INVARIANT TORI FOR AN INFINITE LATTICE OF COUPLED CLASSICAL ROTATORS

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Abstract. We exhibit a class of models in classical hamiltonian mechanics, namely a lattice of classical rotators with short range interactions, for which, provided the interactions are small enough, invariant tori exist in the phase space, describing motions localized in a small region of the lattice. This result constitutes an extension of the Kolmogorov-Arnold-Moser theorem to a class of systems with infinitely many degrees of freedom

Résumé : Nous produisons une classe de modèles en mécanique hamiltonienne classique, formés de rotateurs classiques sur un réseau, couplés par des interactions à courte portée, et pour lesquels pourvu que l'interaction soit assez petite, il existe dans l'espace des phases, des tores invariants décrivant un mouvement localisé dans une petite région du réseau. Ce résultat constitue une extension du théorème de Kolmogorov-Arnold-Moser à une classe de systèmes ayant un nombre infini de degrés de liberté.

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0) INTRODUCTION :

The stability of systems with infinitely many degrees of freedom in classical mechanics is a problem which goes back to the foundation of statistical mechanics a century ago [33]. The ergodic hypothesis by Boltzmann was of central interest since the thermal equilibrium was believed to be a consequence of it. A century of research has not been sufficient to exhaust the subject. Much remains ununderstood yet.

Even for systems with a finite number of degrees of freedom, it was soon realized, that the problem is not simple. The first result of Poincaré [40,41,54] that most of the periodic orbits of an integrable hamiltonian, which are dense in the phase space, are unstable under generic perturbations, together whith the Poincaré recurrence theorem (see for instance [6,16]), were strong arguments favoring the ergodic hypothesis. However the pioneering numerical results of Fermi, Pasta Ulam [17], and simultaneously the mathematical scheme of A.N. Kolmogorov [27], in 1954, showed that in fact, nearly integrable systems with a finite number of degrees of freedom exhibit a non ergodic behavior. The scheme of Kolmogorov, proved to apply in a more general framework by V.I. Arnold [2,3,4] and J. Moser [34], leads to the existence of "a lot" of orbits, quasi-periodic in time, and stable under small perturbations. Each such orbit is dense in an invariant torus. By "a lot", we mean that the set of such tori has a positive Liouville measure in the phase space. Expositions of the proof of the KAM theorem can be found in [7,9,13,20,29,35,36,48].

The extension of the Kolmogorov-Arnold-Moser theorem to a system with an infinite number of degrees of freedom is meaningless in itself, and requires additional informations motivated by the physics of the problem. For instance one important problem is to decide what probability law will be used to measure the phase space. Whereas all the usual choices are equivalent in the finite case, this is far from beeing true in the infinite one. The physics will change drastically under a change of such probability law.

The class of systems we have investigated is a lattice of identical classical rotators, coupled together by short range interactions. Several examples of physical systems are described in a similar way. A classical treatment of a crystal where each atom is described by a point particle, and is coupled to its neighbors via anharmonic forces, would lead to such a model. We can also give a description of a vibrating string or of the oscillations of an electromagnetic field in a cavity, coupled to themselves in a non linear way, leading to the same kind of model; in this latter example, the lattice sites are replaced by the index of the eigenmodes of the harmonic approximation, and the non linear part actually couples the neighboring modes together.

Two physical points of view may be adopted in this problem. The first

one is motivated by the thermodynamics: the classical motion is described through an initial condition which will be chosen randomly according to a Gibbs measure. The main questions to be answered concern (i) the existence of the solutions of the equations of motion and (ii) the invariance and the ergodicity or the non ergodicity of the Gibbs measure. This program was partially fulfilled in other models such as the so called ideal gas [55] or points particles interacting via short range forces [28].

This is not the point of view we shall adopt here. In classical mechanics we are mainly interested by configurations with finite total energy, a point of view which is far more natural for systems like a vibrating string. This is in contrast with the thermodynamics which considers configurations with finite energy per site or per degree of freedom. A finite energy configuration is necessarily localized in the lattice. The question we have investigated here is to know whether it stays localized in the same region for ever under the classical evolution. More precisely the models we consider exhibit many invariant torii even after switching on the coupling between sites. Let us notice that such invariant tori have zero probability with respect to the Gibbs measure. In particular our conclusion is not in contradiction with the common belief that the Gibbs measure is ergodic (which may be false either).

This result may appear surprising since local perturbations should propagate in the crystal. This is what one observes in a nearly harmonic crystal. The reason is that neighboring sites are in strong resonance. This fact produces an instability of the motion of each oscillator and they are likely to exchange their energy. On the other hand, in the general case, almost every configurations with respect to the Gibbs measure exhibit a resonance, and this is probably the mechanism which produces the thermal equilibrium. In contrast, in our case, the crystal is not nearly harmonic, and a wide set of initial localized perturbations satisfies a non resonant condition. The main effect is that for small couplings, the motion is trapped in a local region.

Several mechanisms must be controlled to get such a conclusion. First of all it is known that the higher the number of degrees of freedom the smaller the critical value of the coupling constant below which it is possible to prove the KAM theorem (see below). To get a stability result in the previous case we need to produce a mechanism which prevents most of the degrees of freedom to participate to the motion: somewhere the interaction must act in order to create an effective cut-off in the lattice, decoupling the degrees of freedom lying within a localized region from the outside.

The second mechanism to be controlled concerns the occurrence of resonances between different regions in the lattice: if the typical frequencies of oscillation of two or more sites are commensurate, an instability will appear whatever their distance is. Since these frequencies depend upon the initial conditions, the resonances decompose the phase space into a region of stability and a region of instability. When invariant tori occur, they belong to the stability region, but for more than two degrees of freedom, they do not disconnect the instability region. In particular a typical unstable orbit may wander in the phase space through this subset. This is the so-called Arnold instability [5,14]. As a matter of fact, this phenomena is extremely slow to develop, as the Nekhoroshev theorem shows [37,38], and is usually of no importance in classical mechanics, even in physical phenomena with rapid oscillations like plasma physics. However in our lattice of rotators, for almost all initial configurations with respect to the Gibbs measure, there is a family of sites somewhere in the lattice, producing a resonance. For this reason, the Arnold instability should be an essential fact in the evolution to thermal equilibrium [8]. It is to avoid this phenomena in our situation, that we start with localized configurations. And it is one of the keypoints of our work to recognize that <u>non resonant localized configurations</u> constitute a fairly "large" set. By "large" we mean a set of positive measure with respect to some gaussian measure on the set of localized configurations, which is <u>locally equivalent</u> to the Liouville measure. This remark, together with the existence of an effective spatial cut-off are the main ingredients which allow the conclusion to hold. We emphasize that these restrictions agree actually with physical observations. For instance it is known in quantum chemistry, that large molecules behave in time as if there were made of independent pieces, the radicals, weakly coupled together [43].

The precise model we study in this work, is a D-dimensional infinite lattice of classical rotators described by the following hamiltonian in the action-angle variables:

$$(1) \qquad H(\underline{A}, \underline{\theta}) = \sum_{x\in\mathbf{Z}^D} \{ A_x^2/2 + \varepsilon_0 . E_x(\underline{A}_{x+\Delta}, \underline{\theta}_{x+\Delta}) \}$$

where $\underline{A}=\{A_x; x\in\mathbf{Z}^D\}$ represents the set of action variables, whereas $\underline{\theta}=\{\theta_x; x\in\mathbf{Z}^D\}$ represents the set of angles. In this expression, Δ is a fixed finite subset of the lattice contained in the hypercube of size R centered at the origin: it represents the range of the interactions. On the other hand for Λ a subset of the lattice, we have set $\underline{A}_\Lambda = \{A_x; x\in\Lambda\}$ and $\underline{\theta}_\Lambda = \{\theta_x; x\in\Lambda\}$. The functions E_x are uniformly bounded and holomorphic in the domain $\{D(r_0)\times\mathbf{T}_{\rho_0}\}^\Delta$ where:

$$(2) \qquad D(r_0) = \{A\in\mathbf{C}; |A| < r_0\} \qquad \mathbf{T}_{\rho_0} = \{\theta\in\mathbf{T}+i\mathbf{R}; |\mathrm{Im}(\theta)| < \rho_0\}$$

In addition they satisfy <u>uniformly in</u> $x\in\mathbf{Z}^D$:

$$(3) \qquad E_x(\underline{A}_\Delta, \underline{\theta}_\Delta) = \mathcal{O}(\max_{x\in\Delta} |A_x|^\lambda) \quad \text{as } \underline{A}\to 0 \text{ (with } \lambda\geq 20).$$

At last, ε_0 measure the size of the interactions.

