

SPECTRAL CONTINUITY FOR APERIODIC QUANTUM SYSTEMS I. GENERAL THEORY

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ABSTRACT. How does the spectrum of a Schrödinger operator vary if the the corresponding geometry and dynamics change? Is it possible to define approximations of the spectrum of such operators by defining approximations of the underlying structures? In this work a positive answer is provided using the rather general setting of groupoid C^* -algebras. A characterization of the convergence of the spectra by the convergence of the underlying structures is proven. In order to do so, the concept of continuous field of groupoids is slightly extended by adding continuous fields of cocycles. With this at hand, magnetic Schrödinger operators on dynamical systems or Delone systems fall into this unified setting. Various approximations used in computational physics, like the periodic or the finite cluster approximations, are expressed through the tautological groupoid, which provides a universal model for fields of groupoids. The use of the Hausdorff topology turns out to be fundamental in understanding why and how these approximations work.

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1. INTRODUCTION AND MAIN RESULTS

The present paper delivers a characterization of the convergence of the spectrum of Schrödinger-type operators via the convergence of the underlying structures equipped with a suitable topology. This is the first paper of a series of articles aiming to provide a method to compute the spectrum of a self-adjoint operator on a Hilbert space. Such operators describe the quantum motion of a particle in a solid that might be aperiodic in its structure and/or submitted to a (not necessarily uniform) magnetic field. Special focus of this approach is to deal with aperiodic environments, in particular the ones with long range order, that have been mainly handled so far in one dimensional systems only. Our method delivers a general theory independent of the dimension by using C^* -algebraic techniques. Since this problem arises primarily in Solid State Physics, a special emphasis will be put on periodic approximations. This is because Physicists have at their disposal several numerical codes, based on Floquet-Bloch theory, permitting to compute the band spectrum of a periodic system.

The present article is written for readers who have no expertise in the theory of groupoids and their C^* -algebras. It is expected to serve as an introduction to these techniques for experts in Analysis and Spectral Theory. In order to do so, fundamental and known results in groupoid C^* -algebras are presented and proven for the convenience of the reader, a theory initiated in the classical work by Renault [81]. Some of these results are slightly extended to cover all potential applications. In view of applications to Physics problems, such as the electron motion in an aperiodic solid, submitted to a magnetic field, uniform or not, the case of 1-cocycles depending on a parameter has been added. This addition does not change substantially the techniques used by experts in C^* -algebra theory to prove the main results. But it has not been formally introduced in previous works. The central new concept introduced in this work, is the definition of the *tautological groupoid* of a groupoid Γ (see Section 3) providing the connection of the underlying structures with the spectra of the corresponding operators. It is a universal groupoid in classifying all continuous fields of groupoids with Γ as their enveloping one.

Within the tautological groupoid, the key concept is the use of the Hausdorff topology on the set of closed invariant subsets of the unit space, expressed in its various topological versions due to Vietoris [100], Chabauty [25] and Fell [42]. The power of such a topology in dealing with various approximations has already been illustrated by various previous results obtained by the three authors over the last seven years and that can be found in the PhD Thesis of one of us [9] or in various talks that can be found online [20]. It is expected that these results will be written in forthcoming publications. Specifically, in the upcoming works it will be shown that Delone dynamical systems fall into this setting. These systems include examples in Solid State Physics, such as the Sturmian sequences, the Penrose tiling, and, more generally, the model sets used to describe the structures of quasicrystals. As a warm up, it is shown here that this theory applies to a wide class of effective Hamiltonian and Schrödinger operators associated with dynamical systems. Within this future projects, the Hausdorff topology on the underlying structures is

carefully analyzed explaining when Delone dynamical systems converge. Also the existence of, the construction of, and the obstruction to periodic approximations is on the schedule of this program.

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1.1. Proving the Continuity of the Spectrum: a Strategy. The first step of the strategy was proved in [8] (see also the more detailed account in [1, 9]) and can be expressed as follows: let T be a topological space. For each $t \in T$ let \mathcal{H}_t be a Hilbert space and let A_t be a bounded self-adjoint operator on \mathcal{H}_t . The family $A = (A_t)_{t \in T}$ is called a *field* of self-adjoint operators. It will be called $p2$ -continuous whenever, for any polynomial $p \in \mathbb{R}[X]$ of degree at most 2, the map $t \in T \mapsto \|p(A_t)\| \in [0, \infty)$ is continuous.

Theorem 1 ([8]). *A field $A = (A_t)_{t \in T}$ of self-adjoint operators is $p2$ -continuous if and only if the spectral gap edges of the A_t 's are continuous in t , if and only if the spectrum $\sigma(A_t)$ is continuous as a compact set w.r.t. the Hausdorff metric.*

In [8], the cases of fields of unitary or of unbounded self-adjoint operators are also treated. The results are similar in spirit.

Proving the continuity of the norm of a field of operators might not always be easy though. So, it becomes convenient to enlarge the framework of this approximation method by considering continuous fields of C^* -algebras [57, 96, 97, 34, 35]. If a field of C^* -algebras $\mathcal{A} = (A_t)_{t \in T}$ is continuous then, from the definition, there is a dense algebra of continuous sections for which Theorem 1 applies. If it becomes possible to prove that the field A can be seen as a continuous section of a continuous field of C^* -algebras, then A is automatically $p2$ -continuous and the continuity of its spectrum is proved (see also [57], Lemma 3.3). However, it is not necessarily easy to check that a field of C^* -algebras is actually continuous either.

Thankfully there is a more accessible way to build continuous fields of C^* -algebras. It is sufficient indeed to consider the convolution algebras of a field of groupoids. In [65], Landsman and Ramazan have investigated the definition of continuous field of groupoids and proved that the corresponding C^* -algebras make up a continuous field, at least if the groupoids are amenable. As it turns out, fields of groupoids are much easier to construct essentially because the construction boils down to building genuine topological spaces and dynamical systems. Examples of continuous fields of groupoids useful for Solid State Physics will be given in the second paper of this series. In the present paper, a general construction, valid for a large class of groupoids, will give rise to a *tautological* continuous field, indexed by the set of invariant subsets of the unit space. For example, a periodic orbit can be seen as an invariant subset of a groupoid. The main point will be to endow the set of invariant closed subsets with a suitable topology, which was first defined by Hausdorff for compact subsets of a metric spaces [50, 51], then extended by Vietoris [100] and, more recently, by Chabauty [25] and Fell [42]. It will be called the *Hausdorff*

topology. Then it will be shown that not every invariant set can be approximated by a family of periodic orbits. Characterizing invariant sets that can be approximated by periodic orbits is exactly where the difficulty lies. In the second paper of this series criteria will be provided in various cases.

1.2. The Case of Dynamical Systems. Before going on, let the classical case of dynamical systems be considered. Let (X, G, α) be a topological dynamical system where X is a second countable compact Hausdorff space, G is a discrete countable group and $\alpha : G \times X \rightarrow X$ a continuous group action. In what follows e will denote the neutral element of G . For convenience $\alpha(x, g) = \alpha_g(x)$ will be denoted by gx instead. A subset $Y \subseteq X$ is called invariant if $gY \subseteq Y$ for all $g \in G$. The set of all non-empty closed invariant subsets of X is denoted by $\mathcal{J}(G)$. The set $\mathcal{J}(G)$ equipped with the Hausdorff topology gets the structure of a second countable compact Hausdorff space.

A generalized discrete Schrödinger operator is built as follows. Let $\ell^2(G)$ be the Hilbert space on which this operator acts. Then G is represented by its left regular representation defined by

$$U_h \psi(g) = \psi(h^{-1}g), \quad \psi \in \ell^2(G), g, h \in G.$$

Let $K \subseteq G$ be a finite subset such that $e \notin K$. Then for $k \in K$ let $t_k : X \rightarrow \mathbb{C}$ be a continuous functions. Similarly let $v : X \rightarrow \mathbb{C}$ be continuous as well. For $x \in X$, let the operator $H_x : \ell^2(G) \rightarrow \ell^2(G)$ be defined by

$$(1) \quad (H_x \psi)(g) := \sum_{h \in K} t_h(g^{-1}x) \cdot \psi(gh) + v(g^{-1}x) \cdot \psi(g),$$

where $\psi \in \ell^2(G)$ and $g \in G$. The operator H_x is linear bounded. In order to be self-adjoint the following are required:

- (R1) v is real valued,
- (R2) K is invariant by $k \rightarrow k^{-1}$,
- (R3) the functions t_k satisfies $t_{k^{-1}}(x) = \overline{t_k(k^{-1}x)}$.

The family of operators $H_X := (H_x)_{x \in X}$ is obviously strongly continuous with respect to the variable $x \in X$. In addition, it is G -covariant, namely

$$U(h)H_xU(h)^{-1} = H_{hx}, \quad x \in X, h \in G.$$

In particular, $\sigma(H_x) = \sigma(H_{hx})$ for all $h \in G$. Using strong continuity, it follows that if y belongs to the closure Ω_x of the G -orbit of x , then $\sigma(H_y) \subseteq \sigma(H_x)$. It follows that if the G -action on X is *minimal* (namely any orbit is dense), the previous inclusion holds both ways so that $\sigma(H_y) = \sigma(H_x)$ for any $x, y \in X$. It is worth noticing that such an equality between spectra for all $x, y \in X$ is equivalent to (X, G, α) being minimal (see [9], Theorem 3.6.8, page 109, a result motivated by [66]).

It turns out that the family H_X can be seen as an element (also denoted H_X) of the crossed product C^* -algebra $\mathcal{A} = \mathcal{C}(X) \rtimes_{\alpha} G$ [10]. The spectrum of this element in \mathcal{A} can be shown to be [10]

$$\sigma(H_X) = \overline{\bigcup_{x \in X} \sigma(H_x)}.$$

As will be seen, *amenability* is a crucial property required on the group G (see [48] and Section 2.7) in order to get useful continuous fields. If G is amenable, there is no need to take the closure so that $\sigma(H_X) = \bigcup_{x \in X} \sigma(H_x)$ holds [39, 74].

From the previous construction, if $Y \subseteq X$ is closed and G -invariant, then (Y, G, α) is also a dynamical system, so that H_Y is well defined. The main question investigated in this work is: if Y moves in the set of closed G -invariant subsets of X , how is the spectrum of H_Y evolving? The following result is a consequence of Theorem 3, in Section 1.3

Theorem 2. *Let (X, G, α) be a topological dynamical system where G is an amenable discrete countable group. Let $\mathcal{J}(G)$ denotes the set of closed G -invariant subsets of X equipped with its Hausdorff topology. Let $\mathcal{K}(\mathbb{R})$ denotes the set of compact subsets of \mathbb{R} , equipped with the Hausdorff metric.*

For each $Y \in \mathcal{J}(G)$, the generalized Schrödinger operator $H_Y := (H_x)_{x \in Y}$ defined in eq. (1) and satisfying (R1, R2, R3). Then the map

$$\Sigma_H : \mathcal{J}(G) \rightarrow \mathcal{K}(\mathbb{R}), \quad Y \mapsto \sigma(H_Y),$$

is continuous.

It is worth noticing here that Theorem 2, extends without further assumption to any self-adjoint element of the C^* -algebra, in particular allowing discrete Schrödinger operators with infinite range. Furthermore, the convergence of non-empty invariant closed subsets is characterized by the convergence of the associated spectra for all generalized Schrödinger operators:

Corollary 1. *Let (X, G, α) be a topological dynamical system where G is an amenable discrete countable group. Then the following assertions are equivalent, whenever $Y_n \in \mathcal{J}(G)$, $n \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.*

- (i) *The sequence $(Y_n)_{n \in \mathbb{N}}$ converges to $Y_\infty \in \mathcal{J}(G)$.*
- (ii) *For all generalized Schrödinger operators $H_X := (H_x)_{x \in X}$, the equation*

$$\lim_{n \rightarrow \infty} \sigma(H_{Y_n}) = \sigma(H_{Y_\infty})$$

holds, where the limit is taken with respect to the Hausdorff metric on $\mathcal{K}(\mathbb{R})$.

Proof: This Corollary is a consequence of Corollary 2 in Section 1.3. □

1.3. The Case of Groupoids. The previous examples provide a large class of models liable to represent concrete situations in Condensed Matter Physics. However, a discrete translation group is not necessarily available in general. The most general situation in Condensed Matter Physics is the existence of a dynamical system (X, G, α) in which $G = \mathbb{R}^d$ for some $d \in \mathbb{N}$, called the *space dimension*. More precisely, in practical cases X can be seen as a closed G -invariant subset of the space of Delone sets [17, 19]: each element of X represents the instantaneous position of atomic nuclei and is a typical configuration of a solid or a liquid. Whenever $d = 1$, discretization can be done using the concept of *Poincaré section*: namely it is a closed subset $\Sigma \subseteq X$ such that for any $x \in \Sigma$ the smallest $t \geq 0$ such that $\alpha_t(x) \in \Sigma$ is strictly positive for any x . This concept can be generalized as follows: let (X, G, α) be a topological dynamical system, where X is a compact Hausdorff second countable space, G is a locally compact Hausdorff second countable group (where G may not be discrete), and α is a continuous action of G on X by homeomorphisms. Then a closed subset $\Sigma \subseteq X$ is called a *transversal* if for each $x \in \Sigma$ there is an open neighborhood U of the neutral element $e \in G$ such that for $g \in U$ then $gx \notin \Sigma$. Then

the set Γ_Σ of pairs $(x, g) \in \Sigma \times G$ such that $gx \in \Sigma$ becomes a groupoid (see below) [28, 81]. But in general it is not a topological dynamical system.

To treat these cases, the concept of groupoid is more universal. A precise definition will be provided in Section 2. But a groupoid can be described as a category with all morphisms invertible, for which the family of objects and of morphisms makes up a set. An object of this category is called a *unit* and $\Gamma^{(0)}$ will denote the set of units. A morphism $\gamma : x \rightarrow y$ between the units x and y is called an *arrow* with *range* $r(\gamma) = y$ and *source* $s(\gamma) = x$. Then Γ denotes the set of arrows. By definition, all arrows are invertible. In addition, any unit $x \in \Gamma^{(0)}$ can be associated with a unique arrow e_x such that $\gamma \circ e_x = \gamma = e_y \circ \gamma$ if $\gamma : x \rightarrow y$. Consequently $\Gamma^{(0)}$ can be seen as a subset of Γ . An ordered pair of arrows $(\gamma, \gamma') \in \Gamma \times \Gamma'$ will be called *composable* if $s(\gamma) = r(\gamma')$ (morphism composition) in which case their composition is denoted by $\gamma \circ \gamma'$. The set of composable pairs is denoted by $\Gamma^{(2)}$. At last, given $x \in \Gamma^{(0)}$, then Γ^x will denote the set of arrows with range x and Γ_x will denote the set of arrow with source x .

In this work, all such groupoids will be endowed with a topology, making Γ a *locally compact Hausdorff second countable space*. This topology will be such that the range and the source maps r, s , the inversion map $\gamma \rightarrow \gamma^{-1}$ and the composition map $(\gamma, \gamma') \in \Gamma^{(2)} \mapsto \gamma \circ \gamma' \in \Gamma$ are continuous. It follows that $\Gamma^{(0)}$ and $\Gamma^{(2)}$ are closed. In addition, in this work, all groupoids considered will be assumed to satisfy

- (i) their unit space is compact,
- (ii) the two maps r, s being open.

Such groupoids will be called *handy* (see Section 2.1), because they are convenient and practical. Moreover, they arise naturally in all known realistic problems motivated by Physics. The requirement that the unit space be compact is not necessary as will be seen in Section 2.2. But the space of closed invariant subset of the unit space has a unique Hausdorff topology only if the unit space is compact. Otherwise, the Vietoris and the Chabauty-Fell topologies may differ (see a discussion in [8]).

The analog of the case of a topological dynamical system with a discrete group action is provided by *étale groupoids*. Γ will be called *étale* whenever any $\gamma \in \Gamma$ admits a neighborhood U such that the map $\gamma' \in U \mapsto r(\gamma') \in \Gamma^{(0)}$ is an homeomorphism onto its image.

A typical example of a groupoid is provided by a topological dynamical system as in Section 1.2. In the latter case $\Gamma^{(0)} = X$ and Γ is the set of pairs $\gamma = (x, g) \in X \times G$ with

- (i) $r(x, g) = x, s(x, g) = g^{-1}x,$
- (ii) $(x, g) \circ (g^{-1}x, h) = (x, gh),$
- (iii) $e_x = (x, e)$ if $e \in G$ is the neutral element,
- (iv) $(x, g)^{-1} = (g^{-1}x, g^{-1}).$

The topology on Γ is defined by the product topology of $X \times G$. Such a groupoid is denoted by $X \rtimes_\alpha G$ and is called the *crossed product* of X by G .

Another example is the *groupoid of a transversal*. Namely if $\Sigma \subseteq X$ is a transversal, then the same definition applied to Γ_Σ gives a locally compact groupoid, which is *étale*.

Any locally compact group admits a Haar measure. The convolution of functions allows to define an algebra of functions, and ultimately a C^* -algebra. For groupoid such a concept can be defined and will be called a *Haar system* (see Section 2.2) [81], or a *transverse function* [28]. It is worth mentioning that not all locally compact groupoid admit a Haar system. However, a deep result by Blanchard [22] permits to prove that a groupoid is handy if and only if (i) its unit space is compact and (ii) it admits a Haar system (see Section 2.2, Theorem 4).

Correspondingly, the analog of the convolution algebra, defined with complex valued continuous functions on Γ , defines a C^* -algebra after completion. However, this C^* -algebra is not unique in general. The smallest such C^* -algebra is called the *reduced C^* -algebra* and is denoted by $\mathfrak{C}_{red}^*(\Gamma)$. This algebra is unique if Γ is *amenable* [2, 3].

Given a groupoid Γ an invariant set is a subset $F \subseteq \Gamma^{(0)}$ such that if $x \in F$ then any $\gamma \in \Gamma^x$ satisfies $s(\gamma) \in F$. If Γ is a Hausdorff locally compact groupoid, let $\mathcal{J}(\Gamma)$ denotes the set of closed invariant subsets of $\Gamma^{(0)}$, equipped with the Hausdorff topology. It will be shown that, whenever Γ is handy, $\mathcal{J}(\Gamma)$ is compact. It follows that, for each $F \in \mathcal{J}(\Gamma)$ the set of arrows γ with range in F makes up a closed subgroupoid Γ_F . If $f : \Gamma \rightarrow \mathbb{C}$ is a continuous function with compact support then its restriction to Γ_F will be denoted by f_F . This restriction is continuous with compact support. It follows that the restriction map $f \rightarrow f_F$ extends as a $*$ -homomorphism from $\mathfrak{C}_{red}^*(\Gamma)$ onto $\mathfrak{C}_{red}^*(\Gamma_F)$. It is worth reminding at this point that a *normal* element f , in a C^* -algebra, is an element such that $f^*f = ff^*$.

The main result of this paper can be summarized as follows (see the proof in Section 5):

Theorem 3. *Let Γ be an amenable handy groupoid. For every normal element $f \in \mathfrak{C}_{red}^*(\Gamma)$, the spectral map*

$$\Sigma_f : \mathcal{J}(\Gamma) \rightarrow \mathcal{K}(\mathbb{C}), \quad F \mapsto \sigma(f_F),$$

is continuous.

Proof of Theorem 2: Theorem 2 immediately follows from Theorem 3. For the amenability of the groupoid $X \rtimes G$ follows from the amenability of the group G [2, 3]. \square

Remark 1. The amenability assumption is required only to make sure that the reduced and the full C^* -algebras coincide. In view of [104], this coincidence may occur without the groupoid Γ being amenable, in which case the previous Theorem applies. \square

Corollary 2. *Let Γ be an amenable handy groupoid Γ . Consider a sequence $X_n \in \mathcal{J}(\Gamma)$ of invariant, closed sets for each $n \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Then the following assertions are equivalent.*

- (i) *The sequence $(X_n)_{n \in \mathbb{N}}$ converges to X_∞ in the Hausdorff-topology of $\mathcal{J}(\Gamma)$.*
- (ii) *For all self-adjoint (normal) $f \in \mathcal{C}_c(\mathcal{T}(\Gamma))$, the equation*

$$\lim_{n \rightarrow \infty} \sigma(f_{X_n}) = \sigma(f_{X_\infty}),$$

holds, where the limit is taken with respect to the Hausdorff metric $\mathcal{K}(\mathbb{R})$.

Proof: (i) \Rightarrow (ii) follows by Theorem 3.

(ii) \Rightarrow (i): Assume that $(X_n)_{n \in \mathbb{N}}$ does not converge to $X_\infty \in \mathcal{J}(\Gamma)$. Since $\mathcal{J}(\Gamma)$ is compact, there is a subsequence $(X_{n_k})_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} X_{n_k} =: Y$ exists and $Y \neq X_\infty$. There are two cases, namely (a) $X_\infty \setminus Y \neq \emptyset$ and (b) $Y \setminus X_\infty \neq \emptyset$. Both cases lead as follows to a contradiction.

