{r_\infty}\right)^2\langle\rho_n^4\rangle + \frac{2}{9}\frac{s_n}{r_\infty}\left(\sqrt{2}+\frac{1}{9}\frac{s_n}{r_\infty}\right)$$

$$\delta_2(n) = \frac{2\sqrt{2}}{9}\frac{s_n}{r_\infty}[(\langle\rho_n^4\rangle)^{1/2}+1].$$

Then we get a better estimate which will be useful in § 4.

Lemma 9. If $\langle r_0^2\rangle$ exists and $r_\infty \neq 0$ then
(i) $s_{n+1}^2 \leqslant \frac{1}{2}s_n^2(1+\delta_1(n))$
(ii) $s_{n+1}^2 \geqslant \frac{1}{2}s_n^2(1-\delta_2(n))$.

Proof. In much the same way, using (3.4) instead of (3.2) in the proof of lemma 6 one gets lemma 9.

We will give in the next section some sufficient condition for $\delta_i(n)$ to be a general term for a convergent series.

4. Convergence of the normalised variable

4.1. Existence of the fourth moment of the normalised variable

Let us denote

$$u_1(n) = \tfrac{2}{9} + \tfrac{14}{27}s_n/r_\infty + \tfrac{8}{27}(s_n/r_\infty)^2$$

$$u_2(n) = \tfrac{5}{27} + \tfrac{16}{27}s_n/r_\infty + \tfrac{24}{27}(s_n/r_\infty)^2 + \tfrac{16}{81}(s_n/r_\infty)^4.$$

Thanks to lemma 6, theses quantities converge to their constant term as $n \to \infty$.

Lemma 10. If $r_\infty \neq 0$ then
(i) $s_{n+1}^4\langle\rho_{n+1}^4\rangle \leqslant s_n^4(\langle\rho_n^4\rangle u_1(n) + u_2(n))$
(ii) $s_{n+1}^4\langle\rho_{n+1}^4\rangle \geqslant s_n^4[\langle\rho_n^4\rangle(\frac{121}{648} - \frac{14}{27}s_n/r_\infty) + \frac{25}{216} - \frac{16}{27}s_n/r_\infty - \frac{8}{27}(s_n/r_\infty)^2]$.

Proof. From (2.2) one gets

$$s_{n+1}^4\langle\rho_{n+1}^4\rangle = \langle\bar{\rho}_n^4\rangle - 4\langle\bar{\rho}_n^3 Z_n\rangle + 6\langle\bar{\rho}_n^2 Z_n^2\rangle - 4\langle\bar{\rho}_n Z_n^3\rangle + \langle Z_n^4\rangle.$$

By a direct computation

$$\langle\bar{\rho}_n^4\rangle = s_n^4(\tfrac{43}{216}\langle\rho_n^4\rangle + \tfrac{11}{72}).$$

Using the Holder inequality, lemma 2 and following the proof of lemma 6 one also computes the following inequalities:

$$|4\langle\bar{\rho}_n^3 Z_n\rangle| \leqslant s_n^4\left(\frac{8\sqrt{82}}{27}\frac{s_n}{r_\infty}(\langle\rho_n^4\rangle)^{1/2}+\tfrac{1}{162}\langle\rho_n^4\rangle+\tfrac{1}{54}+\frac{8\sqrt{2}}{27}\frac{s_n}{r_\infty}\right)$$

$$6\langle\bar{\rho}_n^2 Z_n^2\rangle \leqslant s_n^4\{[\tfrac{1}{108}+\tfrac{8}{27}(s_n/r_\infty)^2]\langle\rho_n^4\rangle-\tfrac{1}{108}+\tfrac{12}{27}(s_n/r_\infty)^2\}$$

$$|4\langle\bar{\rho}_n Z_n^3\rangle| \leqslant s_n^4\left[\frac{2}{27}\frac{s_n}{r_\infty}(\langle\rho_n^4\rangle)^{1/2}+\tfrac{1}{162}\langle\rho_n^4\rangle+\tfrac{1}{54}+\frac{2\sqrt{3}}{27}\frac{s_n}{r_\infty}+\tfrac{8}{27}\left(\frac{s_n}{r_\infty}\right)^2\right]$$

$$\langle Z_n^4\rangle \leqslant s_n^4[\tfrac{1}{648}\langle\rho_n^4\rangle+\tfrac{1}{216}+\tfrac{4}{27}(s_n/r_\infty)^2+\tfrac{16}{81}(s_n/r_\infty)^4].$$

The proof from there is straightforward.