In order to measure the size of the configurations in the phase space we introduce the probability μ_σ on $(\mathbf{R}\times\mathbf{T})^{\mathbf{Z}^D}$ defined as follows:

$$(4) \qquad \mu_\sigma(d\underline{A}, d\underline{\theta}) = \otimes_{x\in\mathbf{Z}^D}\ dA_x . d\theta_x . (8\pi^3\sigma_x)^{-1/2} . e^{-(A_x - A_{0,x})^2/2\sigma_x}$$

If the sequence $\underline{\sigma}=\{\sigma_x; x\in\mathbf{Z}^D\}$ decreases sufficiently rapidly at infinity, and if $\underline{A}_0=\{A_{0,x}; x\in\mathbf{Z}^D\}$ satisfies $|A_{0,x}|\leq Z(|x|_1)$ (where Z is a positive decreasing function), then the configurations localized around $\underline{A}_0$, in the sense that, for small enough r:

$$(5) \qquad |A_x - A_{0,x}| < r.Z(|x|_1) \quad \text{with } Z(|x|_1)^2/\sigma_x \geq \text{const. } \mathrm{Ln}(|x|_1)^{1+0},$$

has a positive μ_σ-measure. We remark that, when conditionned on a set of finite degrees of freedom, this measure is equivalent to the Liouville one. This is what we called a <u>locally Liouville measure</u>.

The main result of this paper is summarized in the following theorem:

<u>Theorem</u>: Let us consider the model (1), with the restrictions (2) & (3). Let $\underline{A}_0$ be a configuration such that $|A_{0,x}| \leq Z(|x|_1)$ for all $x\in\mathbf{Z}^D$. Given $\nu\geq 2$ and $\gamma>0$ small enough, there is $\lambda(\nu)\geq(39+5\sqrt{57})/4\approx 19.18$ satisfying:

$$(6a) \qquad \lambda(\nu)\geq 6\nu + 20/3 + O(1/\nu) \qquad \text{as } \nu\to\infty$$

and a closed set Ω localized around $\underline{A}_0$ in the sense of (5) with a μ_σ-measure greater than $1-O(\gamma)$, and for each $\lambda>\lambda(\nu)$ an $\varepsilon_c>0$ such that if $\varepsilon_0<\varepsilon_c$, and $\underline{A}\in\Omega$, the orbit starting at $(\underline{A}, \underline{\theta})$ is almost periodic in time and is dense in an invariant infinite dimensional torus of class $C^{\nu-1}$, provided:

$$(6b) \qquad Z(L)\leq \text{const. } e^{-\text{const. } L^D}$$

In this case, the critical coupling ε_c is bounded by:

$$(6c) \qquad \varepsilon_c \geq \text{const. } \gamma^a . r_0^b . e^{-F(\rho_0)} \quad \text{for some } a,b>0$$

as γ, r_0, or ρ_0 tend to zero, where F is a decreasing function.

This is an extension of the Kolmogorov-Arnold -Moser (KAM) theorem.

In the previous model several changes are allowed without affecting the conclusions of this theorem. First of all the lattice $\mathbf{Z}^D$ can be replaced by any countable set in which it is possible to measure the distance of two points. In particular, we can treat systems for which the lattice is replaced by eigenmodes. On the other hand, in the unperturbed hamiltonian, A_x^2 could be replaced by any function of $\{A_{x'} \in D(r_0); |x'-x| < R_0\}$ (for some $R_0 > 0$), with an invertible second derivative on the domain of definition. This is a serious restriction only for nearly harmonic systems like a vibrating string. It is a well known fact indeed, that perturbating a system of harmonic oscillators requires a special treatment in order to apply the KAM algorithm [20,49].

At last, the importance of the condition (3) must be emphasized. For if $\underline{A}$ is localized in the sense of (5), it means that the effective coupling between faraway sites is extremely weak. This is precisely the condition needed to produce an effective spatial cut-off.

Before ending this introduction, let us point out some works in the litterature which are of interest in the course of the proof.

The KAM algorithm was explained firstly by A.N. Kolmogorov in the International Congress of Mathematics in Amsterdam in 1954 [27], after a paper appeared in russian in 1954 [26]. However, it was not until 1962 that a complete proof of its validity was produced by V.I. Arnold [2,3,4], in the case of analytic perturbations of an integrable hamiltonian with an arbitrary number of degrees of freedom, and by J. Moser [34], in the case of C^k (for $k \geq 333$!) perturbation of an integrable homeomorphism of the annulus (systems with two degrees of freedom). Among the main steps of the proof, one was borowed from an earlier work by N.N. Bogolioubov & N.M. Krylov [10] concerning the use of the first order perturbation theory to produce a canonical transformation changing the original perturbation into a perturbation of smaller order. One other step was borrowed from the famous result of C.L. Siegel [52,53,54] concerning the analysis of the small divisor problem in complex dynamical system. Later on, several major improvement in the quantitative estimates were produced. Let us mention especially the work of H. Rüssmann [44,45,46,47,50] who analyzed in great detail the diophantine condition, a keypoint in getting better estimates. In particular, he succeeded in reducing the differentiability condition in Moser's work to the value $k \geq 5$ [44], and later on, Moser [36] remarked that the Rüssmann proof leaded to $k > 3$ (see also [50]). M. Herman [25], produced a counter example with $k=3-\varepsilon$ (ε arbitrary) and proved that invariant circles having rotation numbers of constant type are C^{k-1} if $k=n+\beta$, n being an integer greater or equal to 3 and $0 < \beta < 1$. For nearly integrable hamiltonians with $\tilde{N}$ degrees of freedom, J. Pöschel [42] founds $k > 3\tilde{N}-1$.

Another type of improvement, more closely related to our work, concerns the dependence upon the critical coupling $\varepsilon(\tilde{N})$ below which the KAM theorem can be proved, as a function of the number $\tilde{N}$ of degrees of freedom. This point did not retain much attention until very recently. In the original V.I. Arnold's work [3,4] one can found the estimate :

$$(7) \qquad \varepsilon(\tilde{N}) \geq \text{const.}\, e^{-a.\tilde{N}^2.\mathrm{Ln}(\tilde{N})} \qquad \text{for some } a>0.$$

More recently, this estimate has been improved to [9,20,21,22,48]:

$$(8) \qquad \varepsilon(\tilde{N}) \geq \text{const.}\, e^{-a.\tilde{N}.\mathrm{Ln}(\tilde{N})}.$$

Then for a chain of oscillators, coupled via short range interactions, without the condition (3) above, E. Wayne [58] proved that it is possible to improve this estimate up to :

$$(9) \qquad \varepsilon(\tilde{N}) \geq \text{const.}\, \tilde{N}^{-a} \qquad \text{for some big } a>1 \text{ [59]}.$$

In a recent work, considering a general model, without the restriction of short range interactions imposed by E. Wayne, one of us (M.V.), using a different kind of diophantine condition, based upon an analysis similar to that of H. Rüssmann, together with a different probability measure on the phase space, equivalent to the Liouville measure, succeeded in improving the result of G. Gallavotti to the form [56]:

$$(10) \qquad \varepsilon(\tilde{N}) \geq \text{const.}\, e^{-\mathrm{Ln}^a(\tilde{N})} \qquad \text{for any } a > 2.$$

Systems with infinitely many degrees of freedom were also considered recently. First of all, there are several results [15,57,61] proving the C.L. Siegel theorem in an infinite dimensional complex space. Moreover, in [56] some nearly integrable hamiltonians with infinitely many degrees of freedom were shown to have invariant tori, the measure of which being positive with respect to a "locally Liouville measure" μ_σ of the type (4). An example of such an hamiltonian is :

$$(11) \qquad H = \Sigma_x A_x^2/2 + \Sigma_{x,y} E_{x,y}(A).\cos(\theta_x - \theta_y)$$

where

$$(12) \qquad \sup_A |E_{x,y}(A)| \leq \alpha.\, e^{-B.(\mathrm{Ln} z)^a}$$

with $z = \max(x,y)$, $a > 2$ (arbitrary), $B>0$, and α is depending upon "a", vanishing when "a" approaches 2. It describes a perturbation localized around the origin.

The next step was to consider an infinite homogeneous chain. The main mechanism to be understood was the existence of an effective cut-off which is achieved thanks to the condition (3).

When finishing to write this paper, we learned from E. Wayne and J. Fröhlich [19], that they proved a very similar result on a model made of a lattice of harmonic oscillators, having a random distribution of eigenfrequencies, coupled via short-range interactions. Eventhough we don't know the details of their proof yet, there is little doubt that ours is very different from it.