- (a) If $X_\infty \setminus Y$ is non-empty, consider an $x \in X_\infty \setminus Y$. Using the Lemma of Urysohn, there exists a continuous function $a : \mathcal{T}(\Gamma) \rightarrow [0, 1]$ with compact support such that $a(x) = 1$ and $a_{\Gamma_Y} \equiv 0$. Without loss of generality we can assume that a is self-adjoint (otherwise consider a^*a). Then $\|a|_{X_\infty}\| > 0$ while $\|a|_Y\| = 0$. Thus, $\lim_{n \rightarrow \infty} \sigma(a_{X_n}) = \sigma(a_{X_\infty})$ does not hold contradicting (ii).
- (b) If $Y \setminus X_\infty \neq \emptyset$, a similar argument like in (a) leads to a contradiction. \square

1.4. How to Construct Approximations ? What is the strategy to built some approximations, using Theorem 3 or the Corollary 2 ? Here are some guesses.

1.4.1. *Periodic Approximations.* Let G be assumed to be \mathbb{Z}^d for instance (or more generally a free Abelian group), and let (Y, \mathbb{Z}^d, β) be a minimal dynamical system, with Y a second countable compact Hausdorff space. Let assume that for each $n \in \mathbb{N}$ there is a minimal periodic dynamical system $(Y_n, \mathbb{Z}^d, \beta_n)$. By periodic, it is meant that there is a subgroup $H_n \subseteq \mathbb{Z}^d$ of finite index fixing each point of Y_n . The minimality of Y_n implies that it is finite then. Let X be the disjoint union of the Y_n 's and of Y . Clearly \mathbb{Z}^d acts on X through the action α such that α coincides with β on Y and with β_n on Y_n . The next step will be to find a topology on X making it a second countable compact Hausdorff space, so that (i) Y and the Y_n are closed \mathbb{Z}^d -invariant subsets, and that (ii) Y_n converges to Y in the Hausdorff topology. In many cases such a strategy works (see for instance [79] for disordered systems and [9] for other examples). If so, then the previous Corollary 1 applies and gives a sequence of periodic approximations to the generalized Schrödinger operator H_Y permitting to approximate its spectrum.

1.4.2. *Finite Cluster Approximations.* Another approximation used in Physics, consists in cutting a finite piece of an infinite distribution of atoms, namely to choose a cluster (see for instance [62]). Let \mathcal{L} be a distribution of atoms: namely it will be a Delone set. In such a set, the Voronoi construction gives rise to a tiling by polyhedrons [19] centered at each atom. Two atoms are nearest neighbors (n.n.) whenever their Voronoi tiles intersect along a facet (face of codimension one). If an edge is a pair of n.n., this gives a graph defining \mathcal{L} . In order to prove that this approximation is appropriate, let $C_1 \subseteq \dots \subseteq C_n \subseteq \dots$ be an increasing family of such clusters such that the union is all the atomic sites. Each such cluster gives rise to a finite groupoid Γ_n , generated by the edges of the graph associated with it. In the infinite volume limit, there is a groupoid Γ describing all such constructions. Then let Γ_{tot} be obtained as the disjoint union of these groupoids, which is also a groupoid. If there is a topology on this union making it a continuous field over the set $\mathbb{N} \cup \{\infty\}$, then the generalization to groupoids of the Corollary 1, will give convergence of the spectrum as well.

1.5. **Methodology: a Review.** The problem of investigating the spectral properties of aperiodic quantum system has a long history, both in Physics and in Mathematics.

1.5.1. *Transfer Matrix Method.* For one-dimensional models, the *transfer matrix* method has been quite successful. The earliest result obtained for Schrödinger's operators with non periodic coefficient is probably the one provided by Dinaburg and Sinai [33], using this method to provide results about the spectral properties of the 1D-Schrödinger operator on the real line with a quasiperiodic potential. The method used for the KAM theorem turns out to be effective. This method was used successfully since the early eighties on similar problems on the discrete line. Eventually, it gave rise to the theory of cocycles. A spectacular achievement, using this method, has been the solution of the *Ten Martini Problem* [6].

However the transfer matrix method is limited in several ways. First it applies only to one dimensional systems. Secondly, it applies only to the case of finite range Hamiltonians. Nevertheless, this method was used successfully to investigate numerically the case of a random potential (Anderson model) using a transfer matrix in high dimension for systems of infinite length but with large finite transverse area [63]. However, all attempt made to use this method for systems in higher dimension in order to get rigorous mathematical results, have failed so far, if one except models that can be decomposed into a direct sum or a product of commuting one-dimensional systems [31].

1.5.2. *Renormalization Group Method.* In the eighties and the early nineties the case of $1D$ -discrete Schrödinger Operators with potential provided by a substitution was illustrated by several examples. The first example was the *Fibonacci Hamiltonian*, also called the *Kohmoto model* [61, 76, 77, 23, 95]. The spectrum could be computed through a Renormalization Group method induced by the substitution applied to the transfer matrix [11, 12, 13]. Since then, the method has been refined and pushed to the limit, thanks to the two decades persistent work of D. Damanik and A. Gorodetski. A spectacular achievement, for instance can be the recent review on the Fibonacci Hamiltonian [32].

An attempt was made to use this approach in higher dimension [94]. For indeed, most quasicrystals studied so far admits a special symmetry (mostly the icosahedral symmetry in $3D$) and, as a consequence admits also an inflation symmetry playing the role of a space Renormalization Map. However, the method turned out to be hopeless unless in the region of energy for which the kinetic energy can be treated as a very small perturbation. Example of Hamiltonian in high dimension that can be considered as Cartesian product of $1D$ -operators, have shown that the qualitative results concerning the spectrum can be surprisingly complicate [93]. In particular the RG-method is unlikely to give so much information.

1.5.3. *Other Numerical Approaches.* Soon after the discovery of quasicrystals [92], one of the earliest numerical calculations were made on a large cluster of the Penrose lattice [27, 75, 62]. The cluster size (varying from 251 to 3806) and the nature of the parameters in the model investigated, may change the results. The existence of inflation rules gives a fast algorithm to grow a cluster from a seed. This gives a sequences C_1, \dots, C_n of cluster with size increasing exponentially fast. However, this inflation rule can hardly been used to implement relations between the various Hamiltonians H_1, \dots, H_n in order to get more rigorous results on its spectral properties [94]. The finite size effect were not systematically estimated, but they are visible. It is expected that the errors, as measured in Density of States (DOS), be of the order of the ratio $\{\text{boundary area}(C_n)\}/\{\text{Volume}(C_n)\}$ namely of the order of the inverse of the cluster diameter. This is a slow convergence.

Another method, promoted in [21], consists in finding a periodic approximation and to use the Bloch Theory to compute the spectrum like in [79]. In most situations investigated so far, it can be proved that the error is much smaller and decay exponentially fast with the diameter of the unit cell of the periodic lattice.

1.5.4. *Defect Creation.* Periodic approximations may come with a prize. Depending upon how the sequence of approximations is built, defects might show up in the limit. This was demonstrated in [77] and proved rigorously in [14] for the case of $1D$ -quasicrystals (also called Sturmian sequences). As it turns out, such defects also occur, the so-called *worms*, in the $2D$ -case for a nearest neighbor Hamiltonian on the quasicrystal with octagonal symmetry. This quasicrystal can be computed by mean of a projection from \mathbb{Z}^4 onto \mathbb{R}^2 respecting the 8-fold symmetry. But the projection 2-plane is not in generic position *w.r.t.* the lattice \mathbb{Z}^4 , precisely because of the symmetry. As a result, generically, a periodic approximation will produce defects [15]. However, there is a way, in this example, to find a sequence of periodic approximation avoiding the occurrence of defects, as was proposed in [36]. In the second paper of this series, this problem will be addressed and solved from using the Anderson-Putnam complex [4].

2. GROUPOIDS: A REMINDER

This Section offers a summary of the definitions and properties of groupoids that will be needed later. Most of this material can be found in [28, 81, 58].

2.1. Definitions. A *groupoid* Γ is a small category in which all morphisms $\gamma : x \rightarrow y$ are invertible. By a small category it is meant that both the families of objects and morphisms are sets. Then Γ will denote the set of morphisms. Let $\Gamma^{(0)}$ denote the set of objects. If $\gamma : x \rightarrow y$, the notation will be $s(\gamma) = x$ which will be called the *source*, and $r(\gamma) = y$ which will be called the *range*. Similarly a morphism γ will be called an *arrow*. Then x can be identified with the morphism $\gamma^{-1} \circ \gamma = e_x$ and y with $\gamma \circ \gamma^{-1} = e_y$. Then $x = s(e_x) = r(e_x)$ and $e_x \circ \gamma = \gamma$ if $r(\gamma) = x$, while $\gamma \circ e_x = \gamma$ when $s(\gamma) = x$. Hence e_x is a unit and is the identity morphism for the object x , so that the map $x \rightarrow e_x$ gives a bijection identifying $\Gamma^{(0)}$ with the *set of units* and will be called the *unit space*, instead, and can then be seen as a subset of Γ . It will be convenient though to separate the concept of unit, the set of which will be sometimes denoted by $\mathcal{U}\Gamma$, from the concept of object in practice, even if the two sets are in bijection. If $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ the composition $\gamma_1 \circ \gamma_2$ makes sense only if $s(\gamma_1) = r(\gamma_2)$. The notation will be

$$\Gamma^{(2)} = \{(\gamma_1, \gamma_2); s(\gamma_1) = r(\gamma_2)\}.$$

For $y \in \Gamma^{(0)}$, its fibers are the r -fiber $\Gamma^y = \{\gamma \in \Gamma; r(\gamma) = y\}$, the s -fiber $\Gamma_y = \{\gamma \in \Gamma; s(\gamma) = y\}$ and the rs -fiber $\Gamma_y^y = \{\gamma \in \Gamma; r(\gamma) = s(\gamma) = y\}$ which is a group. The relation “ x is isomorphic to y ”, namely $\exists \gamma \in \Gamma, \gamma : x \rightarrow y$, is an equivalence relation which will be denoted by $x \sim y$. Given $F \subseteq \Gamma^{(0)}$, its saturation $[F]$ denotes the set $\{x \in \Gamma^{(0)}; \exists y \in F, x \sim y\}$. Given $F \subseteq \Gamma^{(0)}$, Γ_F will denote the subgroupoid $\Gamma_F = \{\gamma \in \Gamma; s(\gamma) \in F, r(\gamma) \in F\}$. A subset $F \subseteq \Gamma^{(0)}$ will be called *invariant* if $F = [F]$. The concept of subgroupoid is straightforward and left to the reader.

Example 1 (Groups). Let G be a group with neutral element $e \in G$. Then $\Gamma := G$ is a groupoid where $\Gamma^{(2)} = G \times G$. In addition, the range $r(g)$ and the source $s(g)$ are equal to e . Thus, $\Gamma^{(0)} = \{e\}$.

Example 2 (Sets). Let X be a set. Then $\Gamma := X$ is a groupoid with $\Gamma^{(2)} := \{(x, y) \in X \times X \mid x = y\}$, composition $x \circ x := x$ and inverse $x^{-1} := x$ for $x \in X$. Thus, the unit space $\Gamma^{(0)} = X$.

Example 3 (Equivalence Relation). Let X be a set and let \sim be an equivalence relation on X . Then the graph of the equivalence relation becomes a groupoid $\Gamma := \{(x, y) \in X \times X \mid x \sim y\}$ with $\Gamma^{(2)} := \{((x_1, x_2), (x_3, x_4)) \mid x_2 = x_3\}$, with the composition $(x, y) \circ (y, z) := (x, z)$ and inverse $(x, y)^{-1} := (y, x)$. Then $\Gamma^{(0)} = X$ with $e_x = (x, x)$.

Example 4 (Dynamical Systems). Let (X, G, α) be a dynamical system with X a set, G a group and $\alpha : (x, g) \in X \times G \rightarrow gx \in X$ a left action of G on X by bijections. Then $\Gamma = X \times G$ becomes a groupoid if $\Gamma^{(0)} = X$, $r(x, g) = x$, $s(x, g) = g^{-1}x$, and the composition is defined by $(x, g) \circ (g^{-1}x, h) = (x, gh)$. Then $(x, g)^{-1} = (g^{-1}x, g^{-1})$. This groupoid will be denoted by $X \rtimes_{\alpha} G$ and will be called *crossed product*. \square

Example 5 (Induced Subgroupoid). If (X, G, α) is a dynamical system, and if $Y \subseteq X$, then the set $\Gamma_Y = \{(y, g) \in Y \times G; g^{-1}y \in Y\}$ is a subgroupoid of $X \rtimes_{\alpha} G$. \square

Definition 1. A groupoid Γ is topological whenever the set of arrows Γ and its unit space $\Gamma^{(0)}$ admit a topology such that

- (i) the range map r , the source map s , are continuous,
- (ii) the composition and the inverse are continuous.

It is worth reminding here that a topological space X is Hausdorff if given any two points $x, y \in X$ such that $x \neq y$, there are open sets $U \ni x$ and $V \ni y$ such that $U \cap V = \emptyset$. Equivalently this condition holds if and only if the diagonal $\Delta_X = \{(x, x) \in X \times X; x \in X\}$ is closed in $X \times X$. In such a case, for $x \in X$ the singleton $\{x\}$ is a closed subset.

Lemma 1. (i) Let Γ be a topological groupoid. If $\Gamma^{(0)}$ is Hausdorff, then the r, s fibers and $\Gamma^{(2)}$ are closed.

(ii) If, in addition Γ is Hausdorff, then the set of units $\mathcal{U}\Gamma$ is closed inside Γ .

Proof: (i) Since $\Gamma^{(0)}$ is Hausdorff, any singleton $\{x\}$ in $\Gamma^{(0)}$ is closed and the diagonal Δ in $\Gamma^{(0)} \times \Gamma^{(0)}$ is closed. Since r, s are continuous, the preimages $r^{-1}(\{x\}) = \Gamma^x$ and $s^{-1}(\{x\}) = \Gamma_x$ are closed, implying that $\Gamma_y^x = \Gamma^x \cap \Gamma_y$ is closed as well. In addition let $\Phi : \Gamma \times \Gamma \rightarrow \Gamma^{(0)} \times \Gamma^{(0)}$ be the map defined by $\Phi(\gamma_1, \gamma_2) = (s(\gamma_1), r(\gamma_2))$. This map is continuous by assumption and the preimage of the diagonal Δ is precisely $\Gamma^{(2)}$, hence it is closed.

(ii) Let $\mathcal{U}\Gamma$ denotes the set of units inside Γ , namely the set of $e \in \Gamma$ such that there is $\gamma \in \Gamma$ for which $\gamma^{-1} \circ \gamma = e$. It follows that $r(e) = s(e)$ and that, from the associativity of the product, $e \circ e = e$ implying $e^{-1} \circ e = e \circ e^{-1} = e$. Consequently e is a unit if and only if $r(e) = s(e)$ and $e \circ e = e$. The map $\xi : \Gamma \rightarrow \Gamma \times \Gamma$ given by $\xi(\gamma) = (\gamma, \gamma \circ \gamma^{-1})$ is continuous. The diagonal $\Delta \subseteq \Gamma \times \Gamma$ is closed if Γ is Hausdorff. It follows that $\mathcal{U}\Gamma = \xi^{-1}(\Delta)$ hence proving that it is closed. \square

Among topological groupoids, the *open* ones will be important. As a reminder, a continuous map $f : X \rightarrow Y$ between two topological spaces is called *open* whenever the image of any open set of X by f is open in Y .

Definition 2. A topological groupoid Γ will be called *open* if its range and source maps are both *open*.

It is worth remarking that because of the continuity of the map $i : \gamma \mapsto \gamma^{-1}$, the range map is open if and only if the source map is. In addition since i is its own inverse, its continuity implies that i is open as well. Being open is actually sufficient to define the *tautological groupoid* of a groupoid Γ (See Section 3). The following result is known in category theory [83], but it is worth giving a more topological proof.

Lemma 2. Let Γ be an open groupoid. Then, the multiplication map $m : \Gamma^{(2)} \rightarrow \Gamma$ is open.

Proof: (i) First let $\Gamma \times_r \Gamma \subseteq \Gamma \times \Gamma$ be the set of pairs (γ_1, γ_2) such that $r(\gamma_1) = r(\gamma_2)$. Let $\pi_i, i = 1, 2$ be the projection onto the i -th coordinate defined by $\pi_i(\gamma_1, \gamma_2) = \gamma_i$. Then π_i is open. For indeed, let U_1, U_2 be two open sets in Γ and let W be the trace of $U_1 \times U_2$ on $\Gamma \times_r \Gamma$. Then $\pi_1(W) = U_1 \cap r^{-1}(r(U_2))$. Since r is both continuous and open, it follows that $r(U_2)$ is open, so that $r^{-1}(r(U_2))$ is open too, showing that $\pi_1(W)$ is open. A similar argument holds for π_2 .

(ii) Let now $\xi : \Gamma \times_r \Gamma \rightarrow \Gamma^{(2)}$ be the map defined by $\xi(\gamma_1, \gamma_2) = (\gamma_1, \gamma_1^{-1} \circ \gamma_2)$. This map is continuous. Moreover it is invertible with inverse $\xi^{-1}(\gamma, \gamma') = (\gamma, \gamma \circ \gamma')$ as can be checked immediately. In particular ξ^{-1} is also continuous, so that ξ is a homeomorphism. In particular, it is clear that $m = \pi_2 \circ \xi^{-1}$, which is open as composition of open maps. \square

Corollary 3. Let Γ be an Hausdorff open groupoid with Hausdorff unit space. Then the map $x \in \Gamma^{(0)} \mapsto e_x \in \mathcal{U}\Gamma$ is a homeomorphism.

Proof: Since Γ is Hausdorff with Hausdorff $\Gamma^{(0)}$, it follows, from Lemma 1 that $\mathcal{U}\Gamma$ and all fibers are closed subsets in Γ . In addition, thanks to Lemma 2, the product map is open. Let $U \subseteq \mathcal{U}\Gamma$ be an open set in the set of units for the induced topology. Let \hat{r} denote the restriction of the range map r to $\mathcal{U}\Gamma$. Then let $\psi : \Gamma \rightarrow \mathcal{U}\Gamma$ be the continuous map defined by $\psi(\gamma) = \gamma \circ \gamma^{-1}$. Hence $W = \psi^{-1}(U)$ is open. In addition $r(W) = \hat{r}(U)$, showing that $\hat{r}(U)$ is open in $\Gamma^{(0)}$. Hence the map $\hat{r} : e \in \mathcal{U}\Gamma \mapsto r(e) \in \Gamma^{(0)}$ is open. Since this map is open, continuous, one-to-one and onto, it follows that its inverse is continuous as well, so that this map is a homeomorphism. \square

The groupoid analog of a discrete group is provided by

Definition 3. *A topological groupoid Γ is étale whenever any $\gamma \in \Gamma$ admits an open neighborhood U such that the map $\gamma' \in U \mapsto r(\gamma') \in \Gamma^{(0)}$ is a homeomorphism onto its image. Equivalently, Γ is étale whenever its topology is generated by a basis of bisections, namely subsets $F \subseteq \Gamma$ such that the restriction of the range map to F is a homeomorphism.*

Thanks to the continuity of the inverse map i , the range map r can be replaced by the source map s as well in the previous definition.

The *canonical transversal* of the *Hull* of a *tiling* in \mathbb{R}^d with finite local complexity gives rise to an étale groupoid with Cantor unit space [59, 18, 54]. Étale groupoid with Cantor unit space have recently been the focus of attention from the group theory community in order to produce groups of intermediate growth with remarkable properties, like simplicity, amenability, finite presentation or finitely generated [73]. This is because the set of bisections of an étale groupoid with range or source given by the whole unit space can be identified with a group of homeomorphisms of the unit space, called the *full group* of Γ . The concept of full group was introduced in [64, 45] for \mathbb{Z}^d -actions on a Cantor set. The full group of a Cantor set was proved to be amenable in a remarkable paper [55].

Lemma 3 ([81], Prop. 2.8). *An étale groupoid Γ has discrete fibers.*

Proof: Let $x \in \Gamma^{(0)}$. If now $\gamma \in \Gamma^x$, let U be an open neighborhood of γ making the map $r_U : U \rightarrow r(U) \subseteq \Gamma^{(0)}$ an homeomorphism. Since r_U is one-to-one, it follows that $\Gamma^x \cap U$ is reduced to γ . Hence γ is isolated inside Γ^x , showing that Γ^x is discrete. The same argument works for the s -fiber. \square

Lemma 4. *Any étale topological groupoid is open.*

Proof: Let Γ be an étale groupoid. Let U be an open subset in Γ . If $x \in r(U)$, there is $\gamma \in U$ such that $x = r(\gamma)$. Since Γ is étale, there is an open set $V_\gamma \subseteq U$, containing γ , on which the restriction of the source map is an homeomorphism on its image. By construction, $x \in r(V_\gamma) \subseteq r(U)$. Hence $r(U)$ is open. The same argument works for the source map. \square

Most groupoids used to describe physical systems have additional properties that make them remarkable and more convenient. In this work they will be called *handy*.