Lemma 11. If $\langle\rho_0^4\rangle<\infty$ and $r_\infty\neq 0$ then for all n there exist a constant $C(r_\infty, s_0, \langle\rho_0^4\rangle)$

$$\langle\rho_n^4\rangle \leqslant C(r_\infty, s_0, \langle\rho_0^4\rangle).$$

Proof. Let us denote

$$v_1(n)=s_n^2\langle\rho_n^4\rangle.$$

Then by lemmas 6 and 9 one shows by induction using lemma 18 (see the appendix) that there exists a constant c_1 such that

$$v_1(n)\leqslant c_1 s_n.$$

Therefore from lemma 9 there exists n_0 such that for all $n\geqslant n_0$

$$\tfrac{1}{2}(1-\delta_2(n))\geqslant\frac{1}{2}-\frac{\sqrt{2}}{9r_\infty}(c_1 s_n)^{1/2}-\frac{\sqrt{2}}{9}\frac{s_n}{r_\infty}>0$$

so that

$$\langle\rho_{n+1}^4\rangle\leqslant\left(\frac{1}{2}-\frac{\sqrt{2}}{9r_\infty}(c_1 s_n)^{1/2}-\frac{\sqrt{2}}{9}\frac{s_n}{r_\infty}\right)^2(\langle\rho_n^4\rangle u_1(n)+u_2(n)).$$

Using lemma 18 again one gets the result.

A direct consequence of the lemma is the following corollary.

Corollary 12. If $\langle\rho_0^4\rangle<\infty$ and $r_\infty\neq 0$ then $\Sigma_{n=1}^{\infty}\delta_i(n)<\infty$ for $i=1, 2$.

Remark 3. From the previous corollary $\Pi(1+\delta_i(n))$ converges. Thus $s_n=(\frac{1}{2})^n c_n s_0$ where c_n converge to a finite non-zero constant as $n\to\infty$.

4.2. Behaviour of the relative fluctuation

The so-called 'relative fluctuation' is defined by

$$S_n=\frac{\sigma_n^2}{\langle R_n\rangle^2}.$$

It is easy to compute from here that the relative fluctuation is related order by order to r_n and s_n by

$$S_n = \frac{s_n^2}{\langle r_n \rangle^2}.$$

From corollary 12 and lemma 9 one finds immediately the next proposition.

Proposition 13. If $\langle r_0^2 \rangle$ exists and $r_\infty \neq 0$ then $\lim_{n\to\infty} 2^n S_n = S_\infty$ exists.

Remark 4. It is clear from propositions 3 and 11 that $S_\infty \neq S_0$. Using the Rammal *et al* association law [11] we would have found $S_n = (\frac{1}{2})^n S_0$, namely $S_\infty = S_0$. The difference between these two terms is due to the fact that $2\langle \bar{\rho}_n Z_n \rangle \neq \langle Z_n^2 \rangle$ in our case. Nevertheless the limit in proposition 13 exists because as we have shown $\langle \bar{\rho}_n Z_n \rangle$ and $\langle Z_n^2 \rangle$ can be neglected with respect to $\langle \bar{\rho}_n^2 \rangle = \frac{1}{2} s_n^2 \approx s_{n+1}^2$ as n goes to infinity.

4.3. Convergence of ρ_n

Let F_n denote the characteristic function of ρ_n and let us set $o_i(n)$ ($i = 1$–3) as follows:

$$o_1(n) = \tfrac{2}{3} s_n / s_{n+1}$$
$$o_2(n) = o_3(n) = \tfrac{1}{6} s_n / s_{n+1}.$$

If we also define δF_n by

$$\delta F_n(t) = F_{n+1}(t) - F_n(o_1(n)t) F_n(o_2(n)t) F_n(o_3(n)t) \tag{4.1}$$

then we get lemma 14.

Lemma 14. If $\langle \rho_0^4 \rangle < \infty$ and $r_\infty \neq 0$ then there exists a constant c such that for all t

$$|\delta F_n(t)| \leq c t^2 (s_n + s_n^2).$$

Proof. From the Taylor expansion one obtains

$$|\delta F_n(t)| \leq t^2 \left[\frac{1}{s_{n+1}} (\langle A_n^2 \rangle)^{1/2} + \langle A_n^2 \rangle \frac{1}{2} \left(\frac{1}{s_{n+1}} \right)^2 \right]$$

from which, using (3.4), one easily obtains lemma 14 by identifying c with

$$\max\left(\sup_n \tfrac{1}{7} (\langle \rho_n^4 \rangle + 1)(1/r_\infty)^2, 1/2 \right)$$

which is finite by lemma 11.

For $j \geq 1$ let B_j be the set $\{1, 2, 3\}^{\times j}$.

Lemma 15. If $\langle \rho_0^4 \rangle < \infty$ and $r_\infty \neq 0$ then

$$\lim_{k\to\infty} \lim_{n\to\infty} \sum_{(i_1, i_2, \ldots, i_k) \in B_k} [o_{i_1}(n+k-1) o_{i_2}(n+k-2) \ldots o_{i_k}(n)]^\alpha = \begin{cases} \infty & \text{if } \alpha < 2 \\ 1 & \text{if } \alpha = 2 \\ 0 & \text{if } \alpha > 2. \end{cases}$$

Proof. One easily gets by induction that

$$\sum_{(i_1,i_2,\ldots,i_k)\in B_k} [o_{i_1}(n+k-1)o_{i_2}(n+k-2)\ldots o_{i_k}(n)]^\alpha = [(\tfrac{2}{3})^\alpha + 2(\tfrac{1}{6})^\alpha](s_n/s_{n+k-1})^\alpha.$$

Using lemma 9 and corollary 12 one obtains the desired result.