We also want to mention that in the recent past, several important numerical studies have been performed in connection with the question of ergodicity. In particular, Froeschlé [18], studied a dynamical system with $\tilde{N}$ degrees of freedom with a varying range R of interaction (and not satisfying (3)!). He showed that the size of the principal island of stability depends only upon R, at least for $\tilde{N} \geq 40$, and shrink to the empty set whenever $R \geq 7$. For a different system with a random range, Gardner and Ashby [23], computed the probability for the greatest Lyapounov exponent to be positive, and proved that it is zero for $R/\tilde{N} \leq .11$, and one for $R/\tilde{N} \geq .15$ independently on $\tilde{N}$. The ratio $R/\tilde{N}$ is called the connectance of the system. Another numerical investigation concerned the test of the ergodic hypothesis. Let us mention the work of Patrascioiu et al. [39] and the important studies of the italian school [24,30,31,32] who exhibited a threshold in the energy per degree of freedom, below which the equipartition of the energy fails to hold. This threshold is independent of $\tilde{N}$ and exists for any R. It seems also independent of the model investigated [32]. Let us however point out that mathematical problems occur with the correct definition of the Lyapounov exponents when the system is infinite dimensional, as remarked by Casati et al. [12], a remark which may force the theoreticians to be careful with the interpretations of the numerical results.

This paper is organized as follows. Section I is devoted to the exposition of the strategy of the proof. We explain here the usual way of proving the KAM theorem together with the necessary changes we have introduced here to achieve the result in our situation. In section II we introduce the technical tools needed for the recurrent estimates, and we perform them without looking at the small divisor problem. Section III concerns the analysis of the small divisor problem. We investigate in particular the Rüssmann approximation function [49] in great detail, together with the properties of the set of non resonant localized configurations. Section IV is devoted to the set of constraints on the different parameters we have introduced before, which allow to achieve the convergence of the recursion.

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I) THE STRATEGY :

Let $H_0(\underline{A})$ be a completely integrable hamiltonian, written in action-angle variables, having only L degrees of freedom. The motion of the corresponding system is then given by :

$$(1) \quad A_x(t) = A_x(0) \qquad \theta_x(t) = \theta_x(0) + \omega_x(\underline{A})\, t \qquad \omega_x(\underline{A}) = \partial H_0/\partial A_x$$

Let $H(\underline{A},\underline{\theta}) = H_0(\underline{A}) + \varepsilon . E(\underline{A},\underline{\theta})$ be a small perturbation of this Hamiltonian. To compute the motion of this new system, we follow Jacobi [20] in trying to find a canonical transformation changing $(\underline{A},\underline{\theta})$ into $(\underline{A}',\underline{\theta}')$, such that in the new coordinates, H depends only upon $\underline{A}'$. From the Poincaré theorem [41,54] on the instability of periodic orbits, we know that this is impossible globally. Therefore we only use first order perturbation theory. It consists in finding a function $G(\underline{A},\underline{\theta})$ such that :

$$(2) \qquad \{G,H\} + E = \langle E \rangle \qquad \langle E\rangle(\underline{A}) = \int \textstyle\prod_x d\theta_x/2\pi \,.\, E(\underline{A},\underline{\theta})$$

where {,} denotes the Poisson bracket. Up to the second order in ε, the hamiltonian reads in the new variables $H_1 = H + \varepsilon\{G,H\} + O(\varepsilon^2)$, and it is therefore completely integrable up to term of orders ε^2.

Following N.N. Bogolioubov and N.M. Krylov [10], we perform the full change of variables given through the infinitesimal canonical transformation G. Namely the new hamiltonian is formally given by :

$$(3) \quad H_1 = \exp(\mathcal{L}_G) H = \Sigma_{n\geq 0}\{G,\{G,\dots\{G,H\}\dots\}\}/n! \quad \text{with} \quad \mathcal{L}_G(H) = \{G,H\}$$

From what has been said, it can be written as :

$$(4) \qquad H_1(\underline{A},\underline{\theta}) = H_0(\underline{A}) + \langle E\rangle(\underline{A}) + E_1(\underline{A},\underline{\theta}) \qquad \text{where} \qquad E_1 = O(\varepsilon^2)$$

Even in this form, the new system is a very good approximation for practical purposes as was remarked in [10].

The next step, a scheme proposed by A.N. Kolmogorov [27], is simply to replace H by H_1, the completely integrable part being now $H_{0,1} = H_0 + \langle E\rangle$, the new interacting term being E_1. Performing this change several times leads to a sequence of hamiltonians $H_k(\underline{A},\underline{\theta}) = H_{0,k}(\underline{A}) + E_k(\underline{A},\underline{\theta})$ with $E_k = O(\varepsilon^{2^k})$, together with a sequence $G_k = O(\varepsilon_{k-1})$ (where we set $\varepsilon^{2^k} = \varepsilon_k$) of infinitesimal

transformations, such that :

$$\exp(\mathcal{L}_{G_k}) H_{k-1} = H_k$$

This should lead to an extremely rapidly convergent algorithm.

However, in the course of this set of transformations there is an essential difficulty connected with solving the equation (2). To get the solution, one usually expands G and E into a Fourier expansion with respect to $\underline{\theta}$. This gives formally :

$$(5) \qquad G(\underline{A},\underline{\theta}) = -i\Sigma_{\underline{n}\in Z^L}\ E_{\underline{n}}(\underline{A}).e^{i\underline{n}.\underline{\theta}}\ .(\omega(\underline{A}).\underline{n})^{-1}$$

$$\text{if} \qquad E(\underline{A},\underline{\theta}) = \Sigma_{\underline{n}\in Z^L}\ E_{\underline{n}}(\underline{A}).e^{i\underline{n}.\underline{\theta}}$$

where $\omega(\underline{A})$ is given by (1). As a matter of fact, the denominator $\omega(\underline{A}).\underline{n}$ which is equal to $\Sigma_x\, \omega(\underline{A})_x.n_x$ can be fairly small for certain values of $\underline{n}$. Indeed either there is some $\underline{n}$ for which it is equal to zero exactly, and this is called a resonance, which prevents (2) to have solutions, or we can apply the Dirichlet box principle [51], according to which there is a sequence $(\underline{n}_j)$ of multi-integers such that $|\omega(\underline{A}).\underline{n}_j| \leq |\underline{n}_j|^{-L}$ where |.| denotes some norm in $\mathbf{R}^L$. As it is well known, the converse type of inequality is not true for any vector. However, given $\sigma > L$, the set Ω of ω's in $\mathbf{R}^L$ for which there is some $\gamma > 0$, such that $|\omega.\underline{n}| \geq \gamma.|\underline{n}|^{-\sigma}$ for any $\underline{n}$ in $\mathbf{Z}^L$, has a full Lebesgue measure [51]. As proposed by V.I. Arnold [2] and also by J. Moser [36] in a different way, this polynomial divergency introduced in (5) can be controlled through an exponential decrease of the Fourier coefficients of E, a property which is equivalent to the analyticity of E with respect to the angle variables in some strip around the real. If we assume :

$$(6) \qquad |E_{\underline{n}}(\underline{A})| \leq \text{const}.e^{-\rho.|\underline{n}|} \qquad \text{for all } \underline{n}\in \mathbf{Z}^L$$

the same type of estimate will follow for G with ρ replaced by $\rho_1 = \rho - \delta_1 < \rho$. In addition, performing this remark at each step of the recursion, we will get for the size ε_k of E_k an estimate of the type :

$$(7) \qquad \varepsilon_{k+1} \leq \varepsilon_k^2/\delta_{k+1} \qquad \text{with} \qquad \Sigma_{k\geq 0}\, \delta_k < \rho$$

Choosing δ_k bigger than ε_k^{2-c} for some $1 < c < 2$, we will get $\varepsilon_k \leq \varepsilon^{c^k}$ and there is enough room for δ_k to satisfy (7), allowing the sequence to converge.