Definition 4. *A topological groupoid Γ will be called handy whenever:*

- (i) *it is locally compact, Hausdorff, second countable,*
- (ii) *its unit set is compact,*
- (iii) *the source map and the range map are open.*

It follows that $\Gamma^{(0)}$ is Hausdorff.

Examples of handy groupoids are obtained from previous examples in an obvious way: (i) Example 1 if the group is locally compact, Hausdorff, second countable; (ii) Example 2 where

X is a compact Hausdorff second countable set; (iii) Example 3 if X is a compact, Hausdorff, second countable space, and if the graph of the equivalence relation is closed.

Example 6 (Topological Dynamical System). Let (X, G, α) be a dynamical system. If (i) X is compact, Hausdorff, second countable, (ii) G is a locally compact, Hausdorff, second countable group, (iii) the action $\alpha : (g, x) \rightarrow gx$ is continuous and the maps $\alpha_g : x \in X \rightarrow gx \in X$ are homeomorphism, (iv) $X \rtimes_\alpha G$ is endowed with the topology of the Cartesian product $X \times G$. Then $X \rtimes_\alpha G$ is a handy groupoid. The proof is a consequence of the Theorem 4 below. \square

Example 7. If Γ is a handy groupoid, any closed sub-groupoid is handy. In particular, if $Y \subseteq \Gamma^{(0)}$ is closed, then the induced groupoid Γ_Y is handy. \square

Remark 2. The condition of second countability allows to use Urysohn's metrization theorem and the Tietze extension theorem. In practice it is not much of a restriction. The condition of local compactness is convenient to construct a C^* -algebra by convolution. Without it, not much is known. The separation axiom is not absolutely needed [29], but it makes the groupoid much easier to handle. The compactness of the unit space is not necessary, but almost all examples used to describe physical systems so far, have this property. The openness of the source and range maps will turn out to be necessary when dealing with the tautological groupoid (see Section 3). \square

2.2. Continuous Haar Systems. A locally compact group G is known to admit a left invariant positive measure, called the *Haar measure* [101], which is unique modulo a normalization factor. It follows that the space $L^1(G)$ can be endowed with an algebraic structure using the convolution product. Similarly an *adjoint* can also be defined. The algebra obtained in this way, denoted by $L^1(G)$, is called the *Wiener algebra*. It also gives rise to a C^* -algebra denoted by $\mathfrak{C}^*(G)$. If G is Abelian, the Gelfand theorem and the Pontryagin duality give the Fourier transform as a $*$ -isomorphism between $\mathfrak{C}^*(G)$ and the space $\mathcal{C}_0(\hat{G})$ of continuous function of the Pontryagin dual \hat{G} vanishing at infinity.

Such a construction can be extended to groupoids. The Haar measure is replaced by what has been called a *Haar system* [102, 89, 90, 81] or a *transverse function* [28]. The latter concept is actually helping in understanding the concept of *transverse measure* [28], which permits to express all positive weights of the C^* -algebra of a groupoid in an analytic way. In particular it gives rise to a noncommutative analog of integration theory.

Definition 5. A *left-continuous Haar system* on a topological groupoid Γ is a family of positive Borel measures $(\mu^x)_{x \in \Gamma^{(0)}}$ on Γ such that the following assertions hold.

- (H1): For each $x \in \Gamma^{(0)}$, the support of the measure μ^x is equal to the r -fiber Γ^x .
- (H2): For each $f \in \mathcal{C}_c(\Gamma)$, the function $\Gamma^{(0)} \rightarrow \mathbb{C}$ defined by

$$x \mapsto \mu^x(f) = \int_{\Gamma^x} f(\gamma) d\mu^x(\gamma).$$

is continuous.

- (H3): These measures are left-invariant, namely if $\gamma : x \rightarrow y$ then

$$\int_{\Gamma^x} f(\gamma \circ \eta) d\mu^x(\eta) = \int_{\Gamma^y} f(\eta') d\mu^y(\eta').$$

for any $f \in \mathcal{C}_c(\Gamma)$.

In this definition, the conditions (H1) and (H3) are similar to the case of a Haar measure on a group. However (H2) is new. It turns out to be necessary and sufficient so that the convolution defined on $\mathcal{C}_c(\Gamma)$ maps into $\mathcal{C}_c(\Gamma)$ see [91] (see also Section 2.5 below).

It is worth mentioning that not all locally compact groupoid admits a Haar system: a simple counter example can be founded in [47], for which the axiom (H2) is violated. However every étale groupoid Γ has one, using the counting measure

$$\mu^x(f) = \sum_{\gamma \in \Gamma^x} f(\gamma), \quad f \in \mathcal{C}_c(\Gamma).$$

Indeed, since the fibers are discrete, if f has compact support, the sum admits only a finite number of nonzero terms. By construction, the support of μ^x is Γ^x . By definition of “étale” the map $x \rightarrow \mu^x(f)$ is continuous. The left invariance comes from the remark that if $\gamma : x \rightarrow y$, then the map $\eta \in \Gamma^x \mapsto \gamma \circ \eta \in \Gamma^y$ is a bijection, since γ is invertible. In particular this applies to Example 2, for the groupoid defined by a compact Hausdorff second countable set X , where $\mu^x(f) = f(x)$ is the Dirac measure. It is worth mentioning that any other Haar system on a handy étale groupoid is given by $\nu^x = g(x)\mu^x$ where $g : \Gamma^{(0)} \rightarrow \mathbb{R}_+$ is a non-negative continuous function such that $x \sim y \Rightarrow g(x) = g(y)$. In particular if $[\{x\}]$ is dense in $\Gamma^{(0)}$, it follows that the counting Haar measure is the unique Haar system on Γ modulo normalization.

Another important example is given by a topological dynamical system (X, G, α) (see Example 6): since G is locally compact it admits a Haar measure denoted by dh here, leading to

$$\mu^x(f) = \int_G f(x, h) dh, \quad f \in \mathcal{C}_c(\Gamma).$$

It is easy to check that the axioms are satisfied. The previous two examples are the most important in view of applications to Physics.

The following characterization is a deep result

Theorem 4 ([102, 22]). *A second countable locally compact Hausdorff topological groupoid admits a left-continuous Haar system if and only if it is open.*

The proof that such a groupoid admitting left-continuous Haar system is open is due to Westman [102]. The proof can be found below. The proof of the converse will be skipped here. But it can be found in [103] and is based upon a deep result by Blanchard [22].

Westman’s Proof: Here Γ is a second countable locally compact Hausdorff topological groupoid admitting a left-continuous Haar system. Using the continuity of the inverse map, it is sufficient to show that the range map is open. Let $U \subseteq \Gamma$ be an open set and let $\gamma \in U$. Since Γ is second countable, locally compact and Hausdorff, Urysohn’s Lemma applies. Thus, there is a continuous function $f \in \mathcal{C}_c(\Gamma)$ satisfying $0 \leq f \leq 1$, $f(\gamma) = 1$ and $\text{supp}(f) \subseteq U$. Let the interior of the support of f be denoted by V_γ . It is an open subset of U by construction. Hence $\mu^x(f) > 0$ holds if and only if $x \in r(V_\gamma)$. Since the map $\Phi_f : x \in \Gamma^{(0)} \mapsto \mu^x(f) \in \mathbb{C}$ is continuous by (H2), the preimage $\Phi_f^{-1}((0, \infty)) \subseteq \Gamma^{(0)}$ is open. By the previous considerations, $\Phi_f^{-1}((0, \infty))$ is equal to $r(V_\gamma)$. Hence, $r(V_\gamma)$ is an open subset of $r(U)$ containing $r(\gamma)$. Altogether, $r(U) = \bigcup_{\gamma \in U} r(V_\gamma)$ is a union of open sets implying that $r(U)$ is open in $\Gamma^{(0)}$. \square

Corollary 4. *Any second countable locally compact topological groupoid, with compact unit space and with a left-continuous Haar system, is handy.*

2.3. Cocycles and Gauge Fields. A fair part of the book by J. Renault [81] is dedicated to the problem of groupoid cohomology. He introduced the concept of n -cocycles. For the purpose of the construction of the convolution algebra only the cases of $n = 0, 1$ are relevant. Here, Γ will be a handy groupoid. It is convenient to introduce $\Gamma^{(n)}$ to be the subset of $\Gamma^{\times n}$ made of families $(\gamma_1, \dots, \gamma_n)$ such that $s(\gamma_k) = r(\gamma_{k+1})$ whenever $1 \leq k < n$. Such a family defines a *path* within Γ . Since Γ is Hausdorff, $\Gamma^{(n)}$ is a closed subset of $\Gamma^{\times n}$, thus a locally compact space. Let L denote an Abelian topological group, which will be called the *group of coefficients*. The composition law in L will be written as an addition and therefore the neutral element will be written as 0. However, in application, L will often be the unit circle \mathbb{S}^1 seen as the multiplicative subgroup of $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$ or the group of unitaries in an Abelian C^* -algebra. *The reader is invited to proceed to the change of notations between an additive and a multiplicative group law.* Then an n -cochain with coefficients in L is a continuous map $\sigma : \Gamma^{(n+1)} \rightarrow L$. The set of such cochains will be denoted by $C^n(\Gamma, L)$. For the pointwise addition it is an Abelian group. The differential map d is defined as a group homomorphism $d : C^n(\Gamma, L) \rightarrow C^{n+1}(\Gamma, L)$ such that

$$(2) \quad d\sigma(\gamma_0, \dots, \gamma_{n+1}) = \sigma(\gamma_1, \dots, \gamma_{n+1}) + \sum_{k=1}^{n+1} (-1)^k \sigma(\gamma_0, \dots, \gamma_{k-2}, \gamma_{k-1} \circ \gamma_k, \gamma_{k+1}, \dots, \gamma_{n+1}) + (-1)^{n+2} \sigma(\gamma_0, \dots, \gamma_n).$$

It can be checked that $d^2\sigma = 0$. An n -cocycle σ is an n -chain with $d\sigma = 0$, namely σ is *closed*. An n -coboundary σ is an n -chain for which there is an $(n-1)$ -chain τ such that $\sigma = d\tau$. For the present purpose, only 0- and 1-cocycles matter.

The case $n = 0$: it can be checked that a 0-cocycle is a groupoid homomorphism $\sigma : \Gamma \rightarrow L$, namely the condition $d\sigma = 0$ becomes $\sigma(\gamma_0 \circ \gamma_1) = \sigma(\gamma_0) + \sigma(\gamma_1)$. By definition, it is a 0-coboundary if and only if it vanishes.

Example 8. Let X be a compact Hausdorff second countable space. Let \mathbb{T} be an homeomorphism of X . Then \mathbb{T} defines a \mathbb{Z} -action. Let $\Gamma = X \rtimes_{\mathbb{T}} \mathbb{Z}$ be the corresponding handy (étale) groupoid. Then a *unitary module* δ is a 0-cocycle with coefficients in $L = \mathbb{S}^1$. Hence it can be checked that $\delta(x, 0) = 1$, while δ is entirely defined by the function $h : x \in X \mapsto \delta(x, 1) \in \mathbb{S}^1$, since, for any $n \in \mathbb{N}$

$$\delta(x, n) = h(x)h(\mathbb{T}^{-1}x) \cdots h(\mathbb{T}^{-n+1}x), \quad \delta(x, -n) = \overline{h(\mathbb{T}x)h(\mathbb{T}^2x) \cdots h(\mathbb{T}^n x)}.$$

□

The case $n = 1$: It will be convenient for the rest of the paper to denote by L a compact Abelian group for which the group law will be denoted multiplicatively. Then a 1-cocycle is a continuous function on $\Gamma^{(2)}$ with values in L such that

$$(3) \quad \frac{\sigma(\gamma_0, \gamma_1 \circ \gamma_2)}{\sigma(\gamma_0 \circ \gamma_1, \gamma_2)} = \frac{\sigma(\gamma_0, \gamma_1)}{\sigma(\gamma_1, \gamma_2)}.$$

In addition σ is a coboundary if and only if there is a continuous function $\tau : \Gamma \rightarrow L$ such that

$$\sigma(\gamma_0, \gamma_1) = \frac{\tau(\gamma_0)\tau(\gamma_1)}{\tau(\gamma_0 \circ \gamma_1)}.$$

Example 9 (Uniform magnetic fields: see [10]). The main example relevant for Physics is the effect of a magnetic field. Let X be a compact Hausdorff second countable space. Let \mathbb{T} be a \mathbb{Z}^2 -action on X and let $\Gamma = X \rtimes \mathbb{Z}^2$ be the corresponding handy (étale) groupoid. Then an arrow can be labeled by a pair (x, n) with $x \in X$ and $n = (n_1, n_2) \in \mathbb{Z}^2$. A unitary 1-cocycle (namely with $L = \mathbb{S}^1$) can be defined by

$$\sigma((x, n), (\mathbb{T}^{-n}x, m)) = e^{iB n \wedge m}, \quad B \in \mathbb{R}, \quad n \wedge m = n_1 m_2 - n_2 m_1,$$

as can be check easily. Here B plays the role of a magnetic field perpendicular to the plane of the physical space in which the lattice \mathbb{Z}^2 lies [10]. Such a cocycle is not a coboundary, except for special values of B . \square

Lemma 5. *Let σ be a 1-cocycle on Γ with values in L . If $x \in \Gamma^{(0)}$ then*

$$\sigma(e_x, \gamma) = \sigma(e_x, e_x) = \sigma(\gamma^{-1}, e_x), \quad \sigma(\gamma, \gamma^{-1}) = \sigma(\gamma^{-1}, \gamma), \quad \forall \gamma \in \Gamma^x.$$

Proof: Use equation (3) with $\gamma_0 = e_x$ and $x = r(\gamma_1)$. It gives

$$\frac{\sigma(e_x, \gamma_1 \circ \gamma_2)}{\sigma(\gamma_1, \gamma_2)} = \frac{\sigma(e_x, \gamma_1)}{\sigma(\gamma_1, \gamma_2)}.$$

Consequently $\sigma(e_x, \gamma_1) = \sigma(e_x, \gamma_1 \circ \gamma_2)$, for any choice of γ_1, γ_2 compatible with the definition. Choosing $\gamma_1 = e_x$ and $\gamma_2 = \gamma$ this gives the first set of identities. Now, choosing $\gamma_1 = e_x$ with $x = s(\gamma_0) = r(\gamma_2)$, the same equation gives

$$\frac{\sigma(\gamma_0, \gamma_2)}{\sigma(\gamma_0, \gamma_2)} = 1 = \frac{\sigma(\gamma_0, e_x)}{\sigma(e_x, \gamma_2)},$$

which gives the second set of identities.

At last, if $\gamma_1 = \gamma$ and $\gamma_0 = \gamma_2 = \gamma^{-1}$, the last equation follows. \square

Corollary 5. *Let σ be a 1-cocycle on Γ with values in L and let $\tau : \Gamma \rightarrow L$ be a continuous function. Then the function $\tilde{\sigma}$ defined by*

$$\tilde{\sigma}(\gamma_0, \gamma_1) = \sigma(\gamma_0, \gamma_1) \frac{\tau(\gamma_0)\tau(\gamma_1)}{\tau(\gamma_0 \circ \gamma_1)}$$

defines also a 1-cocycle. If then τ is chosen so that $\tau(\gamma) = \sigma(e_x, e_x)^{-1}$ where $x = r(\gamma)$, then $\tilde{\sigma}$ is normalized, namely $\tilde{\sigma}(e_x, e_x) = 1$.

2.4. An Example: Non Uniform Magnetic Fields. In a series of articles [68, 69, 70], Măntoiu and his collaborators proposed a pseudo-differential calculus for studying the Schrödinger operator describing the quantum motion of a charged particle submitted to both an electric potential and a non uniform magnetic field. To this end, he built a C^* -algebra in which the product is defined through a cocycle taking values in an Abelian C^* -algebra. The same idea was also used by Rieffel in [84] to build a more general theory of continuous fields of C^* -algebras. In this Section, it will be shown that there is a way around such an extension to get an \mathbb{S}^1 -valued cocycle instead. In order to do so, the idea developed here is the same developed in [10, 16] to build the Hull. The general method is developed first and the specific for magnetic field will be explained afterwards. It is worth mentioning [78] in which the case of a continuous field of cocycles corresponding to a family of magnetic fields, is proved to lead to continuity of the spectrum.

Let G be a locally compact Lindelöf group. For simplicity it will be assumed to be Hausdorff and second countable. The neutral element of G will be denoted by e and the composition law, not necessarily commutative, will be denoted multiplicatively. Then a complex valued function $F : G \rightarrow \mathbb{C}$ is called *uniformly continuous* whenever for any $\epsilon > 0$, there is an open neighborhood W of the neutral element e such that $|F(g) - F(h)| < \epsilon$ for any pair $(g, h) \in G$ such that $g^{-1}h \in W$. Any such function is obviously continuous. Let $\mathcal{C}_u(G)$ denote the set of all complex valued, bounded, uniformly continuous function on G . It can be checked that the uniform limit of a uniformly convergent sequence of uniformly continuous functions is also uniformly continuous. Consequently, equipped with the pointwise addition, product, complex conjugacy and the uniform norm $\|F\| = \sup_{g \in G} |F(g)|$, $\mathcal{C}_u(G)$ becomes a unital, Abelian C^* -algebra. In addition, G acts on $\mathcal{C}_u(G)$ through a group α of $*$ -automorphisms defined by $\alpha_h(F)(g) = F(h^{-1}g)$. Thanks to the uniform continuity it follows that $\lim_{h \rightarrow e} \|\alpha_h(F) - F\| = 0$ for any $F \in \mathcal{C}_u(G)$. Hence the group α is norm-pointwise continuous. Let now $\mathcal{F} \subset \mathcal{C}_u(G)$ be a countable set. Then $A(\mathcal{F}, G)$ will denote the sub- C^* -algebra of $\mathcal{C}_u(G)$ generated by the family $\mathcal{F}(G) = \{\alpha_g(F); g \in G, F \in \mathcal{F}\}$ of G -translated of elements of \mathcal{F} . The C^* -algebra $A(\mathcal{F}, G)$ is unital, Abelian, so, by Gelfand's Theorem (see for instance [5, 30]), it is isomorphic to $\mathcal{C}(\Omega_{\mathcal{F}})$ where $\Omega_{\mathcal{F}}$ is a compact space. Since the family \mathcal{F} is countable and since G is second countable, it follows that $A(\mathcal{F}, G)$ is separable, so that $\Omega_{\mathcal{F}}$ is also second countable. It is Hausdorff by construction. Actually, $\Omega_{\mathcal{F}}$ is built as the *spectrum*, namely the set of *characters* defined as the $*$ -homomorphisms $x : A(\mathcal{F}, G) \rightarrow \mathbb{C}$. It follows that G acts by duality on $\Omega_{\mathcal{F}}$ using $gx = x \circ \alpha_{g^{-1}}$. Consequently, $(\Omega_{\mathcal{F}}, G)$ is a genuine topological dynamical system, where G acts by homeomorphisms. Moreover, the Gelfand isomorphism is defined by the map $F \in A(\mathcal{F}, G) \mapsto \mathcal{G}F \in \mathcal{C}(\Omega_{\mathcal{F}})$ where the Gelfand transform \mathcal{G} is defined by $\mathcal{G}F(x) = x(F)$. In $\Omega_{\mathcal{F}}$ there is a remarkable point namely x_e defined by $x_e(F) = F(e)$. It is straightforward to show that the G -orbit of x_e is dense in $\Omega_{\mathcal{F}}$, so that the Gelfand transform is entirely defined by the equation $\mathcal{G}F(g^{-1}x_e) = F(g)$. Following [10, 16]

Definition 6. *The compact space $\Omega_{\mathcal{F}}$, endowed with its canonical G -action and defined as the Gelfand spectrum of the C^* -algebra generated by the G -translated elements of the family \mathcal{F} , will be called the hull of \mathcal{F} .*

The idea is to use this hull to build a groupoid and a cocycle with values in \mathbb{S}^1 to represent the algebra of observables behind a Schrödinger operator with magnetic field. To illustrate the method, let \mathcal{L} be a Delone set in \mathbb{R}^d , supposed to represent the location of the atomic nuclei of an assembly of atoms making up a condensed material (solid or liquid). If there are several species of atoms, each atom is labeled by a letter l in a finite alphabet \mathcal{A} . Then let $\mathcal{L}_l \subseteq \mathcal{L}$ denote the subset of positions of the atomic nuclei of species l . Let V denote the potential energy seen by an individual valence electron, liable to travel through the material. With a very good approximation V can be described as

$$V(x) = \sum_{l \in \mathcal{A}} \sum_{q \in \mathcal{L}_l} v_l(x - q),$$

where $v_l(x)$ is the effective atomic Coulomb potential created by atoms of the l -species at the position x relative to the position of its nucleus. It includes both the Coulomb attraction by the nucleus and the screening effect of the core electrons. A magnetic field is usually described as a closed two-form $B = \sum_{i,j=1}^d B_{ij}(x) dx_i \wedge dx_j$. In most cases the coefficients B_{ij} are smooth functions of the variable $x \in \mathbb{R}^d$. It will be assumed here that *these coefficients are uniformly continuous* over the whole \mathbb{R}^d . Such a condition is satisfied for instance if the gradient of the

coefficients are uniformly bounded on \mathbb{R}^d . Using the Poincaré construction, B can be locally built from a vector-potential $A = \sum_{i=1}^d A_i(x) dx_i$, a one-form satisfying $dA = B$, that is $B_{ij} = \partial_i A_j - \partial_j A_i$. However, as can be seen from the case of a constant magnetic field, A is not bounded in general. The Schrödinger operator describing the quantum motion of a particle of mass m and charge $-e$ moving in the potential V , is the PDE

$$H = -\frac{\hbar^2}{2m} \sum_{i=1}^d (\partial_i - ieA_i)^2 + V.$$

It is a standard result to show that there is a dense domain \mathcal{D} in $\mathcal{H} = L^2(\mathbb{R}^d)$ such an operator, defined on \mathcal{D} is essentially self-adjoint (see for instance [80]). The kinetic term is the square of the generator of translations. But the presence of the magnetic field induces a change in the translation operators, leading to the concept of *magnetic translations* [105]. Namely the usual translation operators $T(a)$ by the vector $a \in \mathbb{R}^d$, acting on the quantum Hilbert space of states \mathcal{H} , is replaced by

$$(4) \quad U(a)\psi(x) = e^{i(e/\hbar) \oint_{x-a}^x A} \psi(x-a), \quad \psi \in \mathcal{H}.$$

where $\oint_{x-a}^x A$ denotes the line integral of the one-form A along the straight line joining $x-a$ to x . This is the usual rule used in Physics when a gauge field is present, namely to change the momentum operator p , which is the infinitesimal generator of translations, into $p - eA$. The commutation rules for these unitary operators is given by

$$(5) \quad U(a)U(b)U(a+b)^{-1} = e^{i(e/\hbar) \Phi(0,a,b)},$$

where $\Phi(0, a, b)$ represents the map $x \in \mathbb{R}^d \mapsto \Phi_x(0, a, b)$, namely the magnetic flux through the oriented triangle defined by the ordered family of points $(x, x-a, x-a-b)$. If $(0, a, b)$ represents the oriented triangle with vertices $(0, a, a+b)$, it can be written as

$$(6) \quad \Phi_x(0, a, b) = \int_{(0,a,b)} \sum_{i,j=1}^d B_{ij}(x-s) ds_i \wedge ds_j,$$

seen as a multiplication operator acting on \mathcal{H} . To implement a good product on the set of observable, this rule must be used, leading to the appearance of a cocycle. In the case of a uniform magnetic field, the cocycle is described in Example 9. But if B is not uniform, the main difficulty at this point is that the magnetic translation is no longer independent of the position x at which the state is evaluated. The solution to this problem proposed by Măntoiu *et. al.*, consists in seeing the phase factor in the *r.h.s.* of eq. (5), as a bounded continuous function of x , and therefore as an element of a “good” sub-algebra of $\mathcal{C}_b(\mathbb{R}^n)$.