Lemma 16. If $\langle|\rho_n|^3\rangle$ exists then for all t $|F_n(t)-\exp(-t^2/2)| \leqslant \frac{1}{6}|t|^3\langle|\rho_n|^3\rangle + \frac{1}{8}t^4$.

Proof. As ρ_n is normalised one has

$$|F_n(t)-\exp(-t^2/2)| = |\langle\exp(\mathrm{i}t\rho_n) - 1 - \mathrm{i}t\rho_n + \tfrac{1}{2}t^2\rho_n^2 - (\exp(-t^2/2) - 1 + \tfrac{1}{2}t^2)\rangle 1$$

But one shows [23] by integration by parts that

$$|\exp(\mathrm{i}t\rho_n) - 1 - \mathrm{i}t\rho_n + \tfrac{1}{2}t^2\rho_n^2| \leqslant \min\,(t^2\rho_n^2, |\tfrac{1}{6}t^3\rho_n^3|).$$

On the other hand, by the fundamental formula of calculus we get

$$|\exp(-t^2/2) - 1 + \tfrac{1}{2}t^2| \leqslant \min\,(|\tfrac{1}{4}t^3|, \tfrac{1}{8}t^4).$$

The proof is therefore straightforward.

Theorem 17. If $\langle\rho_0^4\rangle < \infty$ and $r_\infty \neq 0$ then the normalised variable of the fluctuation converges in distribution to the standard normal variable.

Proof. It will suffice to show that the characteristic function F_n of the normalised variable tends to $\exp(-t^2/2)$ when n goes to infinity [21–23]. By iterating the procedure begun in (4.1), one computes that for all non-negative k

$$\left|F_{n\bullet k}(t) - \prod_{(i_1,\ldots,i_k)\in B_k} F_n([o_{i_1}(n+k-1)\ldots o_{i_k}(n)]t)\right|$$
$$\leqslant |\delta F_{n+k-1}(t)| + \sum_{j=1}^{k-1} \sum_{(i_1,\ldots,i_j)\in B_j} |\delta F_n([o_{i_1}(n+k-1)\ldots o_{i_j}(n-j)]t)|.$$

By lemmas 9, 10 and 14 and corollary 12, there exists a constant c such that

$$\left|F_{n+k}(t) - \prod_{i_1,\ldots,i_k)\in B_k} F_n([o_{i_1}(n+k-1)\ldots o_{i_k}(n)]t)\right| \leqslant ct^2 \sum_{j=1}^{k} s_{n+k-j}^2 + s_{n+k-j}.$$

Using lemma 19 one also gets

$$\left|\exp[-(t^2/2)] \sum_{(i_1,\ldots,i_k)\in B_k} [o_{i_1}(n+k-1)\ldots o_{i_k}(n)]^2\right.$$
$$\left. - \prod_{(i_1,\ldots,i_k)B_k} F_n([o_{i_1}(n+k-1)\ldots o_{i_k}(n)]t)\right|$$
$$\leqslant \sum_{(i_1,\ldots,i_k)\in B_k} [o_{i_1}(n+k-1)\ldots o_{i_k}(n)]^3 \tfrac{1}{6}|t|^3|\rho_n|^3\rangle$$
$$+ \tfrac{1}{8}t^4[o_{i_1}(n+k-1)\ldots o_{i_k}(n)]^4$$

so that obviously

$$\left|F_{n+k}(t) - \exp[-(t^2/2)] \sum_{(i_1,\ldots,i_k)\in B_k} [o_{i_1}(n+k-1)\ldots o_{i_k}(n)]^2\right|$$
$$\leqslant ct^2 \sum_{j=1}^{k} s_{n+k-j}^2 + \sum [o_{i_1}(n+k-1)\ldots o_{i_k}(n)]^3 \tfrac{1}{6}|t|^3\langle|\rho_n|^3\rangle$$
$$+ \tfrac{1}{8}t^4[o_{i_1}(n+k-1)\ldots o_{i_k}(n)]^4.$$

Now taking the limit first when n goes to infinity, then as k goes to infinity, using lemmas 15 and 9 one finishes the proof.

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Appendix 1

Lemma 18. Let a, b be two positive reals such that $a < 1$ and let U_n be a positive real sequence. If there exist a n_0 and a constant c_1 such that $c_1 = U_{n_0}$ and for all $n > n_0$

$$U_{n+1} \leqslant aU_n + b$$

then there exists a constant c_2 such that for all $n > n_0$ $U_n \leqslant c_2$. Also

$$c_2 \geqslant \max\left(c_1, \frac{b}{1-a}\right).$$

The proof can easily be done by induction.

Appendix 2

Lemma 19. Let z_i and z'_i $(i = 1\text{-}N)$ be two complex sequences such that for all $i|z_i| = |z'_i| = 1$ then

$$\left|\prod_{i=1}^{N} z_i - \prod_{i=1}^{N} z'_i\right| \leqslant \sum_{i=1}^{N} |z_i - z'_i|.$$

This lemma and its proof can be found in [22, 23]. It can also be easily proved by induction.

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