There is, however a new difficulty arising at this point. Namely we need to control the diophantine estimate :

(8) $|\omega(\underline{A}).\underline{n}| \geq \gamma . |\underline{n}|^{-\sigma}$ for all $\underline{n} \in \mathbf{Z}^L$

at each step of the recursion for the free part $H_{0,k}$, and consequently ω_k changes each time we perform a canonical transformation. On the other hand it is not difficult to check that the set of $\underline{A}$'s for which (8) is valid, is nowhere dense. Controlling a function on such a set is not impossible, as was done by E. Borel [11] in his mémoire on monogenic functions. However it is difficult, and any attempt in this direction failed to be convincing up to now. For this reason, V.I. Arnold (see also G. Gallavotti [21,22]), proposed to assume the analyticy of H with respect to $\underline{A}$ in some union of polydiscs $\mathrm{Max}_x |A_x - A_x{}^i| < r$, centered at points satisfying (8). Then provided r is not too large, the diophantine condition (8) is still satisfied uniformly on this domain, provided we assume $|\underline{n}| < N$ for some ultraviolet cut-off N. This can be justified by eliminating from E, in the equation (5), all the Fourier coefficients $E_{\underline{n}}$ where $|\underline{n}|$ is bigger than N. N will be chosen in order that the eliminated terms are small compare to ε^c. This will not affect the conclusion that E_1 is of order ε^c. In order to proceed at the next step, where ω is modified into $\omega_1 = \omega + O(\varepsilon)$ we will diminish a little bit the radius of the polydiscs by changing their center from $\underline{A}^i$ to $\underline{A}_1{}^i$ in such a way that :

(9) $\omega(\underline{A}^i) = \omega_1(\underline{A}_1{}^i)$

a transformation controlled by the implicit function theorem. Similarly, at each step of the recursion we will get a new value r_k of the radius of analyticity of both the integrable part $H_{0,k}$ and the interaction E_k. In particular, the limiting hamiltonian $H_\infty = H_{0,0} + \Sigma_{k \geq 0}(H_{0,k+1} - H_{0,k})$ is the sum of an infinite number of functions, analytic in polydiscs with varying radi . Following G. Gallavotti [21,22], the ν^{th} derivatives of H_∞ is also a convergent sum of similar terms provided $\Sigma_{k \geq 0} \varepsilon_k / r_k{}^\nu$ converges. If this is the case, the invariant tori described through this canonical change of variable, will be of class $C^{\nu-1}$.

If we now come to the model described in the introduction, we need to modify the previous strategy in two respects.

First of all we need to choose an effective spatial cut-off. Thanks to the equation (3) of the introduction, this will be done by neglecting at each

steps those lattice sites for which :

(10) $|x| \geq L$ with L such that $Z(L)^\lambda \leq \varepsilon^c$.

We are now reduced to a finite dimensional hamiltonian system.

The second important change concerns the diophantine condition (8). We could modify it at each step in order to take into account the change in the number of degrees of freedom. We have preferred another strategy. Namely we define a set Ω in $\mathbf{R}^{\mathbf{Z}^D}$ such that for any ω in Ω, the following estimate holds:

(11) $|\omega.\underline{n}| \geq \gamma . \psi(|\underline{n}|_w)$ $|\underline{n}|_w = \Sigma_{x \in \mathbf{Z}^D} |n_x| . w(|x|)$

where the functions ψ and w must be chosen in such a way that Ω has a positive measure with respect to μ_o. This will be studied in detail in section III.

Let us remark that now the change in the analyticity radius at each step, is sensitive to the spatial cut-off L through the decreasing of Z(L). This is the reason we will get restrictions on the rate of decreasing of Z at infinity as indicated in the main theorem.

II) ONE STEP OF RECURSION :

This section will be divided into three parts. In the first one we introduce a set of norms needed for performing the estimates. In the second, we construct the infinitesimal canonical transformation which allows us to go from one step to the next one in the recursion. The last one is devoted to the construction and the estimate of the different parts of the new hamiltonian.

II-1) NORMS :

Let L be a positive integer and Λ be the square in $\mathbf{Z}^D$ centered at zero with radius L, namely :

(1) $$\Lambda = \{ x \in \mathbf{Z}^D ; |x|_1 = \Sigma_{1 \le \mu \le D} |x_\mu| \le L \}$$

Let also ρ be a positive number, and U be an open domain in $\mathbf{C}^\Lambda$. As in the introduction, we set $\mathbf{T}_\rho = \{ \theta \in \mathbf{T} + i\mathbf{R} ; |\mathrm{Im}(\theta)| < \rho \}$. On the set of holomorphic bounded functions on U x $(\mathbf{T}_\rho)^\Lambda$, we define the norm :

(2) $$\| f \|_{U,\rho,L} = \sup_{\underline{A} \in U} \Sigma_{\underline{n} \in \mathbf{Z}^\Lambda} | f_{\underline{n}}(\underline{A}) | . e^{\rho . |\underline{n}|_1}$$

where $f_{\underline{n}}(\underline{A})$ represents the $\underline{n}^{th}$ coefficient of Fourier of the function $\underline{\theta} \to f(\underline{A},\underline{\theta})$. We will denote by $\mathcal{H}(U,\rho,L)$ the Banach space obtained from this norm. The main properties of this space are summarized in the next three propositions.

Proposition 1 : Endowed with the norm given in (2), $\mathcal{H}(U,\rho,L)$ is a Banach algebra for the pointwise multiplication. In particular, if f and g belong to $\mathcal{H}(U,\rho,L)$, one gets :

(3) $$\| f.g \|_{U,\rho,L} \le \| f \|_{U,\rho,L} . \| g \|_{U,\rho,L}$$

◇

Proof : the $\underline{n}^{th}$ Fourier coefficient of f.g is given by :

$$(fg)_{\underline{n}}(\underline{A}) = \Sigma_{\underline{m} \in \mathbf{Z}^\Lambda} f_{\underline{m}}(\underline{A}) . g_{\underline{n}-\underline{m}}(\underline{A})$$

Therefore, if one replaces it into (2) one gets :

$$\| f.g \|_{U,\rho,L} \le \sup_{\underline{A} \in U} \Sigma_{\underline{n},\underline{m} \in \mathbf{Z}^\Lambda} | f_{\underline{m}}(\underline{A}) | . | g_{\underline{n}-\underline{m}}(\underline{A}) | . e^{\rho . |\underline{n}|_1}$$

since $|\underline{n}|_1 \le |\underline{m}|_1 + |\underline{n}-\underline{m}|_1$ one gets, after an obvious change of variables :

$$\| f.g \|_{U,\rho,L} \le \sup_{\underline{A} \in U} \{ \Sigma_{\underline{m} \in \mathbf{Z}^\Lambda} | f_{\underline{m}}(\underline{A}) | . e^{\rho . |\underline{m}|_1} \} . \{ \Sigma_{\underline{n} \in \mathbf{Z}^\Lambda} | g_{\underline{n}}(\underline{A}) | . e^{\rho . |\underline{n}|_1} \}$$

$$\le \| f \|_{U,\rho,L} . \| g \|_{U,\rho,L}$$

◇

The next important property concerns the Poisson bracket. If f and g belongs to the space $\mathcal{H}(U,\rho,L)$, $\{f,g\}$ is the function given by the expression :

(4) $$\{f,g\} = \Sigma_{x \in \Lambda} \{ \partial f / \partial A_x . \partial g / \partial \theta_x - \partial f / \partial \theta_x . \partial g / \partial A_x \}$$

In the following proposition, we denote by $D(\underline{A}^{(o)}, r_i) = U_i$ the polydisc $\mathrm{Max}_{x \in \mathbf{Z}^\Lambda} |A_x - A_x^{(o)}| < r_i . Z_x$, and $Z(L) = \min_{x \in \Lambda} Z_x$. Moreover $\mathcal{H}(r_i,\rho_i,L_i)$ will denote the corresponding space of holomorphic functions. Then we obtain :

Proposition 2 : If $f_1 \in \mathcal{H}(r_1,\rho_1,L_1)$ and $f_2 \in \mathcal{H}(r_2,\rho_2,L_2)$, then $\{f_1,f_2\}$ belongs to $\mathcal{H}(r_3,\rho_3,L_3)$ where :

(5a) $L_3 = \max(L_1,L_2)$, $\rho_3 < \min(\rho_1,\rho_2)$, $r_3 < \min(r_1,r_2)$

and in addition if $\| . \|_i$ denotes the norm in $\mathcal{H}(r_i,\rho_i,L_i)$ (i=1,2,3) we get (where e = 2.718...) :

(5b) $$\| \{f_1,f_2\} \|_3 \le$$
$$\le (e.Z(L_3))^{-1} . \{ (\rho_1-\rho_3)^{-1} (r_2-r_3)^{-1} + (\rho_2-\rho_3)^{-1} (r_1-r_3)^{-1} \} \| f_1 \|_1 . \| f_2 \|_2$$

◇

Proof : It is clear that $\{f_1,f_2\}$ depends upon the variables A_x, θ_x with $x \in \Lambda = \Lambda_1 \cup \Lambda_2$. Therefore $L_3 = \max(L_1,L_2)$. Thanks to (4), the $\underline{n}^{th}$ Fourier coefficient of a term in the Poisson bracket (4) is given by :

$$(\partial f_1/\partial A_x.\partial f_2/\partial\theta_x)_n(\Delta) = \Sigma_{m\in Z^\Lambda}\ i.m_x\ \partial f_{1,n-m}/\partial A_x(\Delta)\ .f_{2,m}(\Delta)$$

If Δ belongs to U_3, one may use the Cauchy formula to estimate the derivative in the right hand side, namely :

$$\partial f_{1,n-m}/\partial A_x(\Delta) = \int_0^{2\pi} d\alpha/2\pi\ .\ e^{i\alpha}.f_{1,n-m}(\Delta+e_x.\delta r.e^{i\alpha})/\ \delta r$$

where e_x denotes the element of $\mathbb{R}^\Lambda$ given by $(e_x)_y = \delta_{x,y}$, and δr is chosen in such a way that $\Delta+e_x.\delta r.e^{i\alpha}$ belongs to U_1. This is true in particular if $\delta r = (r_1-r_3).Z_x$.