This difficulty admits another solution though, using the groupoid approach of the present work. First, if the Delone set \mathcal{L} is not periodic, it has a hull $\Omega_{\mathcal{L}}$, a Hausdorff, second countable compact space. This hull can be defined as the closure of the orbit of \mathcal{L} under the translation group $G = \mathbb{R}^n$, after a genuine topology is defined on the space of Delone sets (see for instance [17]). This Hull allows to see the electric potential V as a function $V_{\omega}(x) = \mathcal{V}(T^{-x}\omega)$ for a continuous function \mathcal{V} on the compact space Ω and an \mathbb{R}^d -action by homeomorphism T . Then, because the magnetic field B is assumed to be uniformly continuous, it admits another hull which will be denoted by Ω_B , another Hausdorff, second countable, compact space, endowed

with an \mathbb{R}^d -action which, for simplicity, will also be denoted \mathbb{T} . This leads to a combined hull $\Omega = \Omega_{\mathcal{L}} \times \Omega_B$ and the combined \mathbb{R}^d -action $\mathbb{T}^x(\omega, \omega') = (\mathbb{T}^x\omega, \mathbb{T}^x\omega')$. The combined hull Ω admits also a remarkable point, ω_0 so that both the actual magnetic field and the electric potential can be written in the form $V(x) = \mathcal{V}(\mathbb{T}^{-x}\omega_0)$ and $B_{ij}(x) = \mathcal{B}_{ij}(\mathbb{T}^{-x}\omega_0)$. The underlying groupoid will then be $\Omega \rtimes_{\mathbb{T}} \mathbb{R}^d$ here. It ought to be reminded here that a typical arrow in this groupoid has the form of a pair $\gamma = (\omega, x)$ with $\omega \in \Omega$ and $x \in \mathbb{R}^d$, and that $r(\gamma) = \omega$ while $s(\gamma) = \mathbb{T}^{-x}\omega$. Then the magnetic translation operator becomes ω -dependent, through the left regular representation of this groupoid. Similarly the cocycle defining the C^* -algebra is given now by

$$\sigma((\omega, x), (\mathbb{T}^{-x}\omega, y)) = e^{ie\Phi_{\omega}(0,x,y)/\hbar},$$

where $\Phi_{\omega}(0, x, y)$ represent the magnetic flux through the oriented triangle $(0, x, y)$ with vertices $(0, x, x + y)$ defined by

$$\Phi_{\omega}(x, y) = \int_{(0,x,y)} \sum_{i,j=1}^d \mathcal{B}_{ij}(\mathbb{T}^{-s}\omega) ds_i \wedge ds_j.$$

A similar situation occurs for the discrete version of the Schrödinger operator. Given the Delone set \mathcal{L} and a point ω in the combined hull Ω , there is \mathcal{L}_{ω} another Delone set such that $\mathcal{L}_{\omega} + x = \mathcal{L}_{\mathbb{T}^x\omega}$. The canonical transversal is the closed subset $\Xi \subset \Omega$ of points such that \mathcal{L}_{ω} contains the origin $0 \in \mathbb{R}^d$. Let Γ_{Ξ} be the groupoid associated with Ξ . Then, the Hilbert space of quantum states will be $\mathcal{H}_{\xi} = \ell^2(\mathcal{L}_{\xi})$ for $\xi \in \Xi$. A typical Hamiltonian will be given by an expression of the form

$$H_{\xi}\psi(a) = \sum_{b \in \mathcal{L}_{\xi}} h(\mathbb{T}^{-a}\xi, b) e^{i(e/\hbar) \int_b^a A_{\xi}} \psi(b), \quad a \in \mathcal{L}_{\xi}, \psi \in \ell^2(\mathcal{L}_{\xi}),$$

where (i) h is a continuous function on Γ_{Ξ} decaying fast enough at infinity to insure that the sum converges and (ii) A_{ξ} is a vector potential associated with the magnetic field $B_{\xi}(x) = \mathcal{B}(\mathbb{T}^{-x}\xi)$. So the treatment will be similar, as indicated in the next Sections.

2.5. Convolution and the Full C^* -algebra. Given a locally compact space Y , let $\mathcal{C}_c(Y)$ denote the space of complex valued continuous functions defined on Y with compact support. This is a topological complex vector space. The topology is defined so that a net $(f_{\alpha})_{\alpha \in A}$ converges to f if and only if (i) there is $K \subseteq Y$ compact and $\alpha \in A$, such that for $\beta \geq \alpha$ $\text{supp}(f_{\beta}) \subseteq K$, and (ii) for any $\epsilon > 0$ there is $\alpha \leq \alpha' \in A$ such that $\sup_{y \in K} |f_{\beta}(y) - f(y)| < \epsilon$ for $\beta \geq \alpha'$.

Let Γ be a handy groupoid. Then Γ admit a Haar system $\mu = (\mu^x)_{x \in \Gamma(0)}$. In addition, let σ be a 1-cocycle with values in the unit circles \mathbb{S}^1 seen as a multiplicative subgroup of \mathbb{C}_* . Then a structure of $*$ -algebra can be defined on the function space $\mathcal{C}_c(\Gamma)$ in complete analogy with the case of locally compact groups (see [37, 35]). If $f, g \in \mathcal{C}_c(\Gamma)$, their convolution is defined by

$$(7) \quad fg(\gamma) = \int_{\Gamma^x} \sigma(\eta, \eta^{-1} \circ \gamma) f(\eta) g(\eta^{-1} \circ \gamma) d\mu^x(\eta), \quad r(\gamma) = x.$$

Following [81], thanks to the cocycle property of σ , it is straightforward to check that this product is associative and bilinear. In addition, the function fg is continuous thanks to (H2) and has compact support [91]. An *adjoint* map is now defined as follows

$$(8) \quad f^*(\gamma) = \overline{\sigma(\gamma, \gamma^{-1})} \overline{f(\gamma^{-1})}.$$

A straightforward calculation shows that $f \rightarrow f^*$ is antilinear. Moreover, thanks to Lemma 5, it is involutive, namely $(f^*)^* = f$. It also satisfies

$$(fg)^* = g^* f^*.$$

In addition, these two operations are continuous with respect to the topology of $\mathcal{C}_c(\Gamma)$ [81].

Definition 7. *The topological $*$ -algebra obtained from $\mathcal{C}_c(\Gamma)$ by using the product (eq. (7)) and the adjoint (eq. (8)) will be denoted by $\mathcal{C}_c(\Gamma, \mu, \sigma)$.*

The topology of $\mathcal{C}_c(\Gamma)$ is not coming from a norm. It is often considered as more convenient to define algebraic norms and to complete this topological algebra in order to get a more practical tool for analysis. In this Section only two examples of such norms will be proposed. The first one is the extension of the Wiener algebra. For this purpose, let $\|f\|_{\infty,1}^{\sim}$ be defined by [49, 81]

$$\|f\|_{\infty,1}^{\sim} = \sup_{x \in \Gamma^{(0)}} \int_{\Gamma^x} |f(\gamma)| d\mu^x(\gamma).$$

It is straightforward to check that this defined a norm on $\mathcal{C}_c(\Gamma)$ such that

$$\|fg\|_{\infty,1}^{\sim} \leq \|f\|_{\infty,1}^{\sim} \|g\|_{\infty,1}^{\sim}.$$

In general though, this norm is not invariant under taking the adjoint. This leads to set

$$\|f\|_{\infty,1} = \max\{\|f\|_{\infty,1}^{\sim}, \|f^*\|_{\infty,1}^{\sim}\}$$

The completion of $\mathcal{C}_c(\Gamma, \mu, \sigma)$ *w.r.t.* this norm will be denoted by $L^{\infty,1}(\Gamma, \mu)$. Using a 3ϵ -argument it is easy to check that any element of this normed $*$ -algebra can be seen as a μ -measurable function which is L^1 along each fiber, such that $x \in \Gamma^{(0)} \rightarrow \mu^x(f)$ is continuous.

In order to define a C^* -algebra, the usual procedure consists in looking at the representations of $\mathcal{C}_c(\Gamma, \mu, \sigma)$ by operators on a Hilbert space. Following [81], a *representation* of $\mathcal{C}_c(\Gamma, \mu, \sigma)$ on a Hilbert space \mathcal{H} is a weakly continuous $*$ -homomorphism $\rho : \mathcal{C}_c(\Gamma, \mu, \sigma) \rightarrow \mathcal{B}(\mathcal{H})$, such that the linear span of the set $\{\rho(f)\psi; f \in \mathcal{C}_c(\Gamma, \mu, \sigma), \psi \in \mathcal{H}\}$ is dense in \mathcal{H} . This representation is called *bounded* whenever $\|\rho(f)\| \leq \|f\|_{\infty,1}$. A C^* -norm can be defined by

$$(9) \quad \|f\| = \sup_{\rho} \|\rho(f)\|, \quad \rho \text{ bounded representation}$$

Definition 8. *The full C^* -algebra of Γ is the completion of $\mathcal{C}_c(\Gamma, \mu, \sigma)$ under the norm $\|\cdot\|$ (eq. (9)). It will be denoted by $\mathfrak{C}^*(\Gamma, \mu, \sigma)$ or $\mathfrak{C}^*(\Gamma, \sigma)$ if there is no ambiguity on μ .*

2.6. The Reduced C^* -algebra. Let G be a locally compact group equipped with a left-invariant Haar measure λ . A *unitary representation* U of G , in a Hilbert space \mathcal{H} , is a group homomorphism $U : G \rightarrow \mathcal{U}(\mathcal{H})$, where $\mathcal{U}(\mathcal{H})$ denotes the set of unitary operators on \mathcal{H} . Such a representation is *strongly continuous* whenever the map $g \in G \mapsto U(g) \in \mathcal{U}(\mathcal{H})$ is strongly continuous. Since $U(g)$ is a unitary operator, strong continuity holds if and only if it is weakly continuous, that is, for any pair ϕ, ψ of elements of \mathcal{H} , the map $g \rightarrow \langle \phi | U(g)\psi \rangle$ is continuous. The *left-regular representation* is an important example. Let $\mathcal{H} = L^2(G)$ be the space of square integrable complex valued functions with respect to λ . Then G acts on \mathcal{H} by left multiplication,

namely if $\psi \in L^2(G)$ and $g \in G$, then $L(g)\psi(h) = \psi(g^{-1}h)$ defines a strongly continuous unitary representation of G in \mathcal{H} . Similarly it acts by right multiplication with $R(g)\psi(h) = \psi(hg)$, giving another strongly continuous unitary representation of G in \mathcal{H} . It can be checked that $L(g)R(g') = R(g')L(g)$. If, in addition, G is a compact group, the Peter-Weyl Theorem expresses that L (*resp.* R) is a direct sum of irreducible unitary representations of G , each such representation occurring with multiplicity equal to its dimension. It is therefore legitimate to ask whether the left-regular (*resp.* the right-regular) representation, is sufficient to recover entirely the group G .

To express this concept, let $\mathfrak{C}_{red}^*(G)$ denotes the C^* -algebra generated by the bounded operators acting on \mathcal{H} of the form $R(f) = \int_G f(g) R(g) d\lambda(g)$ for $f \in C_c(G)$, and let $\|\cdot\|_{red}$ be its norm. It is worth noticing that $L(g)R(f)L(g)^{-1} = R(f)$. In particular, if G is a compact group, $R(f)$ is the direct sum of representative in each irreducible component of L . If the right-regular representation suffices to generate all unitary representation, as suggested by the Peter-Weyl theorem, then any other strongly continuous unitary representation V of G , on some Hilbert space \mathcal{K} , must satisfy $\|\int_G f(g) V(g) d\lambda(g)\| \leq \|f\|_{red}$. As it turns out, this is the case if and only if G is *amenable* [53] (for the concept of amenability, see Section 2.7 and [43, 48]). Any locally compact Abelian group, or any compact group, is amenable. On the other hand, the free group with $n \geq 2$ generators is not amenable and it does not satisfy the previous criterion.

A similar phenomenon occurs for groupoids, for which there is also a concept of amenability [3]. To understand more precisely, the first concept to be defined is the definition of a strongly continuous unitary representation of a groupoid (analogous to the definition of measurable representations [28]). Since a groupoid Γ is a *category*, a unitary representation will be a (covariant) functor from Γ into the category **Hilb** the objects of which are Hilbert spaces, with morphism given by unitary operators. Hence for each object (unit) $x \in \Gamma^{(0)}$, a Hilbert space \mathcal{H}_x should be given so that if $\gamma : x \rightarrow y$, then there is a unitary operator $U(\gamma) : \mathcal{H}_x \rightarrow \mathcal{H}_y$. To be a covariant functor, the additional property should be that $U(\gamma_1 \circ \gamma_2) = U(\gamma_1)U(\gamma_2)$. To express the strong continuity, it will be assumed that the field $(\mathcal{H}_x)_{x \in \Gamma^{(0)}}$ of Hilbert space is *continuous* [35]. Namely there are enough *continuous vector fields* $\psi = (\psi_x)_{x \in \Gamma^{(0)}}$ to generate each of the Hilbert spaces \mathcal{H}_x . By definition of a continuous vector field, the map $x \in \Gamma^{(0)} \mapsto \|\psi_x\| \in [0, \infty)$ is continuous. Hence, by analogy with the case of groups, strong continuity of U can be expressed by demanding that for any pair ϕ, ψ of continuous vector fields, the map $\gamma \in \Gamma \mapsto \langle \phi_{r(\gamma)} | U(\gamma) \psi_{s(\gamma)} \rangle \in \mathbb{C}$ is continuous.

If σ is a 1-cocycle, a σ -representation of Γ will be defined as in the previous construction, with the following modifications

$$U(\gamma_1)U(\gamma_2) = \sigma(\gamma_1, \gamma_2) U(\gamma_1 \circ \gamma_2), \quad U(\gamma)^* = \overline{\sigma(\gamma, \gamma^{-1})} U(\gamma^{-1}).$$

The simplest example is provided by *left regular representation*. Namely, given a continuous Haar system on Γ , then $\mathcal{H}_x = L^2(\Gamma^x, \mu^x)$. If $\gamma : x \rightarrow y$, the unitary operator $U(\gamma)$ is now replaced by

$$L(\gamma)\psi(\eta) = \sigma(\gamma, \gamma^{-1} \circ \eta) \psi(\gamma^{-1} \circ \eta), \quad \psi \in L^2(\Gamma^x, \mu^x).$$

Thanks to the Assumption (H3), it follows that $L(\gamma)$ is a unitary operator. It becomes straightforward to check that $L(\gamma_1)L(\gamma_2) = \sigma(\gamma_1, \gamma_2) L(\gamma_1 \circ \gamma_2)$. It is worth noticing that the subspace of $L^2(\Gamma^x, \mu^x)$ generated by the restriction to Γ^x of elements of $C_c(\Gamma)$ is dense.

The analog of the operators $R(f)$ acting on $L^1(G)$ in the group case, can be also defined in the groupoid case. Indeed, if $f \in \mathcal{C}_c(\Gamma, \mu, \sigma)$, an operator can be defined on $L^2(\Gamma^x, \mu^x)$ as follows

$$(10) \quad \pi_x(f)\psi(\gamma) = \int_{\Gamma^x} f(\gamma^{-1} \circ \eta) \sigma(\gamma, \gamma^{-1} \circ \eta) \psi(\eta) d\mu^x(\eta), \quad \psi \in L^2(\Gamma^x, \mu^x).$$

Standard estimates imply

$$\|\pi_x(f)\| \leq \|f\|_{\infty,1}.$$

showing that $\pi_x(f)$ is a bounded operator. A tedious but simple computation shows that π_x is linear in f , that

$$\pi_x(fg) = \pi_x(f) \pi_x(g), \quad \pi_x(f)^* = \pi_x(f^*), \quad f, g \in \mathcal{C}_c(\Gamma).$$

Therefore π_x is a bounded representation of $\mathcal{C}_c(\Gamma, \mu, \sigma)$. In addition, using the density of $\mathcal{C}_c(\Gamma, \mu, \sigma)$ in $L^2(\Gamma^x, \mu^x)$, the Assumption (H2) and a 3ϵ -argument, it follows that $x \rightarrow \pi_x(f)$ is strongly continuous. At last, if $\gamma : x \rightarrow y$ then, the following *covariance* property holds

$$U(\gamma)\pi_x(f)U(\gamma)^{-1} = \pi_y(f).$$

It then follows that $x \sim y$ implies $\|\pi_x(f)\| = \|\pi_y(f)\|$. Moreover, whenever $f = f^*$ is self-adjoint, then $\pi_x(f)$ and $\pi_y(f)$ have the same spectral measure and the same spectrum. It is also straightforward to check that $\pi_x(f) = 0$ if and only if f vanishes on the fiber Γ^x . Consequently $\pi_x(f) = 0$ for all $x \in \Gamma^{(0)}$ if and only if $f = 0$. This implies that

$$\|f\|_{red} = \sup_{x \in \Gamma^{(0)}} \|\pi_x(f)\|,$$

defines a norm. By construction this norm is algebraic, invariant by the adjoint and satisfies $\|f^*f\|_{red} = \|f\|_{red}^2$. Hence it is a C^* -norm.

Definition 9. *Let Γ be a handy groupoid endowed with a Haar system μ and a 1-cocycle σ . Then the reduced C^* -algebra of Γ is the completion of $\mathcal{C}_c(\Gamma, \mu, \sigma)$ under the C^* -norm $\|\cdot\|_{red}$. This completion will be denoted by $\mathfrak{C}_{red}^*(\Gamma, \mu, \sigma)$.*

In general the left regular representation is not sufficient to encode all representations. A result of Anantharaman-Delaroche and Renault [2] shows that, at least for handy étale groupoids, amenability implies $\mathfrak{C}_{red}^*(\Gamma) = \mathfrak{C}^*(\Gamma)$ (Theorem 6.1.8) and a similar result in a more general situation. However, even in the étale case, R. Willett [104] gave an example of handy étale groupoid that is not amenable and for which $\mathfrak{C}_{red}^*(\Gamma) = \mathfrak{C}^*(\Gamma)$.

