The Gap Labelling Theorem: the case of automatic sequences

Published in "Quantum and non-commutative analysis"

(Kyoto, 1992), pp. 179-181, Math. Phys. Stud., 16, Kluwer Acad. Publ., Dordrecht, 1993.

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Let $\mathcal{H}$ be a separable Hilbert space. Let G be a locally compact group acting on $\mathcal{H}$ by a projective unitary representation U. A selfadjoint operator H acting on $\mathcal{H}$ is called homogeneous with respect to the G action, if the family of its translated is precompact in the strong resolvent topology. The closure Ω of this set is called the "hull" of the pair (H, G), and G acts on it by homeomorphisms. Ω is a compact metrizable space. We will denote by H_ω the self adjoint operator corresponding to $\omega \in \Omega$. It satisfies the covariance relation:

$$U(g)H_\omega U(g)^{-1} = H_{g\omega} \qquad \omega \in \Omega, \qquad g \in G,$$

and the map $\omega \in \Omega \to H_\omega$ is continuous for the strong resolvent topology. It follows that the spectrum of H_ω is contained in the spectrum of H for all $\omega \in \Omega$.

Let $\mathcal{A}$ be the C^*-algebra generated in $\mathcal{H}$ by bounded functions of the H_ω's. Given any pair Γ_1, Γ_2 of gaps in the spectrum of H, the eigenprojection $P(\Gamma_1, \Gamma_2)$ corresponding to the part of the spectrum between these two gaps, defines an element of the countable abelian group $K_0(\mathcal{A})$. This is the "Abstract Gap Labelling Theorem". The labelling is invariant under a norm-resolvent perturbation of H.

Let us consider now the special case for which G is either $\mathcal{R}^D$ or $\mathcal{Z}^D$, and $\mathcal{H}$ is $L^2(G)$. Let us assume in addition that H is bounded from below and that the resolvent $(z - H)^{-1}$ admits a distribution kernel $G_z(x, y)$ which decreases fast enough as $\mid x - y \mid \to \infty$. Then one can show that $\mathcal{A}$ is contained in the crossed-product C^*-algebra $\mathcal{C}(\Omega) \times G$. In addition, given an invariant probability measure $\mathbf{P}$ on Ω the trace per unit volume exists $\mathbf{P}$-almost surely and defines a trace τ on $\mathcal{A}$.

The "integrated density of states" (IDS) of H is defined by:

$$\mathcal{N}(E) = lim_{|\Lambda| \to \infty} \frac{\#\{eigenvalues\ of\ H_\omega \mid_\Lambda \leq E\}}{\mid \Lambda \mid},$$

where $\mid \Lambda \mid$ denotes the volume of Λ, and $H_\omega \mid_\Lambda$ denotes the restriction of the operator H_ω to the (open) set Λ with some boundary condition. In most examples of interest, it can be shown that the limit exists and is independent of ω, $\mathbf{P}$-almost surely. Moreover, the "Shubin formula" holds namely:

$$\mathcal{N}(E) = \tau(\chi(H \leq E)),$$

where $\chi(H \leq E)$ is the eigenprojection of H in the W^*-algebra $\mathcal{L}^\infty(\mathcal{A}, \tau)$ on the part of the spectrum contained in the interval $(-\infty, E]$.

If E belongs to a gap of H, the eigenprojection $\chi(H \leq E)$ belongs to $\mathcal{A}$ and therefore the IDS takes on values in the image by τ of the group $K_0(\mathcal{A})$. This image will be called the set of "gap labels".

A one dimensional chain is described by the discrete Schrödinger operator H_V acting on $l^2(\mathbf{Z})$ by

$$H_V\psi(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \qquad n \in \mathbf{Z}, \qquad \psi \in l^2(\mathbf{Z}),$$

where the "potential" $V = \{V(n)\}_{n\in\mathbf{Z}}$ is a bounded sequence of real numbers. We will assume that V takes on finitely many values in the set $\aleph = \{V_1, V_2, \ldots, V_N\}$, called the alphabet of V. For such a potential, the gap label theorems read:

Theorem 1: *For a one dimensional chain with potential V taking values in a finite alphabet, the set of gap labels of H_V is the $\mathbf{Z}$-module generated by the probabilities of occurence of finite words contained in the sequence $\{V(n)\}_{n\in\mathbf{Z}}$.*

A substitution σ is a map from the finite alphabet $\aleph$ into the set of finite words $\aleph^*$ made with letters of $\aleph$. It can be extended as a map from the set of words into itself by concatenation. We assume that there is a letter denoted by 0 such that $\sigma(0)$ begins by zero, and that the length of the iterated $\sigma^n(0)$ goes to ∞ as $n \to \infty$. In the limit, we get an infinite word called a "substitution sequence". This sequence V will be extended on $\mathbf{Z}$ by symmetry around $n = 0$.

We assume σ to be "primitive", namely given any pair a, b of letters, there is an integer n such that b occurs in $\sigma^n(a)$. The matrix $M(\sigma)$ is indexed by letters and its element $M(\sigma)_{a,b}$ is the number of occurence of b into the word $\sigma(a)$. By the Perron-Frobenius theorem, it has a maximal eigenvalue θ, a positive algebraic number, corresponding to a unique eigenvector $v = \{v_a; a \in \aleph\}$ with positive coordinates and normalized by:

$$\sum_{a\in\aleph} v_a = 1.$$

Given a substitution sequence V, let $\aleph_2$ be the set of words of length 2 occuring in V. It can be considered as a new alphabet on which σ induces a new substitution σ_2 in a canonical way. Correspondingly we get a matrix M_2 and a Perron-Frobenius vector v_2 with the same eigenvalue θ. In this situation, the gap labelling theorem reads:

Theorem 2: *For a potential V given by a substitution sequence σ, the set of gap labels of H_V is the $\mathbf{Z}(\theta^{-1})$-module generated by the coordinates of the two Perron-Frobenius vectors v, v_2 corresponding to the eigenvalue θ of the matrix of that substitution.*

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