From this last remark one obtains :

$$\Sigma_{n,m\in Z^\Lambda}\ e^{\rho_3.|n+m|_1}\ .|\partial f_{1,n}/\partial A_x(\Delta)|\ .|m_x|\ .|f_{2,m}(\Delta)| \leq$$

$$((r_1-r_3)Z_x)^{-1}.\{\Sigma_{m\in Z^\Lambda}\ |m_x|\ .|f_{2,m}(\Delta)|\ .e^{\rho_3.|m|_1}\}.\{\sup_{B\in U_1}\Sigma_{n\in Z^\Lambda}\ |f_{1,n}(B)|\ .e^{\rho_3.|n|_1}\}$$

One can bound Z_x from below by $Z(L_3)$ and use the fact that $\rho_3<\rho_1$. Then summing over x in Λ and using the estimate :

(6) $$\sup_{m\in Z^\Lambda}\ |m|_1.e^{-(\rho_1-\rho_3).|m|_1} \leq (e.(\rho_1-\rho_3))^{-1}$$

we get immediatly the result of the proposition.

◇

The third important result concerns the exponentiation of the Liouville operator associated to some function f and defined formally as (see §I, eq.(3)):

(7) $$\mathcal{L}_f(g) = \{f,g\} \qquad \exp(\mathcal{L}_f)g = \Sigma_{k\geq 0}\ \underbrace{\{f,\{f,\ldots,\{f,g\}\ldots\}\}}_{\text{k-times f}}/k!$$

Proposition 3 : If $f \in \mathcal{H}(r_1,\rho_1,L_1)$ and $g \in \mathcal{H}(r_2,\rho_2,L_2)$, then $\exp(\mathcal{L}_f)g$ belongs to $\mathcal{H}(r,\rho,L)$, where :

(8a) $L_3 = \max(L_1,L_2)$, $\rho<\min(\rho_1,\rho_2)=\rho_0$, $r<\min(r_1,r_2)=r_0$

provided

(8b) $$\|f\|_1 < Z(L).(\rho_0-\rho).(r_0-r)/2.e$$

where $\|.\|_i$ denotes the norm in $\mathcal{H}(U_i,\rho_i,L_i)$ (i=1,2,3) . In addition we get :

(8c) $$\|\mathcal{L}_f^{\,k}(g)/k!\|_{r,\rho,L} \leq \{2e.\|f\|_1/(Z(L).(\rho_0-\rho).(r_0-r))\}^k\ .\|g\|_2$$

(8d) $$\|\exp(\mathcal{L}_f)g\|_{r,\rho,L} \leq [1-2e\|f\|_1/(Z(L).(\rho_0-\rho).(r_0-r))]^{-1}.\|g\|_2$$

◇

Proof : Clearly (8d) is a consequence of (8c), which we now prove. Let us set $\rho_j=\rho_0(1-j/k)+\rho.j/k$ and in much the same way, $r_j=r_0(1-j/k)+r.j/k$. Considering $\mathcal{L}_f^{\,j}(g)$ as an element of $\mathcal{H}(r_j,\rho_j,L_3)$ for each $0\leq j\leq k$ we get from the proposition 2 :

$$\|\mathcal{L}_f^{\,j}(g)\|_j \leq \|\mathcal{L}_f^{\,j-1}(g)\|_{j-1}.\|f\|_1.(Z(L).(\rho_0-\rho).(r_0-r))^{-1}2k^2/j$$

iterating this set of inequalities we get :

$$\|\mathcal{L}_f^{\,k}(g)\| \leq \|g\|_2.[2\|f\|_1.(Z(L).(\rho_0-\rho).(r_0-r))^{-1}]^k\ \ k^{2k}/k!$$

It remains to use the estimate :

(9) $$k^k/k! \leq e^k$$

in order to get the result. ◇

II-2) THE INFINITESIMAL CANONICAL TRANSFORMATION :

In dealing with a step of recursion we need to compute a canonical transformation in the form $\exp(\mathcal{L}_G)$ where G is a solution of the first order equation (2) of section I. In this section, we will give some estimates on G.

Let $E(\underline{A},\underline{\theta})$ be an element of $\mathcal{H}(D(r),\rho,L)$ where D(r) is a union of polydiscs of radi r :

(10) $$D(r) = \bigcup_i D(A^{(i)},r)\ .$$

Moreover , we assume that the n^{th} Fourier coefficient E_n of E , vanishes unless $|\underline{n}|_1 < N$. On the other hand let $H_0(\underline{A})$ be the completely integrable part of the hamiltonian at the step we are considering. We will assume that it satisfies the following non-resonance condition :

$$(11) \qquad |\textstyle\sum_{x\in\Lambda} \omega_x(\underline{A}).n_x| \geq \gamma.\psi(|\underline{n}|_w)/2 \qquad \forall \underline{A}\in D(r^{(1)}),\ \forall |\underline{n}|_1 < N$$

where $0 < r^{(1)} < r$ and :

$$(12) \qquad \omega_x = \partial H_0/\partial A_x \qquad |\underline{n}|_w = \textstyle\sum_{x\in\Lambda} |n_x|.w(|x|_1)$$

and w is some positive increasing function.

Then G is defined as :

$$(13) \qquad G(\underline{A},\underline{\theta}) = -i.\textstyle\sum_{\underline{n}\in Z^{\Lambda},|\underline{n}|_1<N} E_{\underline{n}}(\underline{A}).(\omega(\underline{A}).\underline{n})^{-1}.e^{i\underline{n}.\underline{\theta}}$$

Thus we clearly get :

$$(14) \qquad \|G\|_{r^{(1)},\rho,L} \leq \|E^{<N,L}\|_{r,\rho,L}.2.\{\gamma.\psi(N.w(L))\}^{-1}$$

II-3) THE NEW HAMILTONIAN :

From now on, the sequence $Z=\{Z_x ; x\in \mathbb{Z}^D\}$ will be chosen as follows : there is a continuous monotone decreasing function again denoted by Z on $\mathbb{R}_+$ such that:

(15) (i) $Z_x = Z(|x|_1)$ for all x and $Z(0) = 1$

(ii) $Z(s+2R) > K_0.Z(s)^d$ for all $s \geq 0$ and some $1 < d < 2$ and $K_0 > 1$

(iii) $Z(s) \leq K_1.\exp\{-K_2.s\}$ for some $K_1 > 0$ and $K_2 > 0$.

Let us suppose that, at the initial stage of a recursion step, the hamiltonian $H(\underline{A},\underline{\theta})$ has the form :

(16) $H(\underline{A},\underline{\theta}) =$

$$\textstyle\sum_{x\in\mathbb{Z}^D} A_x^2/2 + h(\underline{A}) + E(\underline{A},\underline{\theta}) + \sum_{L-R\leq|x|_1<L+R} E_x^{(1)}(\underline{A},\underline{\theta}) + \sum_{L+R\leq|x|_1} E_x(\underline{A}_{x+\Delta},\underline{\theta}_{x+\Delta})$$

with the following conditions (here e_0 is some positive constant) :

(i) h and E $\in \mathcal{H}(D(r),\rho,L)$ $\quad \|E\|_{r,\rho,L} \leq \varepsilon < e^{-1}$ (e=2.718...)

(ii) $E_x^{(1)} \in \mathcal{H}(D(r),\rho,|x|_1+R)$ and $\|E_x^{(1)}\|_{r,\rho,L+2R} \leq 2e_0.Z(|x|_1)^\lambda$.

(iii) Δ is the square $|x|_1 \leq R$, $E_x(\underline{A}_{x+\Delta},\underline{\theta}_{x+\Delta}) \in \mathcal{H}(D(r),\rho,|x|_1+R)$ and: $\|E_x\|_{r,\rho,|x|_1+R} \leq e_0.Z(|x|_1)^\lambda$ for all x.