2.7. Amenability. At this point it might help the reader to give some hint about the concept of amenability. The most elementary way to address this problem is as follows: consider a *time-dependent* function f . Here the time is represented by a variable $t \in \mathbb{R}$. The average of f over time is simply defined as

$$\langle f \rangle = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{+T} f(t) dt.$$

If f is unbounded such a limit might not even exist. But even if f is bounded and continuous, *there might be several limits*. In this precise example, all possible limits are classified by ultrafilters, or, equivalently by the Čech compactification of \mathbb{R} . Then any of these limits are

translation-invariant, namely if $s \in \mathbb{R}$ and if $f_s(t) := f(t - s)$, then $\langle f_s \rangle = \langle f \rangle$ for all s 's. This is because, if $\|f\| = \sup_t |f(t)|$,

$$\left| \frac{1}{2T} \int_{-T}^{+T} f(t - s) dt - \frac{1}{2T} \int_{-T}^{+T} f(t) dt \right| \leq \frac{|s| \|f\|}{T} \xrightarrow{T \uparrow \infty} 0$$

A similar problem might occur in higher dimension, if the time axis \mathbb{R} is replaced by the position space \mathbb{R}^d , in order to average over space. This problem occurs in Thermodynamics for which an observable can be defined as a position dependent function, $f(x)$, $x \in \mathbb{R}^d$. Then its space-average can be defined by

$$\langle f \rangle = \lim_{\Lambda \uparrow \mathbb{R}^d} \frac{1}{|\Lambda|} \int_{\Lambda} f(x) d^d x,$$

where $|\Lambda|$ denotes the volume of Λ , namely its Lebesgue measure. Again in this case, at least if f is continuous and bounded, the Čech compactification of \mathbb{R}^d classifies all limits and it is possible to show that, under suitable conditions on the choice of Λ , these limits are translation invariant. Such a limit defines what is called an *invariant mean*. In dealing with translation invariance, it becomes easy to realize that the ratio of the area $|\partial\Lambda|$ of the boundary of Λ should become negligible *w.r.t.* the volume of Λ as this volume grows to become infinite. This concept is crucial in Thermodynamics or in Statistical Mechanics, where the limit above is controlled by the concept of *Van Hove* limit (see [98] Section 2 Axiom (e)).

If \mathbb{R}^d is replaced by a locally compact group G , the same problem can be addressed in similar terms. A mean is defined as a state on the the Abelian C^* -algebra $\mathcal{C}_b(G)$ of complex valued, continuous, bounded functions on G . That is to say

- (i) $m : \mathcal{C}_b(G) \rightarrow \mathbb{C}$ is linear,
- (ii) m is positive, namely if $f \in \mathcal{C}_b(G)$ is non-negative everywhere in G , then $m(f) \geq 0$,
- (iii) m is normalized, namely $m(1) = 1$.

Given $s \in G$ and $f \in \mathcal{C}_b(G)$, let f_s be defined by $f_s(t) = f(s^{-1}t)$ for $t \in G$. Then a mean m is G left-invariant whenever $m(f_s) = m(f)$ for all $s \in G$.

However, there are examples of groups for which there is no invariant mean ! Such is the case for the free group with $n \geq 2$ generators. In general, a locally compact group G will be called *topologically amenable* whenever there is a G -invariant mean. Characterizing amenable groups has been a challenge for a long time. The concept of Følner sequence became central [43, 48]: it expresses the idea that in taking the average over a subset $\Lambda \subseteq G$, the boundary of Λ has a Haar-measure negligible compared to the Haar measure of Λ . More precisely, if λ is a left-invariant Haar measure on G , a sequence $(\Lambda_n)_{n \in \mathbb{N}}$ of λ -measurable subsets of G is called a Følner-sequence whenever

- (i) $\lambda(\Lambda_n) < \infty$ for all $n \in \mathbb{N}$,
- (ii) $\lambda(\Lambda_n) \rightarrow \infty$ as $n \uparrow \infty$,
- (iii) for any $s \in G$, the symmetric difference $s\Lambda_n \Delta \Lambda_n$ satisfies

$$\lim_{n \rightarrow \infty} \frac{\lambda(s\Lambda_n \Delta \Lambda_n)}{\lambda(\Lambda_n)} = 0.$$

Remark 3. A Følner sequence might escape to infinity, far from the “origin” (namely the neutral element) of the group. In particular the construction of a Følner sequence is less constrained than the construction of a Van Hove sequence that is commonly used in Physics for the purpose of controlling the infinite volume limit [98, 99]. \square

Given a Følner sequence and given an ultrafilter \mathfrak{F} on the set \mathbb{N} of natural integers, let m be the mean defined by

$$m(f) = \lim_{\mathfrak{F}} \frac{1}{\lambda(\Lambda_n)} \int_{\Lambda_n} f(t) d\lambda(t), \quad f \in \mathcal{C}_b(G).$$

That such a limit is well defined comes from the fact that sequence on the *r.h.s.* is bounded by $\|f\|$. Moreover,

$$|m(f_s) - m(f)| \leq \lim_{\mathfrak{F}} \|f\| \frac{\lambda(g\Lambda_n \Delta \Lambda_n)}{\lambda(\Lambda_n)} = 0,$$

showing that this mean is G -invariant. Hence, the existence of a Følner sequence implies the existence of an invariant mean. It turns out that the converse is true [48].

Eventually, at least for locally compact Hausdorff and second countable groups, amenability could be described in terms of the equality $\mathfrak{C}_{red}^*(G) = \mathfrak{C}^*(G)$ [53]. For a handy groupoid Γ , the concept of amenability is much more involved. Anantharaman-Delaroche and Renault expressed it in terms of positive definite functions in [3]. They distinguish between several concepts of amenability, topological, Borel or measurable.

As a consequence, a dynamical system (X, G, α) where G is a locally compact, Hausdorff, second countable and amenable group and X a compact Hausdorff, second countable space, gives rise to an amenable groupoid $\Gamma = X \rtimes_{\alpha} G$. If $Y \subseteq X$ is closed, any sub-groupoid $\Gamma_Y \subseteq X \rtimes_{\alpha} G$ is also amenable. And this is the only property that will be needed in the present work. However, there are dynamical systems for which G is not amenable but the action on X admits an invariant probability measure, leading to an amenable groupoid. The typical example is the action of the free group with 2-parameters acting on its boundary (see [2], Ex. 3.8).

3. THE TAUTOLOGICAL GROUPOID

In all this Section Γ denotes a handy groupoid. The set of closed invariant sets $\mathcal{J}(\Gamma)$ of a handy groupoid Γ is introduced. Among these subsets, some are minimal, like in the case of dynamical systems. In many examples of applications to Physics, minimal *finite* invariant subsets can be identified with periodic orbits. Using the Hausdorff topology (as defined either by Vietoris [100], by Chabauty [25] or by Fell [42]), the space $\mathcal{J}(\Gamma)$ becomes a compact Hausdorff second countable space. This leads to the definition of the *tautological groupoid* of Γ , denoted by $\mathcal{T}(\Gamma)$. Consequently, at least whenever Γ is étale and amenable, thanks to a Theorem by Landsman and Ramazan [65], the C^* -algebra of the tautological groupoid appears as the enveloping algebra of a continuous field of C^* -algebras indexed by $\mathcal{J}(\Gamma)$. This machinery permits to control whether or not a closed invariant subset can be approximated by a sequence of periodic orbits. If so, a genuine Schrödinger operator H , defined on the minimal invariant subset, can be approximated by periodic operators for which Bloch Theory applies, leading to a convergent sequence of approximations of the spectrum of H .

3.1. Invariant Sets. If $F \subseteq \Gamma^{(0)}$, its saturated is defined by $[F] = \{x \in \Gamma^{(0)}; \exists y \in F, x \sim y\}$. F is called invariant whenever $F = [F]$. Equivalently, F is invariant if and only if, whenever $x \in F$ and $y \sim x$ then $y \in F$. Hence an invariant set is the union of the equivalence classes of its elements. If $x \in \Gamma^{(0)}$, $[x]$ is its equivalence class, namely it is the saturated of $\{x\}$. If Γ is provided by a dynamical system (X, G, α) (see Example 6), and if $x \in X = \Gamma^{(0)}$, then $[x]$ is nothing but the set $\{g^{-1}x; g \in G\}$, namely it is the G -orbit of x . By analogy, $[x]$ will be also

called the Γ -orbit of x (or simply “orbit” if there is no ambiguity) and the notation $[x] = \text{Orb}(x)$ will be used. The first result is the following

Proposition 1. *Let $F \subseteq \Gamma^{(0)}$ be invariant. Then its closure \overline{F} is also invariant.*

Proof: Let $x \in \overline{F}$. For $y \sim x$ let $\gamma : y \rightarrow x$. Let then V be an open neighborhood of y . Since the source map is continuous, it follows that $s^{-1}(V)$ is an open set in Γ containing γ . Since the range map is open, $U = r(s^{-1}(V))$ is also open in $\Gamma^{(0)}$ and contains $r(\gamma) = x$. Since $x \in \overline{F}$ the intersection $U \cap F$ is not empty. Let $x' \in U \cap F$. Then there is $\gamma' \in s^{-1}(V)$ such that $r(\gamma') = x'$, so that $s(\gamma') \in V$. Since $x' \in F$ and F is invariant $s(\gamma') = y' \in F$. Hence $V \cap F \neq \emptyset$. Since V is arbitrary, it follows that $y \in \overline{F}$. \square

Definition 10. *The set of closed invariant subset in $\Gamma^{(0)}$ is denoted by $\mathcal{J}(\Gamma)$.*

Using the analogy with topological dynamical systems [46], the following result holds

Theorem 5. *The set $\mathcal{J}(\Gamma)$, ordered by inclusion, has minimal elements. A closed invariant subset $M \in \mathcal{J}(\Gamma)$ is minimal if and only if any orbit is dense in M .*

Proof: (i) Any intersection of closed invariant subsets $(F_j)_{j \in J}$ of $\Gamma^{(0)}$ is closed and invariant: (a) it is closed since any intersection of closed sets is closed, (b) it is invariant because if F denotes the intersection, if $x \in F$ then, for all $j \in J$, $x \in F_j$. Hence if $y \sim x$, it follows that $y \in F_j$ as well for all j 's. Hence $y \in F$ and F is closed and invariant. In particular, the intersection of a decreasing family of elements of $\mathcal{J}(\Gamma)$ belongs to $\mathcal{J}(\Gamma)$. By Zorn's Lemma, $\mathcal{J}(\Gamma)$ admits minimal elements.

(ii) Let M be minimal. If $x \in M$ then $[x] \subseteq M$ since M is invariant. In addition the closure $\overline{[x]}$ is also included in M since M is closed, and is invariant by Proposition 1. Since M is minimal $\overline{[x]} = M$.

(iii) Conversely, let $M \in \mathcal{J}(\Gamma)$ be such that every of its orbit is dense. If $N \subseteq M$ is closed and invariant, then given $x \in N$ it follows that $M = \overline{[x]} \subseteq N$, so that $M = N$ and M is minimal. \square

If (X, G, α) is a topological dynamical system giving rise to a handy groupoid, then for $x \in X$, $[x]$ is the G -orbit of x . In particular, G acts *transitively* on $[x]$. Let H_x be the *stabilizer* of x in G , namely $H_x = \{h \in G; h^{-1}x = x\}$. It follows that the $g^{-1}x$ depends only on the right coset $[g] = H_x g$, so that $H_x g \in H_x \backslash G \mapsto g^{-1}x \in [x]$ is a bijection. In addition, it is continuous. The following result, initially proved by Freudenthal [44] gives a condition under which this map is an homeomorphism.

Theorem 6 (Open Action [44]). *Let G be a group acting on a space X in a transitive way (the orbit of any point is X). If G and X are locally compact and if G is a Lindelöf space, then the map $\phi_x : G \rightarrow X$, defined by $\phi_x(g) = g^{-1}x$ is open.*

It is worth reminding that a topological space is Lindelöf whenever every open covers contains a countable sub-cover. In particular all compact space are Lindelöf. The proof below is provided because it will be used later in the case of groupoid.

Sketch of the Proof of Theorem 6: Let $K \subseteq G$ be any compact symmetric (namely K is invariant by $g \mapsto g^{-1}$) neighborhood of the identity in G . Then G can be seen as the countable union of sets of the form $g^{-1}K$. Let $\overset{\circ}{A}$ denote the interior of a subset $A \subseteq X$. By Baire's theorem, some of the $\phi_x(g^{-1}K)$ has a closure with non empty interior. By transitivity of the action, there is $h \in K$ such that $\phi_x(h) \in \overline{\phi_x(K)}$. In particular $x = \phi_x(e) \in \overline{\phi_x(h^{-1}K)} \subseteq \overline{\phi_x(K^2)}$.

Let now $U \subseteq G$ be open and let $g \in U$. Then K can be chosen small enough so that $gK^2 \subseteq U$. Therefore $\phi_x(g) \in \overline{\phi_x(gK^2)} \subseteq \phi_x(U)$. Hence $\phi_x(U)$ is open. \square

Corollary 6. *Let (X, G, α) be a topological dynamical system giving rise to a handy groupoid. Let $x \in X$ have a closed G -orbit. If $H_x \subseteq G$ is the stabilizer of x , then $[x]$ is homeomorphic to $H_x \backslash G$ and the groupoid $\Gamma_{[x]}$ is isomorphic to the dynamical system $(H_x \backslash G, G, \beta)$ where β is the canonical right G -action on $H_x \backslash G$.*

Proof: If $[x]$ is closed, then it is compact (since X is compact). In particular the condition that the groupoid $X \rtimes_{\alpha} G$ is handy, implies that G is Lindelöf. Of course so is $[x]$ then. The map $g \in G \rightarrow g^{-1}x \in [x]$ being open, it follows from the definition of the quotient topology that the induced map $[g] \in H_x \backslash G \rightarrow [x]$ is also open. Hence the inverse of this map is also continuous, meaning that this map is actually an homeomorphism. Consequently, $H_x \backslash G$ is compact, namely H_x is co-compact in G . The last statement is an exercise left to the reader. \square

In the specific case for which $G = \mathbb{R}^d$, a co-compact subgroup is isomorphic to $H = \mathbb{Z}^n \times \mathbb{R}^{d-n}$ for some $0 \leq n \leq d$. The special case $n = d$ corresponds to a non-singular *periodic dynamical system*. So, it is natural to study points $x \in \Gamma^{(0)}$ with closed Γ -orbit. If so $[x]$ is automatically minimal in $\mathcal{J}(\Gamma)$. The following definition and the arguments after it are a generalization of Corollary 6 to the groupoid case:

Definition 11. *A unit $x \in \Gamma^{(0)}$ is called *periodicoid* if its Γ -orbit is closed.*

Let x be periodicoid unit of Γ . We want to know what is the structure of the sub-groupoid $\Gamma_{[x]}$. In order to do so, let Γ^x be the r -fiber of x and let $H_x = \Gamma_x^x$ its rs -fiber. Then Γ^x and H_x are both locally compact, Hausdorff and second countable. In particular they are Lindelöf. H_x acts on Γ^x on the left, namely if $\eta \in H_x$ then $\gamma \rightarrow \eta \circ \gamma$ gives this action. Similarly, H_x acts diagonally on the product space $\Gamma^x \times \Gamma^x$ namely through $(\gamma, \gamma') \mapsto (\eta \circ \gamma, \eta \circ \gamma')$. Let then $S\Gamma^x = H_x \backslash (\Gamma^x \times \Gamma^x)$ be the set of diagonal left H_x -cosets of $\Gamma^x \times \Gamma^x$. The equivalence class of a pair $(\gamma, \gamma') \in \Gamma^x \times \Gamma^x$ will be denoted by $[(\gamma, \gamma')]$.

A groupoid structure can be defined on $S\Gamma^x$ as follows:

- (i) *Range map:* $r[(\gamma, \gamma')] = s(\gamma)$,
- (ii) *Source map:* $s[(\gamma, \gamma')] = s(\gamma')$

Since $s(\eta \circ \gamma) = s(\gamma)$ for all $\eta \in H_x$, these definitions make sense in the coset space. Then $[(\gamma_1, \gamma'_1)]$ and $[(\gamma_2, \gamma'_2)]$ are composable if and only if $s(\gamma'_1) = s(\gamma_2)$, namely if and only if $\eta = \gamma_2 \circ \gamma'_1{}^{-1} \in H_x$. Hence, $\gamma_2 = \eta \circ \gamma'_1$. Therefore there is a unique $\gamma_3 \in \Gamma^x$ such that $(\gamma_2, \gamma'_2) = (\eta \circ \gamma'_1, \eta \circ \gamma_3)$. This gives the product rule

- (iii) *Product:* $[(\gamma_1, \gamma'_1)] \circ [(\eta \circ \gamma'_1, \eta \circ \gamma_3)] = [(\gamma_1, \gamma_3)]$,
- (iv) *Inverse:* $[(\gamma, \gamma')]^{-1} = [(\gamma', \gamma)]$
- (v) *Unit:* if $y \in [x]$ and $\gamma : y \rightarrow x$ then $e_y = [(\gamma, \gamma)]$

If $\gamma, \gamma' \in \Gamma^x$ it follows that $\gamma^{-1} \circ \gamma'$ is an element of $\Gamma_{[x]}$. This is because $r(\gamma^{-1} \circ \gamma') = s(\gamma) \sim x$ and $s(\gamma^{-1} \circ \gamma') = s(\gamma') \sim x$ as well.

Proposition 2. *Let Γ be a handy groupoid. Let $x \in \Gamma^{(0)}$ be a periodicoid element. Then there is a topological groupoid isomorphism $\phi : S\Gamma^x \rightarrow \Gamma_{[x]}$ defined by*

$$\phi([(\gamma, \gamma')]) = \gamma^{-1} \circ \gamma' ,$$

Proof: (i) Let $\varphi : \Gamma^x \times \Gamma^x \rightarrow \Gamma_{[x]}$ be defined by $\varphi(\gamma, \gamma') = \gamma^{-1} \circ \gamma'$. Clearly φ is continuous. In addition, $\varphi(\gamma, \gamma') = \varphi(\eta \circ \gamma, \eta \circ \gamma')$ for all $\eta \in H_x$. Hence φ defines a map $\phi : S\Gamma^x \rightarrow \Gamma_{[x]}$. Using the quotient topology implies that ϕ is continuous as well.

(ii) The map ϕ is onto: if $\delta : z \rightarrow y$ belongs to $\Gamma_{[x]}$, then there are $\gamma : y \rightarrow x$ and $\gamma' : z \rightarrow x$ so that $\eta = \gamma \circ \delta \circ \gamma'^{-1} \in H_x$. Therefore $\delta = \gamma^{-1} \circ \eta \circ \gamma' = \varphi(\gamma, \eta \circ \gamma')$.

(iii) The map ϕ is one-to-one. This is because $\gamma_1^{-1} \circ \gamma'_1 = \gamma_2^{-1} \circ \gamma'_2$, if and only if $\gamma'_2 \circ \gamma_1'^{-1} = \gamma_2 \circ \gamma_1^{-1} = \eta \in H_x$, namely if and only if $(\gamma_2, \gamma'_2) = (\eta \circ \gamma_1, \eta \circ \gamma'_1)$.

(iv) Under the composition ϕ satisfies

$$\begin{aligned} \phi([\!(\gamma_1, \gamma_3)\!]) &= \gamma_1^{-1} \circ \gamma_3 \\ &= \gamma_1^{-1} \circ \gamma'_1 \circ \gamma_1'^{-1} \circ \eta^{-1} \circ \eta \circ \gamma_3 \\ &= \phi([\!(\gamma_1, \gamma'_1)\!]) \circ \phi([\!(\eta \circ \gamma'_1, \eta \circ \gamma_3)\!]). \end{aligned}$$

Showing that ϕ is a groupoid isomorphism.