We define a spatial cut-off L^* through the condition, where $d \leq c < 2$

$$(17) \qquad Z(L^* - R) = Z(L-R)^c$$

From the hypothesis on Z it follows that $L^* > L+2R$. Then we set :

(18) $E^{(1)}(\underline{A},\underline{\theta}) =$

$$E(\underline{A},\underline{\theta}) + \textstyle\sum_{L-R\leq|x|_1<L+R} E_x^{(1)}(\underline{A},\underline{\theta}) + \sum_{L+R\leq|x|_1<L^*-R} E_x(\underline{A}_{x+\Delta},\underline{\theta}_{x+\Delta})$$

Lemma : If we assume $L > e = 2.718...$, and $Z(L-R)^\lambda \leq \varepsilon$, then :

$$(19) \qquad \varepsilon^{(1)} = \|E^{(1)}\|_{r,\rho,L^*} \leq C_0.\varepsilon.\mathrm{Ln}^D(1/\varepsilon)$$

where C_0 is some positive constant ◇

Proof : From (16) & (18) one gets immediatly :

$$\varepsilon^{(1)} = \|E^{(1)}\|_{r,\rho,L^*} \leq \varepsilon + 2.e_0.Z(L-R)^\lambda.K(D).(L^*-R)^D$$

From (15 (iii)), one deduces :

$$(L^*-R)^D \leq \mathrm{Ln}^D(K_1/Z(L^*-R))/K_2 = \mathrm{Ln}^D(K_1/Z(L-R)^c)/K_2$$

The hypothesis of the lemma imply

$$Z(L-R)^\lambda.\mathrm{Ln}^D(K_1/Z(L^*-R))/K_2 \leq \mathrm{const}.\varepsilon.\mathrm{Ln}^D(1/\varepsilon) \quad \text{for } \varepsilon < e^{-1}$$

and therefore we get the result. ◇

Let us choose $0 < \delta < \rho$, and define N^* as the number such that:

$$(20a) \qquad \varepsilon^{(1)}.e^{-\delta.N^*} = \varepsilon^c/2 .$$

We decompose $E^{(1)}$ as : $E^{(1)} = \delta h + E^{(1)<N} + E^{(1)\geq N}$, with :

$$(20b)\quad \delta h(\Delta) = \int \prod_{x\in\Lambda} d\theta_x/2\pi \,.\, E(\Delta,\theta) \qquad E^{(1)<N} = \Sigma_{0<|n|_1<N}\, E_n(\Delta)\,.\,e^{i\,n.\theta}$$

where the E_n's are the Fourier coefficients of $E^{(1)}$. G is now defined as in section II-2 with E replaced by $E^{(1)<N}$. We assume that $0<r^{(1)}<r$ exists such that the non-resonant condition (11) occurs. We set $r^* = r^{(1)}/2$. Then we get the following result :

Proposition 4 : If, in addition to the previous hypothesis (15), (16), (17), ϵ satisfies :

$$(21)\quad (i)\quad \epsilon^{2-c}.Ln^{2D}(1/\epsilon) \le (16.C_0.e)^{-1}.\gamma.\psi(N^*.w(L^*)).Z(L^*).r^*.\delta$$

$$(ii)\quad Z(L-R)^\lambda \le \epsilon$$

the transformed hamiltonian $H_* = \exp(\mathcal{L}_G)\,H$ is well defined and can be written in the same form as H (eq.16), where :

$$(22)\quad (i)\quad \rho^* = \rho - 2.\delta$$

$$(ii)\quad h_* = h + \delta h \quad \text{and} \quad |\delta h(\Delta)| \le \epsilon^{(1)} \qquad \forall \Delta \in D(r)$$

$$(iii)\quad \| E_* \|_{r^*,\rho^*,L} \le \epsilon^c$$

$$(iv)\quad \| E_{*,x} \|_{r^*,\rho^*,L^*+2R} \le 2.e_0.Z(|x|_1)^\lambda \qquad \forall\, |x|_1 < L^*+R$$

Setting $\epsilon^* = \epsilon^c$ we also have :

$$(v)\quad Z(L^*-R)^\lambda \le \epsilon^*$$

◇

Proof : In order to exponentiate $\mathcal{L}_G$ it is necessary to satisfy the estimates (8) of proposition 3. In our case, if it acts on $\mathcal{H}(r',\rho',L)$, with $r^{(1)} \le r' \le r$, and $\rho-\delta \le \rho' \le \rho$, this gives :

$$\| G \|_{r^{(1)},\rho,L} < (2e)^{-1}.Z(L).r^*.\delta \quad .$$

Since one has :

$$\| G \|_{r^{(1)},\rho,L} < 2.\,\epsilon^{(1)}/\{\gamma.\psi(N^*.w(L^*))\}$$

the condition (21) gives :

$$2e\,.\,\| G \|_{r^{(1)},\rho,L}\,.\{Z(L^*).r^*.\delta\}^{-1} \le 1/4 < 1/2$$

which implies that the norm of the operator $\exp(\mathcal{L}_G)$ acting on $\mathcal{H}(r',\rho',L)$ with values in $\mathcal{H}(r^*,\rho^*,L)$, is dominated by 2 . Now, we have :

$$(23)\quad \exp(\mathcal{L}_G)H = H_0 + \{G,H_0\} + E^{(1)<N} + (\exp(\mathcal{L}_G)-1-\mathcal{L}_G)H_0 + \ldots$$

$$\ldots(\exp(\mathcal{L}_G)-1)E^{(1)<N} + \exp(\mathcal{L}_G)E^{(1)\ge N} + \ldots$$

$$\ldots\Sigma_{L^*-R \le |x|_1 < L^*+R} \exp(\mathcal{L}_G)E_x + \Sigma_{L^*+R \le |x|_1} E_x$$

for the last terms do not depend upon the variables in G. From the first order equation, the sum of the second and third term in the right hand side of the previous equality is equal to δh. We get E_* by adding the fourth, fifth and sixth terms. For instance :

$$(\exp(\mathcal{L}_G)-1-\mathcal{L}_G)H_0 = ((\exp(\mathcal{L}_G)-1)/\mathcal{L}_G - 1)(\delta h - E^{(1)<N}) =$$
$$= \Sigma_{k\ge 1}\, \mathcal{L}_G^{\,k}\,(\delta h - E^{(1)<N})/(k+1)!$$

Thanks to the previous remark, and to the proposition 3, if (21) holds one gets :

$$\| (\exp(\mathcal{L}_G)-1-\mathcal{L}_G)H_0 \|_{r^*,\rho^*,L^*} \le 8(\epsilon^{(1)})^2.\{Z(L^*).\gamma.\psi(N^*w(L^*)).r^*.\delta\}^{-1} \le \epsilon^c/2$$

The fifth term in (21) can be combined with the previous one to get some cancelation, and actually the previous estimate holds also for the sum of the fourth and the fifth term. The next one is estimated in the same way if we remark that :

$$\| E^{(1)\ge N} \|_{r,\rho-\delta,L^*} \le e^{-\delta.N^*}.\epsilon^{(1)} \le \epsilon^c/2$$

Therefore (iii) is satisfied. The $E_{*,x}$'s are also estimated in the same way. At last, for $|x|_1 \ge L^*+R$, E_x does not depend upon the degrees of freedom inside Λ. For this reason, $\exp(\mathcal{L}_G)$ acts as the identity operator and this term remains unchanged. The proof of (v) is immediate. ◇

III) SMALL DIVISORS :

In this section we will study especially all the technicalities needed for controlling the non-resonance condition that we introduced in the previous chapter.

III-1) MEASURING LOCALIZED CONFIGURATIONS :

Let $A_x^{(o)}$ be a configuration satisfying :

(1) $$|A_x^{(o)}| \leq Z_x \quad \forall x \in \mathbf{Z}^D$$

Then, for $r_0 < 1$, we define the polydisc $D(r_0)$ as follows :

(2) $$D(r_0) = \{\underline{A} \in \mathbf{C}^{\mathbf{Z}^D} ; |A_x - A_x^{(o)}| < r_0 . Z_x \quad \forall x \in \mathbf{Z}^D\}$$

On the other hand, if $\underline{\sigma} = \{\sigma_x ; x \in \mathbf{Z}^D\}$, is a sequence of positive real numbers, we define on the space $\mathbf{R}^{\mathbf{Z}^D}$ the gaussian probability measure $\mu_{\underline{\sigma}}$ as :

(3) $$\mu_{\underline{\sigma}} = \otimes_{x \in \mathbf{Z}^D} \, dA_x / \sqrt{2\pi\sigma_x} \, . \, e^{(A_x - A_x^{(o)})^2 / 2\sigma_x}$$

Then we get :

Proposition 5 : Under the condition :

(4) $$\sigma_x / Z_x^2 = o(\mathrm{Ln}(|x|_1)^{-1})$$

the polydisc $D(r_0)$ has a positive $\mu_{\underline{\sigma}}$-measure.