(v) The map ϕ is open: if $U \subseteq S\Gamma^x$ is open, then, by the very definition of the quotient topology, there is $V \subseteq \Gamma^x \times \Gamma^x$ open and H_x -invariant such that $[V] = U$. Hence, because the topology on $\Gamma^x \times \Gamma^x$ is induced by the topology on $\mathcal{P}\Gamma = \Gamma \times_r \Gamma$, there is $W \subseteq \mathcal{P}\Gamma$ which is open and satisfies $W \cap \Gamma^x \times \Gamma^x = V$. Now, let \hat{r} be the map defined on $\mathcal{P}\Gamma$ by $\hat{r}(\gamma_1, \gamma_2) = r(\gamma_1) = r(\gamma_2)$. It is clearly continuous. Thanks to Lemma 2, it follows that \hat{r} is also open. Consequently $\widehat{W} = \hat{r}^{-1}(\hat{r}(W))$ is open. It follows also, from the proof of Lemma 2, that the map $\hat{m} : \mathcal{P}\Gamma \rightarrow \Gamma$ defined by $\hat{m}(\gamma_1, \gamma_2) = \gamma_1^{-1} \circ \gamma_2$ is continuous and open. Consequently $\hat{m}(\widehat{W})$ is open in Γ . It should be remarked, at this point, that φ coincides with the restriction of \hat{m} on $\Gamma^x \times \Gamma^x$. Thus $\varphi(\widehat{W} \cap \Gamma^x \times \Gamma^x)$ is obviously included in $\hat{m}(\widehat{W}) \cap \Gamma_{[x]}$. This inclusion is actually an equality: this is because, if $\gamma \in \hat{m}(\widehat{W})$, there is a pair $(\gamma_1, \gamma_2) \in \widehat{W}$, with $\gamma_1^{-1} \circ \gamma_2 = \gamma$. Hence $u = \hat{r}(\gamma_1, \gamma_2) \in [x]$. By the transitivity property, it follows that there is $\delta : u \rightarrow x$ such that replacing γ_i by $\delta \circ \gamma_i$ $\hat{m}(\delta \circ \gamma_1, \delta \circ \gamma_2) = \gamma$. Since \widehat{W} is \hat{r} -invariant, it follows that $(\delta \circ \gamma_1, \delta \circ \gamma_2) \in \widehat{W} \cap \Gamma^x \times \Gamma^x$ as well. Therefore $\phi(U) = \varphi(\widehat{W} \cap \Gamma^x \times \Gamma^x) = \hat{m}(\widehat{W}) \cap \Gamma_{[x]}$, proving that $\phi(U)$ is open. Since this conclusion holds for any open set $U \subseteq S\Gamma^x$, ϕ is open. Hence, ϕ is a homeomorphism. \square

Proposition 3. *Let Γ be a handy étale groupoid. Then a periodicoid element $x \in \Gamma^{(0)}$ has a finite orbit. Conversely any finite invariant subset is a disjoint union of periodicoid orbits.*

Proof: (i) Let X, Y be topological spaces and let $\phi : X \rightarrow Y$ be a surjective open map. If X is discrete, then $\{x\}$ is open for any $x \in X$. It follows that $\phi(\{x\}) = \{\phi(x)\}$ is also open. Since ϕ is onto, it implies that Y is discrete as well. On the other hand, if X is also compact the family $\mathcal{U} = \{\{x\}; x \in X\}$ is an open cover. Since X is compact one can extract a finite subcover. But \mathcal{U} is also a partition, therefore it must be finite, namely X is finite.

(ii) Let $x \in \Gamma^{(0)}$ be periodicoid, namely the equivalence class $[x]$ is closed. Since Γ is handy, $\Gamma^{(0)}$ is compact, and therefore $[x]$ is compact as well. Thanks to Proposition 2, the topological groupoid $\Gamma_{[x]}$ is isomorphic to $S\Gamma^x$. Since Γ is étale, the r -fiber Γ^x is discrete, so that $\Gamma^x \times \Gamma^x$ is discrete as well. The quotient map $\pi : \Gamma^x \times \Gamma^x \rightarrow S\Gamma^x$ is always open by definition of the quotient topology. It follows that $S\Gamma^x$ is discrete, and thus $\Gamma_{[x]}$ is discrete. Since Γ is handy, the source and the range maps are open. Hence $[x] = r(\Gamma_{[x]}) = s(\Gamma_{[x]})$ is also discrete. Since $[x]$ is compact, it must be finite.

(iii) Let $F \subseteq \Gamma^{(0)}$ be finite and invariant. Then it is closed, since $\Gamma^{(0)}$ is Hausdorff. The equivalence classes $\{[x]; x \in F\}$ make up a finite partition of F . By construction each $x \in F$ is periodicoid. \square

Remark 4. If the groupoid Γ is proper, in addition to being handy, namely if the source and range maps are proper, then every unit is periodicoid (see [26]). \square

3.2. The Hausdorff Topology. In his famous 1914 memoir [50, 51, 52], Hausdorff defined the foundations of topology. He also defined a topology on the set of closed subsets of a complete metric space (X, d) using the so-called *Hausdorff metric*, defined as follows: if A, B are two subset of X , then

$$\delta(A, B) = \sup_{x \in A} \text{dist}(x, B), \quad d_H(A, B) = \max\{\delta(A, B), \delta(B, A)\}.$$

By definition $d_H(A, B) = d_H(B, A)$. In addition, it is straightforward to check that $d_H(A, B) \leq d_H(A, C) + d_H(C, B)$ for any C . In general $d_H(A, B)$ may not be finite. In addition $d_H(A, B) = 0$ if and only if A and B have the same closure. But if both A, B are compact, d_H is finite and defines a metric on the space $\mathcal{K}(X)$ of non empty compact subsets of X . If X is complete, so is $\mathcal{K}(X)$. If X is compact, so is $\mathcal{K}(X)$ (see [71, 72, 7] for instance).

In 1922, Vietoris [100] published a paper in which he defined a topology on the set $\mathcal{C}(X)$ of nonempty closed subsets of X , whenever X is only a topological space and showed that whenever X is metric and complete, his definition coincides, on $\mathcal{K}(X)$ with the topology defined by the Hausdorff metric [24]. In 1950, Chabauty [25] revisited the problem in the framework of locally compact Abelian groups, and in 1962, Fell [42] did the same in the context of C^* -algebras. Both defined a modification of the Vietoris topology leading to $\mathcal{C}(X)$ being compact, but not necessarily Hausdorff. However, both topology coincide if X is compact.

A basis for the Vietoris topology on a topological space X is defined by the family $\mathcal{U}(F, \mathcal{F})$ defined below, where $F \subseteq X$ is closed and \mathcal{F} is a finite family of open sets:

$$\mathcal{U}(F, \mathcal{F}) = \{A \in \mathcal{C}(X); A \cap F = \emptyset \ \& \ \forall O \in \mathcal{F}, A \cap O \neq \emptyset\}.$$

Such a topology is also known under the name of *hit-and-miss* [67], since elements of $\mathcal{U}(F, \mathcal{F})$ must hit each open set in \mathcal{F} and miss F . It is tedious but elementary to check that this family defines indeed a basis for a topology. The Fell topology is defined in a similar way by restricting F to be compact. If X is compact the Fell topology and the Vietoris topology coincide and will be called the *Hausdorff topology*. The following result is a consequence of the results obtained by Vietoris, Michael and Fell [100, 71, 42]

Theorem 7. *Let X be a compact Hausdorff second countable space. Then, equipped with the Hausdorff topology, $\mathcal{C}(X)$ is Hausdorff, compact, and second countable.*

If now Γ is a handy groupoid, the following holds

Proposition 4. *Let Γ be a handy groupoid. Then, if $\mathcal{C}(\Gamma^{(0)})$ is equipped with the Hausdorff topology, the subset $\mathcal{J}(\Gamma)$ is closed. Consequently it is compact Hausdorff and second countable.*

Proof: It is sufficient to prove that $\mathcal{J}(\Gamma)$ is closed, since the rest follows from Theorem 7. Let A be a closed subset of $\Gamma^{(0)}$ belonging to the Vietoris closure of $\mathcal{J}(\Gamma)$. If, by contradiction, A is not an element of $\mathcal{J}(\Gamma)$, then there is a unit $x \in A$, and an arrow $\gamma \in \Gamma$, such that $r(\gamma) = x$, $s(\gamma) = y$, and $y \notin A$. For any open neighborhood O_x of x the preimage $V = r^{-1}(O_x)$ is an open

neighborhood of γ . Similarly, given any open neighborhood O_y of y the preimage $W = s^{-1}(O_y)$ is an open neighborhood of γ . Let then U be an open neighborhood of γ contained in $V \cap W$. Since Γ is handy, $x \in r(U) \subseteq O_x$ is open and $y \in s(U) \subseteq O_y$ is also open. The separability of Γ allows to choose O_y such that $A \cap \overline{O_y} = \emptyset$. Let F, \mathcal{F} be defined by $F = \overline{s(U)}$ and $\mathcal{F} = \{r(U)\}$. It follows that F is closed and does not intersect A . Then $\mathcal{U}(F, \mathcal{F})$ is an open neighborhood of A in the Hausdorff topology. Therefore it intersects $\mathcal{J}(\Gamma)$ by assumption. In particular there is a closed invariant subset B of $\Gamma^{(0)}$ such that $B \cap F = \emptyset$ and $B \cap r(U) \neq \emptyset$. Therefore, if $x' \in B \cap r(U)$, there is $\gamma' \in U$ such that $x' = r(\gamma') \in B$. As B is invariant, it follows that $s(\gamma') \in s(U) \cap B \subseteq F \cap B = \emptyset$, a contradiction. Hence $A \in \mathcal{J}(\Gamma)$. \square

The question addressed in this paper is to control the approximation of an invariant subset by a sequence of periodic orbits. In the general framework we chose to use here, periodic orbits are, in general only periodicoid elements. In the case of étale groupoids, these are finite invariant subsets. So the question is “*Can any invariant subset be approximate by a sequence of periodicoid orbits ?*”. The following example shows that the answer is: NO.

Example 10 (Periodic Orbits are not Hausdorff-dense). Let \mathcal{A} be a finite set, called an *alphabet*. \mathcal{A} will be seen as equipped with its discrete topology. Let $X = \mathcal{A}^{\mathbb{Z}}$ be equipped with the product topology. By Tychonov’s Theorem, X is compact. It is also clearly Hausdorff and second countable. The *shift* defines a \mathbb{Z} -action by

$$(Sx)_n = x_{n-1}, \quad x = (x_n)_{n \in \mathbb{Z}} \in X.$$

A periodic point, of period $p \in \mathbb{N}$, is a sequence x such that $x_{n+p} = x_n$ for all $n \in \mathbb{Z}$. Hence if $w = x_0 x_1 \cdots x_{p-1}$, then $x = w^\infty$ is the concatenation of the word w infinitely many times on both sides. Given $a, b \in \mathcal{A}$ with $a \neq b$, let $y^{ab} = (y_n)_{n \in \mathbb{N}}$ be the sequence defined by $y_n = a$ if $n \geq 0$ and $y_n = b$ if $n < 0$. Then $y^{ab} = b^\infty \cdot a^\infty$ where the \cdot indicates the position of the separation between negative and nonnegative integers. Then,

- (i) The set of periodic point is dense in X , a property of the product topology,
- (ii) The closure Y of the orbit of y^{ab} cannot be approximated, in the Hausdorff topology by a sequence of periodic orbits.

To see this it is sufficient to equip X with a compatible metric defining the topology, and to check that the Hausdorff distance between Y to any periodic orbit is bounded from below by a constant independent of the periodic orbit (see [9] Example 5.2.8). \square

3.3. The Tautological Groupoid: Definition. Let Γ be a handy groupoid. Let then $\mathcal{T}(\Gamma) \subseteq \mathcal{J}(\Gamma) \times \Gamma$ be the set of pairs (M, γ) such that $r(\gamma) \in M$. Since M is a closed invariant subset of $\Gamma^{(0)}$, it follows that $s(\gamma) \in M$ as well. Equivalently $\gamma \in \Gamma_M$. Then $\mathcal{T}(\Gamma)$, equipped with the topology induced by the product topology of $\mathcal{J}(\Gamma) \times \Gamma$ is closed, so that it is a locally compact Hausdorff, second countable space. In addition, it becomes a groupoid if endowed with the following structure

- (T1):** the set of units $\mathcal{T}(\Gamma)^{(0)}$ is the set of pairs $(M, x) \in \mathcal{J}(\Gamma) \times \Gamma^{(0)}$ such that $x \in M$;
- (T2):** the range and the source maps are defined by $r(M, \gamma) = (M, r(\gamma))$ and $s(M, \gamma) = (M, s(\gamma))$;
- (T3):** the set $\mathcal{T}(\Gamma)^{(2)}$ of composable pairs of arrows is the set of pairs $((M, \gamma), (M, \eta))$ such that $s(\gamma) = r(\eta)$;
- (T4):** The composition is then given by $(M, \gamma) \circ (M, \eta) = (M, \gamma \circ \eta)$; consequently, the inverse map is given by $(M, \gamma)^{-1} = (M, \gamma^{-1})$.

Proposition 5. *Let Γ be a handy groupoid. Then, if $\mathcal{J}(\Gamma)$ is equipped with the Hausdorff topology, the set $\mathcal{T}(\Gamma)$, endowed with the product topology of $\mathcal{J}(\Gamma) \times \Gamma$ and with the groupoid structure defined by (T1)-(T4), is a handy groupoid as well, which will be called the tautological groupoid of Γ .*

The term *tautological* reflects the fact that it is a universal object for all continuous fields of groupoids modeled on Γ , as will be seen in the next Section 4, in Theorem 8.

Proof: (i) it is straightforward to check that (T1)-(T4) defines a groupoid structure. In addition, the range map is continuous: any open set in $\mathcal{T}\Gamma^{(0)}$ is containing an open set of the form $\mathcal{U} \times V \cap \mathcal{T}\Gamma^{(0)}$, the preimage of which by r is $\mathcal{U} \times r^{-1}(V) \cap \mathcal{T}(\Gamma)$, which is an open set in $\mathcal{T}(\Gamma)$. Similarly, the source map is continuous. Since Γ is handy, $\Gamma^{(0)}$ is Hausdorff, so is $\mathcal{J}(\Gamma)$. Hence, since Γ is also Hausdorff, it follows that $\mathcal{T}(\Gamma)$ is Hausdorff as well. Hence $\mathcal{T}\Gamma^{(2)}$ is closed in $\mathcal{T}(\Gamma) \times \mathcal{T}(\Gamma)$. By the same argument, the groupoid product is continuous and the inverse map also. Since $\mathcal{T}\Gamma^{(0)} \subseteq \mathcal{J}(\Gamma) \times \Gamma^{(0)}$ is closed, it is compact. At last the range map is open because if \mathcal{U} is an open set in $\mathcal{J}(\Gamma)$ and if $U \subseteq \Gamma$ is open, then $\mathcal{U} \times U \cap \mathcal{T}(\Gamma)$ is open in $\mathcal{T}(\Gamma)$ and such sets generate the topology. Therefore $r(\mathcal{U} \times U \cap \mathcal{T}(\Gamma)) = \mathcal{U} \times r(U) \cap \mathcal{T}\Gamma^{(0)}$ as can be checked easily, which is open. Similarly, the source map s is open. \square

4. CONTINUOUS FIELDS

4.1. Continuous Fields of Groupoids. The definition below was proposed and studied in [65] as a way to construct continuous fields of C^* -algebras, in connection with the concept of deformation quantization. The latter concept has a long history (see [65]), but Rieffel deserves the credit for providing a rigorous satisfactory definition for deformations of C^* -algebras [84, 85, 86, 87, 88].

Definition 12 ([65]). *A field of groupoids is a triple (Γ, T, p) where Γ is a groupoid, T a set and $p : \Gamma \rightarrow T$ is a map such that, if p_0 is the restriction of p to the unit space $\Gamma^{(0)}$, then $p = p_0 \circ r = p_0 \circ s$.*

If, in addition, Γ is a topological groupoid, T a topological space and if p is continuous and open, then (Γ, T, p) is called a continuous field of topological groupoids.

If, moreover, Γ is locally compact (resp. handy) and if T is Hausdorff, and if p is continuous and open, then (Γ, T, p) is called a continuous field of locally compact (resp. handy) groupoids.

As remarked in [65], given $S \subseteq T$, the set $M = p_0^{-1}(S) \subseteq \Gamma^{(0)}$ is invariant. If, in addition, the field (Γ, T, p) is continuous and S is closed, so is M . In particular, if Γ is handy, since p_0 is continuous, the map $\widehat{p}_0 : M \in \mathcal{J}(\Gamma) \mapsto p_0(M)$ takes on values in the compact subsets of T , and is continuous if the set of closed subsets of T is endowed with Vietoris' topology (see [8], Proposition 2). If, in addition, T is compact, then \widehat{p}_0 is Hausdorff continuous. The following Theorem provides a description of all possible field of handy groupoids based on Γ

Theorem 8. *Let Γ be a handy groupoid.*

(i) Let $p_\Gamma : \mathcal{T}(\Gamma) \rightarrow \mathcal{J}(\Gamma)$ be the map defined by $p_\Gamma(M, \gamma) = M$ for $M \in \mathcal{J}(\Gamma)$ and $\gamma \in \Gamma_M$. Then $(\mathcal{T}(\Gamma), \mathcal{J}(\Gamma), p_\Gamma)$ is a continuous field of handy groupoids, called the tautological field of Γ .

(ii) If (Γ, T, p) is continuous field of handy groupoids, with T compact and Hausdorff, then the map $\widehat{p}_0 : \mathcal{J}(\Gamma) \rightarrow \mathcal{C}(T)$ is continuous for the Hausdorff topologies, and $p = \widehat{p}_0 \circ p_\Gamma$. Hence the tautological field of Γ serves as an initial object for any continuous field based on Γ .

Proof: (i) Since, by Proposition 4, $\mathcal{J}(\Gamma)$ is compact and second countable, it is sufficient to prove that p_Γ is both continuous and open. Since the topology on $\mathcal{T}(\Gamma)$ is induced by the product topology of $\mathcal{J}(\Gamma) \times \Gamma$ and since p is the first projection, the claim is obvious by definition.

(ii) the relation $p = \widehat{p}_0 \circ p_\Gamma$ is just another way to write $\widehat{p}_0(M) = p_0(M) = p(M, \gamma)$. \square

Lemma 6. *Let Γ be a handy groupoid equipped with a left-continuous Haar system μ . Then the restriction $\mu_M = x \in M \mapsto \mu^x$ is a well-defined left-continuous Haar system on Γ_M . In addition, if $(M, x) \in \mathcal{T}\Gamma^{(0)}$ and if δ_M denotes the Dirac measure supported by M on $\mathcal{J}(\Gamma)$, then $\mu^{(M,x)} = \delta_M \times \mu^x$ on $\{M\} \times \Gamma^x$, defines a left-continuous Haar system on $\mathcal{T}(\Gamma)$*

Proof: for each $x \in \Gamma^{(0)}$, μ^x is a positive Borel measure on the r -fiber Γ^x satisfying the conditions described in Definition 5. If $M \in \mathcal{J}(\Gamma)$ it follows that for any $x \in M$ and any $\gamma \in \Gamma^x$, then $s(\gamma) \in M$ as well. In particular $\Gamma^x \subseteq \Gamma_M$, namely $\Gamma_M^x = \Gamma^x$. Therefore the restriction $\mu_M = x \in M \mapsto \mu^x$ can be seen as a left-continuous Haar system on Γ_M . The rest is left to the reader. \square

4.2. Continuous Field of C^* -Algebras: a Reminder. The concept of continuous field of C^* -algebras has been investigated at least since Kaplansky in [57]. In particular he gave the first proof that a self-adjoint continuous section of such a field has a spectrum varying continuously with the parameter. This concept was developed and used by Fell [40, 41], Tomiyama [96, 97] and later studied by Dixmier and Douady [34, 35], who gave a complete definition. It is worth noticing that continuous fields are sometimes called “*bundle*” [60]. But this name does not corresponds to the concept of bundle in Differential Geometry, as continuous fields might not be locally trivial.

Let T be a topological space. For each $t \in T$ let \mathcal{A}_t be a C^* -algebra. The family $\mathcal{A} = (\mathcal{A}_t)_{t \in T}$ is called a field. A *section* or a *field of operators* is a family $A = (A_t)_{t \in T}$ for which $A_t \in \mathcal{A}_t$ for all $t \in T$. Given a set Υ of sections, Υ_t will denote the set of $A_t \in \mathcal{A}_t$ such that $A \in \Upsilon$.

Definition 13. *A field \mathcal{A} of C^* -algebras will be called continuous whenever there is a set Υ of sections with the following property*

(CF1): *for each $t \in T$, then Υ_t is a dense $*$ -subalgebra of \mathcal{A}_t ,*

(CF2): *for each $A \in \Upsilon$, the map $t \in T \mapsto \|A_t\|$ is continuous,*

(CF3): *a section B belongs to Υ if and only if for each $t \in T$, for each $\epsilon > 0$, there is an open neighborhood U of t in T and a section $A \in \Upsilon$, such that $\|A_s - B_s\| < \epsilon$ for $s \in U$.*

Then an element of Υ is called a continuous section.

The set Υ , endowed with the pointwise addition, scalar multiplication, product and adjoint, becomes a $*$ -algebra. However, in general, this is not a normed algebra. A continuous section $A \in \Upsilon$ will be called bounded whenever $\|A\| = \sup_{t \in T} \|A_t\|$ is finite. In this case, $\|\cdot\|$ is a C^* -norm and the set of bounded continuous sections becomes a C^* -algebra, which will be denoted by $\mathcal{C}_b(T, \mathcal{A})$. Then the pointwise multiplication by a complex valued bounded continuous function on T makes it a $\mathcal{C}_b(T)$ - C^* -module [22]. If, in addition, T is locally compact, then the set of continuous sections vanishing at infinity is also a C^* -algebra when endowed with that norm, which will be denoted by $\mathcal{C}_0(T, \mathcal{A})$. Again, it can be seen as a $\mathcal{C}_0(T)$ - C^* -module.

It is not difficult in practice to check that a field of C^* -algebra satisfies (CF1). The axiom (CF3) looks hard to check, but, as can be seen in [35], Υ can be replaced by a smaller set generating it, and in most situations, it is not difficult to built such a generating set. The tricky part is the axiom (CF2). Proving the continuity of the norm is always a problem. For this reason,

Rieffel defined the concept of upper and lower semi-continuous field, by replacing the continuity property (CF2) by the corresponding semi-continuity.

Following Rieffel [84], if T is locally compact and if \mathcal{B} is a C^* -algebra, it will be called a $\mathcal{C}_0(T)$ - C^* -module whenever there is an injective $*$ -homomorphism defined on $\mathcal{C}_0(T)$ with values in the center of the multiplier algebra $M(\mathcal{B})$ (see Section 5.1 for a definition of the multiplier algebra). Then \mathcal{B} can be seen as the set of continuous sections, vanishing at infinity, of an upper semi-continuous field of C^* -algebras [84]. The lower semi-continuity requires a specific property, for instance the existence of a suitable representation of \mathcal{B} . In the case of groupoid C^* -algebras, the left regular representation does the job. But then, the difference between the reduced and the full algebra may prevent those fields to be more than only semi-continuous, one way or the other.

It is also worth noticing that the tensor product of two continuous fields may not be continuous. This is due to the non uniqueness of the tensor product of C^* -algebras, more precisely, on the algebraic tensor product of two C^* -algebras there are, in general, several C^* -norms, one being minimal and one maximal. The point of view of $\mathcal{C}(T)$ -modules turns out to be very useful to investigate the tensor product [22, 60].

Building upon such a concept, Landsman and Ramazan [65] showed that continuous field of groupoids lead to semi-continuous fields of C^* -algebras depending upon whether the full or the reduced C^* -algebra are used. For convenience, this theorem will be reformulated for handy groupoids, since the purpose of this paper is to consider applications rather than a general mathematical result. In [65], only the case of a trivial cocycle is considered

Theorem 9 (see [65], Section 5). *Let (Γ, T, p) be a continuous field of handy groupoids. Let μ be a Haar system on Γ . Then, the field $(\mathfrak{C}(\Gamma_t, \mu))_{t \in T}$ is upper semi-continuous and the field $(\mathfrak{C}_{red}(\Gamma_t, \mu))_{t \in T}$ is lower semi-continuous.*

As a result, if the field has the “*Amenability Property*” (see [84], Hypothesis 2.6), namely if each fiber satisfies $\mathfrak{C}_{red}(\Gamma_t, \mu) = \mathfrak{C}(\Gamma_t, \mu)$, then the field obtained this way is continuous. Here some remark is in order: since the fiber Γ_t is a sub-groupoid of Γ , it follows that μ restricts to Γ_t is a left-continuous Haar system as well.

4.3. Continuous Fields of 1-Cocycles. In the previous result, the C^* -algebras are defined without reference to a cocycle, meaning that $\sigma = 1$. Adding a cocycle on a handy groupoid Γ , defines a field of cocycles σ_M on each fiber of the tautological groupoid. However, in view of applications to Physics, it might be convenient to also consider the continuity *w.r.t.* the cocycle itself. Such a problem arises in Physics precisely whenever a magnetic field is added, which might be uniform or not. In the study of the Heisenberg group [85], which corresponds to the Weyl quantization procedure, a similar problem occurs where the magnetic field is replaced by a *semiclassical* parameter, generalizing the concept of *Planck constant*, leading to a continuous field of cocycles. It becomes important, in view of potential applications, to include the possibility of changing the cocycle along the way and to get results about the continuity of the field of C^* -algebras.

In the case for which Γ is actually a discrete group, the space of all possible 1-cocycles can be equipped with a topology making it a compact space ([84], Corollary 2.8). But in the case of a handy groupoid, even an étale one, this is not the case. To see the difficulty, let the case of 0-cocycles, called modules, be considered as in Example 8: a module over the groupoid $X \rtimes \mathbb{Z}$ is entirely defined by a continuous function $h : X \rightarrow \mathbb{S}^1$. To put a topology on the set of such modules, it is sufficient to define a topology on the space $\Omega = \mathcal{C}(X, \mathbb{S}^1)$ of such functions. The

problem is that compactness can be achieved using the pointwise convergence, but then Ω is not closed, since discontinuous function will show up in the closure. If then the uniform convergence is defined, the space Ω is closed, but not compact, not even locally compact. In the definition of Dixmier-Douady [35], the space of parameters needs not be locally compact in general, so that it is not an obstacle. However, the argument provided by Rieffel [84] to prove the upper semi-continuity fails if Ω is not locally compact, so that the definition of Dixmier-Douady cannot be used since the upper semi-continuity of the norm cannot be proved. As a compromise, Rieffel propose to use a *continuous field of 1-cocycles* instead. Adjusting his definition to the groupoid case is provided below. The notations are the following: let $\mathrm{TF}^{(2)} \subseteq \Gamma^{(2)} \times T$ denote the set of (γ, η, t) , such that $(\gamma, \eta) \in \Gamma_t^{(2)}$. Clearly, since the map p is continuous, it follows that $\mathrm{TF}^{(2)}$ is closed in $\Gamma^{(2)} \times T$. Then it will be endowed with the induced topology.

Definition 14. *Let (Γ, T, p) be a continuous field of handy groupoids. Then a continuous field $\sigma = (\sigma_t)_{t \in T}$ of 1-cocycles is a continuous function $\sigma : \mathrm{TF}^{(2)} \rightarrow \mathbb{S}^1$ such that, for each $t \in T$, the map $\sigma_t : (\gamma, \eta) \in \Gamma_t^{(2)} \mapsto \sigma(\gamma, \eta, t) \in \mathbb{S}^1$ is a 1-cocycle.*

5. PROOF OF THEOREM 3

In this Section all the previous concepts and results will be used to prove Theorem 3. The assumptions made in the statement of this theorem imply that the groupoid Γ is handy, thanks to Theorem 4. The strategy for the proof is not new. It follows the ideas first used by J. Renault [81], by G. Elliott [38], by M. Rieffel [84] and by N. P. Landsman and B. Ramazan [65]. However, none of these results include the case for which there is a continuous field of cocycles defining the C^* -algebra. Hence, a new proof is in order, to show that adding such a cocycle does not change the strategy nor the result. The proof is organized in two parts. The first, entitled “*Upper Semi-Continuity*” follows the strategy proposed by Rieffel: if $A = (A_t)_{t \in T}$ is a continuous field of C^* -algebras, then J_t denotes the closed two sided ideal generated by continuous sections vanishing at t . If it can be proved that A_t is $*$ -isomorphic to the quotient algebra A/J_t , then the norm on A_t is a quotient norm, namely the infimum over continuous semi-norms, leading to the upper semi-continuity (see [84], Proposition 1.2). The second part, entitled “*Lower Semi-Continuity*”, uses the left regular representation to show that the norm on the reduced C^* -algebra is the supremum of continuous functions, leading to the lower semi-continuity.

In this Section, Γ is a handy groupoid, equipped with a left-continuous Haar system, μ . Then for $M \in \mathcal{J}(\Gamma)$, μ_M will denote the restriction of μ to the subgroupoid Γ_M (see Lemma 6). For T a locally compact second countable space, let $p_0 : \mathcal{J}(\Gamma) \rightarrow T$ be a continuous open map. It induces a continuous field of groupoid $(\mathcal{J}(\Gamma), T, p)$ if p is the map $p = p_0 \circ r = p_0 \circ s$. Then let σ be a continuous field of 1-cocycles over T with values in \mathbb{S}^1 . Such a field can be seen as a continuous field over $\mathcal{J}(\Gamma)$ instead by setting $\sigma_M \stackrel{\text{def}}{=} \sigma_{p_0(M)}$. Hence there is no loss of generality in assuming that $T = \mathcal{J}(\Gamma)$. For convenience, it will always be assumed that σ is *normalized* (see Corollary 5).

5.1. The Algebra \mathcal{A} and its Ideals. In this Section \mathcal{A} will denote the $*$ -algebra $\mathcal{A} = \mathcal{C}_c(\mathcal{J}(\Gamma), \sigma, \mu)$. It is reminded that its topology is defined as the inductive limit topology, namely a net $(a_\alpha)_{\alpha \in A}$ in \mathcal{A} converges to $a \in \mathcal{A}$ if there is a compact subset $K \subseteq \mathcal{J}(\Gamma)$ supporting each a_α and a and $\sup_{(M, \gamma) \in K} |a(M, \gamma) - a_\alpha(M, \gamma)| \rightarrow 0$. With this topology the product and the adjoint are continuous.

Let \mathcal{C} denote the space $\mathcal{C}(\mathcal{J}(\Gamma))$ of complex valued continuous functions on the space of closed invariant subsets of the groupoid Γ . Since $\mathcal{J}(\Gamma)$ is compact, it follows that \mathcal{C} is a unital Abelian C^* -algebra when endowed with the uniform norm

$$\|h\| = \sup_{M \in \mathcal{J}(\Gamma)} |h(M)|.$$

For $F \in \mathcal{J}(\Gamma)$, let \mathcal{I}_F denotes the set of functions $h \in \mathcal{C}$ such that $h(F) = 0$. This is a maximal (closed, two-sided) ideal of \mathcal{C} and any maximal ideal is of this type.

As a reminder, a multiplier of \mathcal{A} is a pair (L, R) of linear continuous operators on \mathcal{A} such that, for any $a, b \in \mathcal{A}$

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad R(a)b = aL(b) \quad (\text{compatibility}).$$

If $M = (L, R)$, it is common to write $Ma = L(a)$ and $aM = R(a)$. The relation above are expressing the associativity of such a product. If $b \in \mathcal{A}$ then b can be seen as a multiplier through

$$b = (L_b, R_b), \quad L_b(a) = ba, \quad R_b(a) = ab.$$

If (L, R) and (L', R') are two multipliers then $(L + L', R + R')$ is a multiplier, called their sum. If $\lambda \in \mathbb{C}$ is a scalar then $\lambda(L, R) = (\lambda L, \lambda R)$ is also a multiplier. The product is defined by $(L, R)(L', R') = (LL', R'R)$, and it can be checked that this is a multiplier. At last, if $S : \mathcal{A} \rightarrow \mathcal{A}$ is linear and continuous, the operator S^* is defined by $S^*(a) = (S(a^*))^*$. Then the adjoint of a multiplier is defined by $(L, R)^* = (R^*, L^*)$. It can be checked that the adjoint is also a multiplier. Consequently, endowed with these algebraic operations, the set of multipliers is a $*$ algebra. An elementary exercise shows that given any two multipliers (L, R) and (L', R') , then $LR' = R'L$ and, similarly $L'R = RL'$. In particular if $L = R$ then $(L, R)(L', R') = (LL', R'R) = (RL', R'L) = (L'R, LR') = (L'L, RR') = (L', R')(L, R)$.

Lemma 7. For $h \in \mathcal{C}$ and $a \in \mathcal{A}$ let $m_h : \mathcal{A} \rightarrow \mathcal{A}$ be the linear operator defined by

$$(m_h a)(M, \gamma) = h(M) a(M, \gamma), \quad M \in \mathcal{J}(\Gamma), \gamma \in \Gamma_M.$$

Then, m_h defines a multiplier of \mathcal{A} commuting with all multipliers of \mathcal{A} .

Proof: It is immediate to check that the pair (m_h, m_h) of linear operators on \mathcal{A} satisfies these algebraic relations defining a multiplier (L, R) . The relation $L = R$ is equivalent to this multiplier commuting with elements of \mathcal{A} . In addition, for $a \in \mathcal{A}$ and $h \in \mathcal{C}$

$$\text{supp}(m_h a) \subseteq \text{supp}(a), \quad |m_h a(M, \gamma)| \leq \|h\| |a(M, \gamma)|,$$

as can be checked by inspection. It follows that the map $a \in \mathcal{A} \mapsto m_h a \in \mathcal{A}$ is continuous. \square

Definition 15 (See for instance [22]). The multiplier m_h will be written as $m_h(a) = ha = ah$. Then \mathcal{A} becomes a $*$ - \mathcal{C} -modules.

Let $n \in \mathbb{N}$ be an integer. Then $M_n(\mathcal{A})$ denotes the $n \times n$ matrices with elements in \mathcal{A} . The algebraic operations, sum, scalar multiplication, product and adjoint are defined in the usual way. An element $T \in M_n(\mathcal{A})$ will be called *positive* whenever there is $S \in M_n(\mathcal{A})$ such that $T = S^*S$. Then the relation $T \leq T'$ means $T' - T$ is positive. If $h \in \mathcal{C}$ then the multiplier m_h extend to $M_n(\mathcal{A})$ by multiplying h to each matrix element.

Proposition 6. *Given $h \in \mathcal{C}$, then, for any $S \in M_n(\mathcal{A})$,*

$$S^*h^*hS \leq \|h\|^2 S^*S.$$

*In particular, given $\{a_1, \dots, a_n\}$ in \mathcal{A} , the matrix $T_{k,l}^h = a_k^*h^*ha_l$ satisfies*

$$T^h \leq \|h\|^2 T^1, \quad \text{where} \quad 1(M) = 1 \quad \forall M \in \mathcal{J}(\Gamma).$$

Proof: Let g be the function defined by

$$g(M) = \sqrt{\|h\|^2 - |h(M)|^2}, \quad |h(M)|^2 = (h^*h)(M).$$

Then as a multiplier $g^* = g \geq 0$. Moreover, $g^*g + h^*h = \|h\|^2 1$ where $1(M) = 1$ for all $M \in \mathcal{J}(\Gamma)$. Thus

$$\|h\|^2 S^*S = S^*h^*hS + S^*g^*gS,$$

proving the inequality. If $a = \{a_1, \dots, a_n\}$ in \mathcal{A} , let S_a be the matrix defined by

$$S_a = \begin{bmatrix} a_1 & \cdots & a_n \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathcal{A}).$$

Then $T^h = S_a^*h^*hS_a$ and the second inequality follows from the first. \square

Corollary 7. *Let ρ be a representation of \mathcal{A} in the Hilbert space \mathcal{H} . Then for each $h \in \mathcal{C}$ there is a bounded operator $\rho(h)$, commuting with the operators $\rho(a)$ for $a \in \mathcal{A}$ such that ρ becomes a \mathcal{C} -module $*$ -homomorphism.*

Consequently, the \mathcal{C} -action extends by continuity to both the full and the reduced C^ -algebras of $(\mathcal{J}(\Gamma), \mu, \sigma)$.*

Proof: If ρ is a representation, by definition the map $a \rightarrow \rho(a)$, is a weakly continuous $*$ -homomorphism such that the linear span of the vectors of the form $\rho(a)\xi$ with $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$ is dense in \mathcal{H} . Then, if ξ_1, \dots, ξ_n are vectors in \mathcal{H} and if a_1, \dots, a_n are elements of \mathcal{A} it follows from Proposition 6 that

$$\sum_{k,l} \langle \rho(h a_k) \xi_k | \rho(h a_l) \xi_l \rangle \leq \|h\|^2 \sum_{k,l} \langle \rho(a_k) \xi_k | \rho(a_l) \xi_l \rangle.$$

Hence if $\psi = \sum_k \rho(a_k) \xi_k$ and if $T\psi = \sum_k \rho(h a_k) \xi_k$, then $\|T\psi\| \leq \|h\| \|\psi\|$, and the map $\psi \rightarrow T\psi$ is well defined, linear by construction and bounded. This map is denoted by $\rho(h)$ and it is an exercise that this defines ρ as a weakly continuous \mathcal{C} -module $*$ -homomorphism. The second part of the statement is just a consequence of the definition of the reduced and the full C^* -algebras. \square

The following Lemma will be used in the proof of the next result

Lemma 8. *Let X be a compact Hausdorff space. Let $f : X \rightarrow [0, 1]$ be a function with the following properties*

- (a) *f is lower semi-continuous,*
- (b) *there is $x_0 \in X$ such that $f(x_0) = 0$ and f is continuous at $x = x_0$.*

Then there exists a continuous function $h : X \rightarrow [0, 1]$ such that $h(x_0) = 0$ and $f(x) \leq h(x)$ for $x \in X$.

Proof: (i) Since f is continuous at x_0 , for any natural integer n there is an open set U_n containing x_0 such that if $x \in U_n$ then $0 \leq f(x) < 1/n$. Let then V_n be defined by

$$V_n = \bigcup_{m \geq n} U_m.$$

By construction V_n contains x_0 , it is open and $V_{n+1} \subseteq V_n$ for all n 's. Similarly let W_n be the open set defined by

$$W_n = V_n \setminus \overline{V_{n+2}}.$$

Then, for each n , $x_0 \notin W_n$. Let W denote the union of all the W_n 's (which coincides with $V_1 \setminus \{x_0\}$). Its complement $K = X \setminus W$ is a compact set, containing x_0 . By compactness it is possible to extract a finite subset $\{x_0, x_1, \dots, x_l\} \subseteq K$ and for each $0 \leq k \leq l$ an open neighborhood O_k of x_k such that $K \subseteq \bigcup_{k=0}^l O_k$. It is even possible to choose this open cover so that $x_0 \notin O_k$ for $k \geq 1$. Then let W_0 denotes the open set $W_0 = O_0 \cup O_1 \cup \dots \cup O_l \setminus \{x_0\}$. In addition the family $\mathcal{W} = \{W_n; n \in \mathbb{Z}_+\}$ is an open cover of the locally compact normal space $X \setminus \{x_0\}$. By construction, it is locally finite. Using the Urysohn's Lemma, let $(\phi_n)_{n \in \mathbb{Z}_+}$ be a partition of unity subordinate to the open cover \mathcal{W} of $X \setminus \{x_0\}$: namely for each $n \geq 0$, ϕ_n is continuous, $0 \leq \phi_n(x) \leq 1$, the support of ϕ_n is included in W_n , and for any $x \in X \setminus \{x_0\}$, $\sum_{n=0}^{\infty} \phi_n(x) = 1$. It is easy to check, using the definition of the W_n 's, that if $x \neq x_0$ is contained in W_n but not in W_{n-1} , then it is not contained in W_k if $k \neq n, n+1$. Therefore the previous infinite sum has at most two nonzero terms, so it is finite.

(ii) For $N \in \mathbb{N}$ let h_N be the function defined by

$$h_N(x) = \phi_0(x) + \sum_{n=1}^N \frac{1}{n} \phi_n(x).$$

By construction, h_N is continuous, it vanishes at $x = x_0$, and $0 \leq h_N(x) \leq h_{N+1}(x)$. In addition

$$h_N(x) \leq \sum_{n=0}^N \phi_n(x) \leq 1,$$

Consequently the sequence $(h_N)_{N \in \mathbb{N}}$ converges monotonically pointwise to a function h , such that $h(x_0) = 0$ and $0 \leq h(x) \leq 1$. However, for $1 \leq M < N$ two integers

$$0 \leq h_N(x) - h_M(x) = \sum_{n=M+1}^N \frac{1}{n} \phi_n(x) \leq \frac{1}{M+1} \sum_{n=M+1}^N \phi_n(x) \leq \frac{1}{M+1}.$$

Consequently the sequence $(h_N)_{N \in \mathbb{N}}$ is Cauchy for the uniform topology, so that $\sup_{x \in X} |h(x) - h_N(x)| \rightarrow 0$ and therefore h is continuous. By construction, if $x \in W_n$, $0 \leq f(x)\phi_n(x) < \phi_n(x)/n$ for $n \geq 1$, while for $n = 0$, $0 \leq f(x)\phi_0(x) \leq \phi_0(x)$. Consequently, summing over n and using the pointwise convergence, this gives $0 \leq f(x) \leq h(x)$. \square

Let now $F \in \mathcal{J}(\Gamma)$ be a closed invariant subset. Then \mathcal{I}_F will denote the set of functions $h \in \mathcal{C}$ such that $h(F) = 0$. Thus \mathcal{I}_F is a maximal (closed two-sided) ideal of \mathcal{C} . Similarly \mathcal{J}_F will denote the set of $a \in \mathcal{A}$ such that $a(F, \gamma) = 0$ for $\gamma \in \Gamma_F$. The following result holds

Proposition 7. (a) *The set \mathcal{J}_F is a closed two-sided ideal of \mathcal{A} which coincides with the closed two-sided ideal generated by $\mathcal{I}_F \mathcal{A}$.*

(b) *The quotient algebra $\mathcal{A}_F = \mathcal{A}/\mathcal{J}_F$ is $*$ -isomorphic to the algebra $\mathcal{C}_c(\Gamma_F)$.*

Proof: (i) That \mathcal{J}_F is a two-sided ideal can be checked by inspection: namely if $a \in \mathcal{J}_F$ and if $b \in \mathcal{A}$ then $ab(F, \gamma) = 0 = ba(F, \gamma)$ for $\gamma \in \Gamma_F$. That it is closed comes from the fact that if a is the limit of a net a_α of elements of \mathcal{J}_F then, in particular, $a(F, \gamma) = \lim_\alpha a_\alpha(F, \gamma) = 0$ for any $\gamma \in \Gamma_F$.