◇

Proof : Let us note first the estimate :

$$J(u) = \int_u^\infty dx/\sqrt{2\pi} \, . \, e^{-x^2/2} \leq e^{-u^2/2} / u . \sqrt{2\pi}$$

Then, we get :

$$\mu_{\underline{\sigma}}(D(r_0)) = \prod_{x \in \mathbf{Z}^D} \{1 - 2 . J(r_0 . Z_x / \sqrt{\sigma_x})\}$$

This infinite product converges to a non zero number, if and only if :

$$\sum_{x \in \mathbf{Z}^D} J(r_0 Z_x / \sqrt{\sigma_x}) < \infty$$

thanks to the previous estimate this is true whenever $\sigma_x / Z_x^2 = o(\mathrm{Ln}(|x|_1)^{-1})$.

◇

Remark : It is easy to show that this condition is also sufficient for we have the estimate :

$$u^{-1} . e^{-u^2/2} \geq \int_u^\infty dx \, . e^{-x^2/2} \geq u . (u^2+2)^{-1} . e^{-u^2/2}$$

◇

III-2) A SET OF NON-RESONANT VECTORS :

Let w be a positive increasing function on $\mathbf{R}_+$ such that $w(0) = 1$, and we consider the function :

(5) $$\psi(s)^{(1-g/2)} = \int_0^\infty dt \, . \, e^{-\varphi(t) - ts}$$

where $0 < g < 2$, and φ is a decreasing positive function to be determined later. For $\underline{n}$ in $\mathbf{Z}^{(\mathbf{Z}^D)}$, we define the norm $|\underline{n}|_w$ as :

$$|\underline{n}|_w = \sum_{x \in \mathbf{Z}^D} |n_x| . w(|x|_1)$$

We now introduce the set of non-resonant vectors Ω as the susbet $\underline{\omega}$'s of $\mathbf{R}^{\mathbf{Z}^D}$ such that :

(6) $$|\underline{\omega}\underline{n}| \geq \gamma . \psi(|\underline{n}|_w) \quad \text{for all} \quad \underline{n} \in \mathbf{Z}^{(\mathbf{Z}^D)} \setminus \{0\}$$

where $\underline{\omega}\underline{n} = \sum_{x \in \mathbf{Z}^D} \omega_x . n_x$

Then we get :

Proposition 6 : Under the assumption (4) of proposition 5, if γ is small enough and if :

(7a) $$C_1 = \int_{0<t} dt . e^{-\varphi(t)} \{\prod_{x \in \mathbf{Z}^D} \coth(t . w(|x|_1)) - 1\} < \infty$$

(7b) $$\psi(w(|x|_1))^g \leq C_2^2 . 2\pi . \sigma_x \quad \text{for all } x\text{'s, with } C_2 < \infty$$

then $\Omega \cap D(r_0)$ is non empty, and has a positive μ_g-measure.

◇

Proof : from a well known property of probabilities one has :

$$\mu_g(\Omega \cap D(r_0)) \geq \mu_g(D(r_0)) - \mu_g(\Omega^c)$$

where Ω^c is the complement of Ω. Now we have :

$$\mu_g(\Omega^c) \leq \Sigma_{n \neq 0}\, \mu_g\{\omega ; |\omega.n| \leq \gamma.\psi(|n|_w)\} \leq$$
$$\Sigma_{n \neq 0} \int_{|\xi| \leq \gamma.\psi(|n|_w)} d\xi .(2\pi\sigma(n))^{-1/2}.e^{-(\xi-\xi(n))^2/2\sigma(n)}$$

where $\xi(n)$ and $\sigma(n)$ are the mean and the covariance of the random variable $\omega.n$, namely :

$$\xi(n) = \Sigma_{x \in Z^D} A^{(0)}{}_x.n_x \qquad \sigma(n) = \{\Sigma_{x \in Z^D} |n_x|^2.\sigma_x\}$$

From these estimates and (5) we obtain :

$$\mu_g(\Omega^c) \leq \gamma.\Sigma_{n \neq 0}\, \psi(|n|_w) / (2\pi.\sigma(n))^{1/2} \leq \gamma.C_2.\Sigma_n\, \psi(|n|_w)^{(1-g/2)} =$$

$$= \gamma.C_2.\Sigma_{n \neq 0} \int_{t>0} dt.e^{-\varphi(t)} \Pi_x\, e^{-t|n_x|.w(|x|_1)}$$

$$= \gamma.C_2.\int_{t>0} dt.e^{-\varphi(t)}\{\Pi_x \Sigma_{n \in Z}\, e^{-t|n|.w(|x|_1)} - 1\} = \gamma.C_2.C_1$$

for $\Sigma_{n \in Z}\, e^{-t|n|} = \coth(t/2)$.

Therefore, provided γ is small enough, we have :

$$\mu_g(\Omega \cap D(r_0)) \geq \mu_g(D(r_0)) - \gamma.C_2.C_1 > 0$$

◇

Proposition 7 : Let now $H(A) = \Sigma_{x \in Z^D} A_x^2/2 + h(A)$ be a function on $D(r_0)$ such that (for some $\beta_0 < 1/2$):

$$(8)\ |\Sigma_{x,y \in Z^D} f(x)^*.f(y).\sigma_x.\partial^2 h/\partial A_x \partial A_y| \leq \beta_0.\Sigma_{x \in Z^D} \sigma_x.|f(x)|^2$$

for all sequence f of complex numbers. If we denote by ω the function defined as :

$$(9) \quad \omega_x(A) = \partial H/\partial A_x = A_x + \partial h/\partial A_x$$

then provided that $\varepsilon < 1$, and that (7a, 7b) hold, one has the estimate :

$$(10) \quad \mu_g(\omega^{-1}(\Omega) \cap D(r_0)) \geq \mu_g(D(r_0)) - 2\gamma.C_1.C_2$$

and the left hand side is positive for γ small enough.

◇

Proof : As in the previous proposition, it is sufficient to estimate the measure of the set of A's for which $|\omega(A).n| \leq \gamma.\psi(|n|_w)$. For a given $n \in Z^D$ let us decompose A into :

$$A_x = a.n_x.\sigma_x / \sigma(n)^{1/2} + B_x \qquad \text{with} \qquad \Sigma_{x \in Z^D} B_x.n_x = 0$$

Then if $\xi = \omega(A).n$, one has :

$$\xi = a.\sigma(n)^{1/2} + \partial h/\partial A.n$$

This defines "a" as a function of ξ and B which satisfies :

$$1 = |\partial a/\partial\xi \sigma(n)^{1/2}\{1 + 1/\sigma(n).\Sigma_{x,y \in Z^D} \partial^2 h/\partial A_x \partial A_y.n_x.n_y.\sigma_x\}|$$
$$> |\partial a/\partial\xi|.\sigma(n)^{1/2}(1-\beta_0)$$

The change of variable $A \to (\xi, B)$ gives the following result (since $\beta_0 < 1/2$).

$$\mu_g\{A; |\omega(A).n| \leq \gamma.\psi(|n|_w)\} \leq 2\int_{|\xi| \leq \gamma\psi(|n|_w)} d\xi/(2\pi\sigma(n))^{1/2}\, e^{-(a-a^{(0)})^2/2}\, d\mu_g(B)$$

where $d\mu_g(B)$ is the conditionned probability with respect to the data of a. The right hand side can be estimated as :

$$\mu_g\{A; |\omega(A).n| \leq \gamma.\psi(|n|_w)\} \leq 2\gamma.\psi(|n|_w).(2\pi\sigma(n))^{-1/2}$$

which is the same as in the previous proposition up to the multiplication by 2.

◇

III-3) APPROXIMATED NON-RESONANCE :

Given $r_0' < r_0$, let D_ω be the set $\omega^{-1}(\Omega \cap D(r_0 - r_0'))$. For $r < r_0'$ we define $D_\omega(r)$ as the set of points lying within the distance r from D_ω in the following sense :

(11) $$D_\omega(r) = \{\underline{A}\,;\,\exists \underline{B} \in D_\omega,\ |A_x - B_x| < r.Z_x,\ \text{for all } x\}$$

By construction, $D_\omega(r)$ is included in $D(r_0)$. Let now $H(\underline{A})$ be as in proposition 7, with the additional requirements that $h(\underline{A})$ depends only upon the sites such that $|x|_1 \le L$, and that it is holomorphic in $D(r)$ for some $0 < r \le r_0'$. Then we get the following result :

Proposition 8 : Let H as before and let $\omega = \partial H/\partial A$ be defined by (9); we also assume that $y.\psi(0)/Z_0 < 1$. For any real N bigger than 1, there is $0 < r' < r$ such that for all $\underline{A}$ in $D_\omega(r')$ and $\underline{n}$ in $\mathbf{Z}^D$, $|\underline{n}|_1 \le N$, the following non resonance condition holds.