(ii) Given $a \in \mathcal{J}_F$ there are a_0, a_1, a_2, a_3 such that $a = a_0 - a_2 + \iota(a_1 - a_3)$ and $a_l(M, \gamma) \geq 0$ for all l 's and (M, γ) . This is because a can be decomposed into real and imaginary part. In addition, each continuous real valued function f on a topological space X can be written as $f = f_+ - f_-$, with $f_\pm(x) = \max\{\pm f(x), 0\}$ for $x \in X$. Hence f_\pm are continuous with support contained in the support of f . In particular each a_l is continuous and supported in the support of a . In addition, each a_l vanishes at F so that $a_l \in \mathcal{J}_F$. Thus, there is no loss of generality in assuming that $a(M, \gamma) \geq 0$ for all $(M, \gamma) \in \mathcal{T}(\Gamma)$. Similarly, changing a into λa if necessary, where $\lambda \in \mathbb{R}_+$ there is no loss of generality in assuming that $0 \leq a(M, \gamma) \leq 1$. Let $f : \mathcal{J}\Gamma \rightarrow [0, 1]$ be defined by $f(M) = \sup_{\gamma \in \Gamma_M} a(M, \gamma)$. Then f is lower semi-continuous, $f(F) = 0$ and $0 \leq f(M) \leq 1$.

(iii) Let K denote the support of a . By assumption, K is compact in $\mathcal{J}(\Gamma)$. It follows that f is continuous at $M = F$. Indeed, thanks to the Bolzano-Weierstrass theorem, given $M \in \mathcal{J}(\Gamma)$, there is $\gamma_M \in \Gamma_M$ such that $a(M, \gamma)$ reaches its supremum at (M, γ_M) . Hence $f(M) = a(M, \gamma_M)$. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence converging to F in $\mathcal{J}(\Gamma)$. Thanks to the compactness of K , by extracting a sub-sequence if necessary, the corresponding γ_{M_n} converge to a point $\eta \in \Gamma_F$. Since a is continuous it follows that $\lim_{n \rightarrow \infty} f(M_n) = \lim_{n \rightarrow \infty} a(M_n, \gamma_{M_n}) = a(F, \eta) = 0$. Since Γ is second countable, it follows, from the properties of the Hausdorff topology (see Proposition 4), that $\mathcal{J}(\Gamma)$ is second countable as well, so that this convergence is sufficient to prove the continuity of f at $M = F$.

(iv) Thanks to Lemma 8, let then $h \in \mathcal{I}_F$ be such that $h(M) \geq f(M)$. Then given $0 < \epsilon < 1$, let a_ϵ be defined by

$$a_\epsilon(M, \gamma) = h(M)^\epsilon a(M, \gamma)^{1-\epsilon}.$$

Hence $a_\epsilon \in \mathcal{I}_F$. It follows that if R denotes the ratio $h(M)/a(M, \gamma)$ then $R \geq 1$ (it can be infinite) and

$$0 \leq (a_\epsilon - a)(M, \gamma) = (R^\epsilon - 1)a(M, \gamma) = \int_0^\epsilon \ln(R) R^s a(M, \gamma) ds \leq \epsilon R^{\epsilon-1} \ln(R) h(M).$$

where $a(M, \gamma) = R^{-1}h(M)$ is used. The support of a_ϵ is contained in the support of a . Moreover, $R^{\epsilon-1} \ln(R) \leq (e(1-\epsilon))^{-1}$. Therefore $\lim_{\epsilon \rightarrow 0} a_\epsilon = a$ in the topology of \mathcal{A} . Hence $a \in \mathcal{I}_F \mathcal{A}$.

(v) For $a \in \mathcal{A}$ let a_F denote the function $a_F(\gamma) = a(F, \gamma)$ for $\gamma \in \Gamma_F$. The map $p_F : a \rightarrow a_F$ is a continuous $*$ -homomorphism with kernel \mathcal{J}_F by construction. Hence it defines an injective $*$ -homomorphism from \mathcal{A}_F into $\mathcal{C}_c(\Gamma_F)$. Conversely, if $b \in \mathcal{C}_c(\Gamma_F)$, with support in the compact subset $K \subseteq \Gamma_F$ it can be seen as a function defined on the compact subset $\{F\} \times K \subseteq \mathcal{T}(\Gamma)$. Since Γ is handy, the Tietze Extension Theorem applies, so that there is a function $\hat{b} \in \mathcal{A}$ such that $\hat{b}_F = b$. Hence p_F is onto. \square

5.2. The Full C^* -algebra. The definition of the full C^* -algebra was provided in Section 2.5. As a preparation, the following result is elementary but is needed.

Lemma 9. *Let $0 \rightarrow J \xrightarrow{i} A \xrightarrow{p} B \rightarrow 0$ be a sequence of C^* -algebras with i, p being $*$ -homomorphisms such that i is injective, p is surjective and $p \circ i = 0$. If, for any $*$ -representation*

π of A vanishing on $\text{Im}(i)$, there is a $*$ -representation π_B of B such that $\pi_B \circ p = \pi$, then the previous sequence is exact at A , namely $\text{Ker}(p) = \text{Im}(i)$.

Proof: From the assumptions it follows easily that $\text{Im}(i) \subseteq \text{Ker}(p)$. If $i(J) \neq \text{Ker}(p)$, there is $a \in \text{Ker}(p)$ such that $a \notin \text{Im}(i)$. Consequently there is a $*$ -representation π of A vanishing on $\text{Im}(i)$ such that $\pi(a) \neq 0$. By the assumption, there is a representation π_B of B such that $\pi(a) = \pi_B \circ p(a) = 0$, leading to a contradiction. \square

Like in Section 5.1 let $\mathcal{A} = \mathcal{C}_c(\mathcal{J}\Gamma, \sigma, \mu)$. Given F a closed invariant subset of $\Gamma^{(0)}$, let \mathcal{J}_F denotes the ideal of functions $a \in \mathcal{A}$ such that $a(F, \gamma) = 0$ for all $\gamma \in \Gamma_F$. In addition, \mathcal{A}_F denotes the $*$ -algebra $\mathcal{C}_c(\Gamma_F, \sigma_F, \mu_F)$ where σ_F represents the restriction of the cocycle σ on Γ_F and, similarly, μ_F denotes the restriction of the Haar system μ to Γ_F . Then the following result holds

Lemma 10. *Let $i_F : \mathcal{J}_F \rightarrow \mathcal{A}$ be the injection map and let $p_F : \mathcal{A} \rightarrow \mathcal{A}_F$ be the evaluation map at F . Then:*

(a) *The sequence of topological $*$ -algebras $0 \rightarrow \mathcal{J}_F \xrightarrow{i_F} \mathcal{A} \xrightarrow{p_F} \mathcal{A}_F \rightarrow 0$ is exact.*

(b) *The maps i_F and p_F extends as $*$ -homomorphisms to the completions of the previous algebras for both the reduced and the full norms. In both case i_F is injective and $p_F \circ i_F = 0$.*

Proof: (i) The map i_F is defined as $i_F(f)(M, \gamma) = f(M, \gamma)$, for $f \in \mathcal{J}_F$, which makes sense since $\mathcal{J}_F \subseteq \mathcal{A}$. The map p_F is defined by $p_F(a)(\gamma) = a(F, \gamma)$ for any $\gamma \in \Gamma_F$. The result of Proposition 7 is equivalent to $\text{Ker}(p_F) = \text{Im}(i_F)$.

(ii) The map p_F is onto which will prove the exactness at \mathcal{A}_F . Namely if $b \in \mathcal{A}_F$, it defines a continuous function with compact support on Γ_F , which can be seen as a continuous function with compact support on $\{F\} \times \Gamma_F \subseteq \mathcal{J}\Gamma$. Thanks to the Tietze extension theorem, there is $a \in \mathcal{A}$ coinciding with b on $\{F\} \times \Gamma_F$. Then, by definition $p_F(a) = b$. Together with (i) this proves (a).

(iii) Given a bounded, $*$ -representation π of \mathcal{A} , the map $f \in \mathcal{J}_F \mapsto \pi \circ i_F(f)$ gives a bounded, $*$ -representation of \mathcal{J}_F . In particular, if $M \in \mathcal{J}\Gamma$ and if $x \in M$, let $\pi_{M,x}$ denote the left regular representation of $\mathcal{J}\Gamma$ on the fiber space $L^2(\mathcal{J}\Gamma^{M,x}, \mu^{M,x})$. Then the reduce norm $\|a\|_{red}$ of $a \in \mathcal{A}$ is obtained by taking the supremum over (M, x) of $\|\pi_{M,x}(a)\|$. It then follows that $\|i_F(f)\|_{red} = \sup_{M,x} \|\pi_{M,x}(i(f))\| \leq \|f\|_{red}$. Similarly, taking the supremum over all bounded, $*$ -representations π of \mathcal{A} , this argument gives $\|i_F(f)\| = \sup_{\pi} \|\pi(i_F(f))\| \leq \|f\|$. Hence i_F extends as a $*$ -homomorphism to the completions of \mathcal{J}_F and \mathcal{A} w.r.t. both the reduced and the full norm. By construction $i_F(f) = f$ for $f \in \mathcal{J}_F$, therefore the same identity holds in both completions. Hence i_F is injective and therefore it is an isometry in both completions.

(iv) A similar argument holds for p_F . If ρ is a bounded, $*$ -representation of \mathcal{A}_F , then $\rho \circ p_F$ is also a bounded, $*$ -representation of \mathcal{A} as can be checked by inspection, using the invariance of the closed set F . As a result the same argument shows that p_F extends to both the reduced and the full completions of \mathcal{A} and \mathcal{A}_F as a $*$ -homomorphism. The relation $p_F \circ i_F = 0$ holds on \mathcal{J}_F so it must hold on both completions as well by continuity. \square

The following Lemma is using an argument due to Renault (see [81], p. 102) used in a different context.

Lemma 11. *Let A denotes the full C^* -algebra $A = \mathfrak{C}^*(\mathcal{J}(\Gamma), \sigma, \mu)$ and let J_F be the closure of \mathcal{J}_F in A . Similarly let A_F be the full algebra $\mathfrak{C}(\Gamma_F, \sigma_F, \mu_F)$. If π a $*$ -representation of A vanishing on J_F , there is a $*$ -representation ρ of A_F such that $\rho \circ p_F = \pi$.*

Proof: The restriction of π on \mathcal{A} is a bounded $*$ -representation of \mathcal{A} . Since π vanishes on J_F , it also vanishes on \mathcal{J}_F . Let then ρ be defined, for $b \in \mathcal{A}_F$, by $\rho(b) = \pi(a)$ for any $a \in \mathcal{A}$ coinciding with b on Γ_F , namely for any $a \in \mathcal{A}$ such that $a(F, \gamma) = b(\gamma)$ for $\gamma \in \Gamma_F$. This definition makes sense since changing a into a' implies $a - a' \in \mathcal{J}_F$ namely $\pi(a) = \pi(a')$. This construction gives $\rho \circ p_F = \pi$ when restricted to \mathcal{A} . By completion, ρ extends to a $*$ -representation of the full algebra A_F . The relation $\rho \circ p_F = \pi$, valid on \mathcal{A} , extends to A since both π and p_F are $*$ -homomorphisms. \square

Corollary 8. *The sequence of C^* -algebras $0 \rightarrow J_F \xrightarrow{i_F} A \xrightarrow{p_F} A_F \rightarrow 0$ is exact. In particular, A_F is $*$ -isomorphic to the quotient algebra A/J_F .*

Proof: Thanks to Lemma 9, it follows from Lemma 11 that $\text{Ker}(p_F) = \text{Im}(i_F)$ in this sequence. Hence it is exact at A . It remains to prove that $p_F : A \rightarrow A_F$ is surjective. Since p_F is a $*$ -homomorphism, the image $p_F(A)$ is closed ([56], Theorem 4.1.9). Since \mathcal{A} is dense in A and since $p_F(\mathcal{A}) = \mathcal{A}_F$ (thanks to Lemma 10) is dense in A_F , it follows that $\mathcal{A}_F = p_F(\mathcal{A}) \subseteq p_F(A) \subseteq A_F$, so that $A_F = \overline{\mathcal{A}_F} = \overline{p_F(\mathcal{A})} \subseteq p_F(A) \subseteq A_F$. Hence $p_F(A) = A_F$ and p_F is surjective. \square

The following result is using an argument that can be found in Rieffel ([84], Proposition 1.2) and the author credits Varela for it.

Proposition 8. *For any $a \in A$, the map $M \in \mathcal{J}\Gamma \mapsto \|p_M(a)\| \in \mathbb{R}_+$ is upper semi-continuous.*

Proof: Given $a \in A$, let $F \in \mathcal{J}\Gamma$ and let $\epsilon > 0$. Using the exactness of the sequence in Corollary 8, it follows that p_F defines a $*$ -isomorphism from the quotient algebra A/J_F into A_F . By definition of the quotient norm, $\|p_F(a)\| = \inf\{\|a - b\|; b \in J_F\}$. Now, J_F is the closure of \mathcal{J}_F . Moreover, thanks to Proposition 7, $\mathcal{J}_F = \mathcal{I}_F \mathcal{A}$. Hence there is a finite sum $b = \sum_{n=1}^N f_n b_n$ with $f_n \in \mathcal{I}_F$ and $b_n \in \mathcal{A}$, such that

$$\|p_F(a)\| \geq \|a - b\| - \epsilon.$$

Let $g \in \mathcal{C}(\mathcal{J}\Gamma)$ be chosen such that $0 \leq g(M) \leq 1$ and such that $g(M) = 1$ in a neighborhood U of F . In particular $\|ga\| \leq \|a\|$. Choosing U and the support of g small enough, it is even possible to assume that $\|gb\| < \epsilon$, since $g(F)f_n(F) = 0$ and since f_n is continuous for $1 \leq n \leq N$. Consequently,

$$\begin{aligned} \|p_F(a)\| &\geq \|a - b\| - \epsilon \geq \|g(a - b)\| - \epsilon > \|ga\| - 2\epsilon \\ &= \|a - (1 - g)a\| - 2\epsilon \end{aligned}$$

Since $1 - g$ vanishes at $M \in U$, the element $(1 - g)a \in J_M$ for $M \in U$, so that $\|a - (1 - g)a\| \geq \|p_M(a)\|$, leading to

$$\|p_F(a)\| \geq \|p_M(a)\| - 2\epsilon, \quad \forall M \in U.$$

\square

5.3. The Reduced C^* -algebra. Using the previous notations, let $a \in \mathcal{A}$. Then $p_F(a) \in \mathcal{A}_F$ is the function $\gamma \in \Gamma_F \mapsto a(F, \gamma)$.

Proposition 9. *Given $a \in \mathfrak{C}_{red}^*(\mathcal{J}\Gamma, \sigma, \mu)$ the map $F \in \mathcal{J}\Gamma \mapsto \|p_F(a)\|_{red}$ is lower semi-continuous.*

Proof: (i) The left-regular representation of \mathcal{A}_F is acting on the Hilbert space $\mathcal{H}_{F,x} = L^2(\Gamma^x, \mu^x)$ whenever $x \in F$. The corresponding operator is given by the following formula (see Section 2.6, eq. (10)), where $\psi \in \mathcal{H}_{F,x}$

$$\pi_{F,x}(a)\psi(\gamma) = \int_{\Gamma^x} a(F, \gamma^{-1} \circ \eta) \sigma_F(\gamma, \gamma^{-1} \circ \eta) \psi(\eta) d\mu^x(\eta).$$

In particular, if $\phi \in \mathcal{H}_{F,x}$ then the inner product is given by

$$\langle \phi | \pi_{F,x}(a)\psi \rangle_{\mathcal{H}_{F,x}} = \int_{\Gamma^x \times \Gamma^x} \overline{\phi(\gamma)} a(F, \gamma^{-1} \circ \eta) \sigma_F(\gamma, \gamma^{-1} \circ \eta) \psi(\eta) d\mu^x(\gamma) d\mu^x(\eta).$$

(ii) At this point, it is worth reminding that $\mathcal{C}_c(\Gamma^x) \subseteq \mathcal{H}_{F,x}$ and the former is dense in the later. Since $\mathcal{J}\Gamma$ is normal, using the Tietze Extension Theorem, any continuous function with compact support defined on Γ^x , seen as the subset $\{F\} \times \Gamma^x \subseteq \mathcal{J}\Gamma$, can be extended as a continuous function with compact support in $\mathcal{J}\Gamma$. For such a function, say f , its restriction $f_{F,x}$ to $\{F\} \times \Gamma^x$ is an element of $\mathcal{H}_{F,x}$, while f itself belongs to \mathcal{A} . Therefore if $f, g \in \mathcal{A}$ the following formula holds

$$\langle f_{F,x} | \pi_{F,x}(a)g_{F,x} \rangle_{\mathcal{H}_{F,x}} = \int_{\Gamma^x \times \Gamma^x} \overline{f(F, \gamma)} a(F, \gamma^{-1} \circ \eta) \sigma_F(\gamma, \gamma^{-1} \circ \eta) g(F, \eta) d\mu^x(\gamma) d\mu^x(\eta).$$

From the definition of a Haar system, in particular the axiom (H2), it follows immediately that the map $(F, x) \in \mathcal{J}\Gamma^{(0)} \mapsto \langle f_{F,x} | \pi_{F,x}(a)g_{F,x} \rangle_{\mathcal{H}_{F,x}}$ is continuous whenever $f, g, a \in \mathcal{A}$. Similarly, the map $(F, x) \in \mathcal{J}\Gamma^{(0)} \mapsto \|f_{F,x}\|_{\mathcal{H}_{F,x}} \in \mathbb{R}_+$ is also continuous. Using the density result it follows that

$$\|\pi_{F,x}(a)\| = \sup_{f, g \in \mathcal{A}} \left| \frac{\langle f_{F,x} | \pi_{F,x}(a)g_{F,x} \rangle_{\mathcal{H}_{F,x}}}{\|f_{F,x}\|_{\mathcal{H}_{F,x}} \|g_{F,x}\|_{\mathcal{H}_{F,x}}} \right|.$$

In addition, the reduced C^* -norm of $p_F(a)$ is given by

$$\|p_F(a)\|_{red} = \sup_{x \in F} \|\pi_{F,x}(a)\|.$$

Consequently the map $F \in \mathcal{J}\Gamma \mapsto \|p_F(a)\|_{red}$ is lower semi continuous, as it is the supremum over a family of continuous functions.

(iii) If now $a \in \mathfrak{C}_{red}^*(\mathcal{J}\Gamma, \sigma, \mu)$, it can be approximated by a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of \mathcal{A} in the reduced norm. Hence, given $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for $n \geq N$, $\|a - a_n\|_{red} < \epsilon/3$. Since p_F extends as a $*$ -homomorphism from \mathcal{A} to the reduced C^* -algebra, it follows immediately that $\|p_F(a) - p_F(a_n)\|_{red} < \epsilon/3$ for $n \geq N$. Now choosing $n \geq N$, by lower semi-continuity, there is a neighborhood U of F in $\mathcal{J}\Gamma$ such that for $M \in U$

$$\|p_F(a_n)\|_{red} \leq \|p_M(a_n)\|_{red} + \epsilon/3, \quad \Rightarrow \quad \|p_F(a)\|_{red} \leq \|p_M(a)\|_{red} + \epsilon.$$

This finishes the proof. \square

Proof of Theorem 3: The continuity of the norm $t \mapsto \|\phi(f)\|$ of a normal element $f \in \mathfrak{C}_{red}^*(\mathcal{J}\Gamma, \sigma, \mu)$ for any continuous $\phi : \mathbb{C} \rightarrow \mathbb{C}$ follows directly from Propositions 8 and 9 whenever the reduced and full C^* -algebras agree. A sufficient condition for this is that the groupoid is amenable [2, 3]. However, there is a subtlety depending upon whether or not the reduced C^* -algebra is unital or not unital.

(i) If the reduced C^* -algebra is unital, and if f is self-adjoint, the continuity of the spectrum follows by Theorem 1 proven in [8]. For normal elements, see [35] and a more precise statement in [9, Thm. 2.7.9], implying the continuity of the spectra.

(ii) If the reduced C^* -algebra is not unital, the usual way consists in adding a unit (in the Abelian case it consists in replacing the locally compact spectrum by its Alexandrov compactification adding one point at infinity). By convention then, the spectrum $\sigma(f_F)$ contains always 0. Then the continuity of the spectrum follows by [9, Thm. 2.7.9] but only functions with $\phi(0) = 0$ can be used [35, Prop. 10.3.3]. \square

Remark 5. If Γ is a handy étale groupoid, the reduced C^* -algebra is unital, where the characteristic function $\chi_{\Gamma(0)}$ of the unit set is the unit. Note that $\chi_{\Gamma(0)} \in \mathcal{C}_c(\Gamma)$ since the unit set is compact and open [82]. \square

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