(12) $$|\omega(\underline{A}).\underline{n}| \ge 1/2.y\,\psi(|\underline{n}|_w)$$

Moreover r' satisfies :

(13) $$r' \le r.y.\psi(N.w(L))/\{4(1+\eta).Z_0.N\}$$

where :

(14) $$\eta = \sup_{\underline{A}\in D_\Omega(r)} \sup_x\ |\partial h/\partial A_x|/Z_0$$

◇

Proof : if $\underline{A} \in D_\omega(r')$, with $r' \le r$, there is $\underline{B}$ in D_ω such that $|A_x - B_x| < r'Z_x$ for all x 's. On the other hand since h is holomorphic in $D_\omega(r)$, the fundamental calculus formula yields :

$$\omega(\underline{A}).\underline{n} = \omega(\underline{B}).\underline{n} + \int_0^1 d\sigma.\partial/\partial\sigma\left[\omega(\underline{A}(\sigma)).\underline{n}\right]$$

where, $\underline{A}(\sigma) = \sigma\underline{A} + (1-\sigma)\underline{B}$. Using the Cauchy formula, the remainder becomes:

$$\int_0^1 d\sigma.\partial/\partial\sigma\left[\omega(\underline{A}(\sigma)).\underline{n}\right] = \int_0^1 d\sigma \int_0^{2\pi} d\alpha/2\pi\,.\omega(\underline{A}(\sigma+se^{i\alpha})).\underline{n}/s.e^{i\alpha}$$

where s is a positive number to be chosen. Since $\underline{A}$ belongs to $D_\omega(r')$ the integrand is well defined provided $r'.(1+s) < r$, namely : $s < (r-r')/r'$. Therefore, since $\omega(\underline{A}) = \underline{A} + \partial h/\partial A$, the remainder is dominated by :

$$\int_0^1 d\sigma.\partial/\partial\sigma\ \omega(\underline{A}(\sigma)).\underline{n} \le r'/(r-r').Z_0.(1+\eta).N$$

If the right hand side is smaller than or equal to $1/2.y.\psi(|\underline{n}|_w)$ we get :

$$r' \le 1/2.r.y.\psi(|\underline{n}|_w).\{Z_0.(1+\eta).N + 1/2.y.\psi(|\underline{n}|_w)\}^{-1}$$

From our hypothesis, we have $1/2.y.\psi(|\underline{n}|_w)/Z_0 \le 1$ for all $\underline{n}$, and obviously $N \ge 1$. This gives the result.

◇

Let us now consider $h_1(\underline{A}) = h(\underline{A}) + \delta h(\underline{A})$ where again δh satisfies the same conditions as h. We set :

(15) $$\xi = \sup_{\underline{A}\in D_\Omega(r)} \sup_y \sum_x Z_x.|\partial^2 h/\partial A_x \partial A_y|/Z_y$$

$$\varepsilon = \sup_{\underline{A}\in D_\Omega(r)} |\delta h(\underline{A})| \qquad \delta\xi = \sup_{\underline{A}\in D_\Omega(r)} \sup_y \sum_x Z_x.|\partial^2 \delta h/\partial A_x \partial A_y|/Z_y$$

In much the same way we also define $\omega_1 = \omega + \partial\,\delta h/\partial A$.

Proposition 9 : Under the hypothesis of proposition 8, and if :

(16) $$\varepsilon \le 2/5.Z(L)^2.(r-r').r' \qquad \text{and} \qquad \xi + \delta\xi < 1/2$$

then, if $r_1 = r'/5$, $D_{\omega_1}(r_1)$ is contained into $D_\omega(r')$.

◇

Proof : Let us consider the polydisc $D(\underline{B}, r')$ centered at $\underline{B} \in D_\omega$, of radius r'. We set $\omega_\sigma = \omega + \sigma.\partial\delta h/\partial A$,and we define a path $\underline{B}(\sigma)$ via the implicit equation

$$\omega_\sigma(\underline{B}(\sigma)) = \omega(\underline{B}) \qquad 0 < \sigma < 1$$

If ε is small enough the implicit function theorem implies that such a path exists and is differentiable. Actually differentiating this equation yields :

$$\Sigma_y (\delta_{x,y} + \partial^2 h(\sigma)/\partial A_x \partial A_y).dB_y / d\sigma = -\partial\, \delta h/\partial A_x$$

Since ξ is defined via an algebraic norm on the matrix-valued functions indexed by the sites of our lattice, we get the estimate :

$$\sup_x Z_x^{-1}.| dB_x/d\sigma | \leq (1-\xi-\delta\xi).\sup_{\Delta\in D_\Omega(r'),x} Z_x^{-1}.| \partial\, \delta h/\partial A_x |$$

From the Cauchy formula, since δh is holomorphic in $D_\omega(r)$, and $Z_x \geq Z(L)$, we get :

$$\sup_{\Delta\in D_\Omega(r'),x} Z_x^{-1}.| \partial\, \delta h/\partial A_x | \leq \varepsilon . Z(L)^{-2}.(r-r')^{-1}$$

In particular, the distance between $\underline{B}$ and $\underline{B}(1)$ is smaller than or equal to $4/5\, r'$, and the polydisc $D(\underline{B}(1), r_1)$ centered at $\underline{B}(1)$ of radius $r_1 = r'/5$ is included in the previous one.

Now, we remark from the previous argument, that both ω and ω_1 are local isomorphisms, since they are small perturbations of the identity. However they are defined only on a subdomain of the original one $D(r_0)$. Since by hypothesis, the perturbations h and δh depends only upon a finite number of variables, we can use the Whitney theorem [60] to extend them as C^2 functions on $D(r_0)$ with the same bounds ξ and $\delta\xi$ of the second derivatives on $D(r_0)$. If ω and ω_1 denote these extension, they are also invertible on $D(r_0)$. The implicit equation defining $\underline{B}(1)$ can be reduced to the following identity :

$$\underline{B}(1) = \omega_1^{-1} \circ \omega(\underline{B})$$

and from the definition of D_ω we also get $\omega_1^{-1} \circ \omega(D_\omega) = D_{\omega_1}$. Therefore :

$$\bigcup_{\underline{B}\in D_\omega} D(\underline{B}(1), r_1) = \bigcup_{\underline{B}'\in D_{\omega 1}} D(\underline{B}', r_1) = D_{\omega_1}(r_1) \subset D_\omega(r')$$

◇

IV) RECURSIVE CONSTRAINTS :

IV-1) PARAMETERS OF THE RECURSION :

In this section, we collect all the constraints needed to perform the recursion process. Then we will analyze them in order to check under what conditions on the parameters they are satisfied.

In order to fix the notations, we define a priori the sequence $\underline{Z} = (Z_x\, ; x\in \mathbb{Z}^D)$ as in §II-3, the radius r_0 of the polydisc of analyticity of the original interaction in the action variable, and ρ_0 the width of its strip of analyticity in the angle variable. Now let ε_0 be the size of the original interaction namely :

(1) $$\varepsilon_0 = \sup_x \| E_x \|_{r_0, \rho_0, R}$$

where R is the range of the interaction (see Introduction eq. (1)). At the k^{th} step of the recursion, we will denote by ε_k the size of the interaction and correspondingly r_k, ρ_k, L_k will denote the radius of analyticity in the action variables, the width of the strip of analyticity in the angle variables, and the spatial cut-off, of the new hamiltonian. At this step the hamiltonian is given by (see §II-3):

(2) $$H_k = \Sigma_{x\in Z^D} A_x^2/2 + h_k(\underline{A}) + E_k(\underline{A},\underline{\theta}) + \ldots$$
$$\ldots \Sigma_{L_k - R \leq |x|_1 < L_k + R} E_{k,x}(\underline{A},\underline{\theta}) + \Sigma_{L_k + R \leq |x|_1} E_x(\underline{A}_{x+\Delta}, \underline{\theta}_{x+\Delta})$$

where h_k and E_k depends only upon the variables located on the sites x with $|x|_1 < L_k$, whereas the $E_{k,x}$'s depends on the sites x such that $|x|_1 < L_k + R$.

At the next step, we first define the new datas as :

(3) $$L_{k+1} = L_k^* \quad r_{k+1} = r_k^* \quad \rho_{k+1} = \rho_k^*$$
$$h_{k+1} = h_k + \delta h_k \quad E_{k+1} = E_{*k} \quad E_{k+1,x} = E_{*,k,x} = \exp(\mathcal{L}_{G_{k+1}}) E_{k,x}$$

where G_{k+1} is the infinitesimal canonical transformation defined when passing from k to k+1. From the eq. (17) of §II-3 L_{k+1} is defined through :

(4) $$Z(L_{k+1} - R) = Z(L_k - R)^c$$

We define also $Z_k = Z(L_k - R)$ and we get (§II-3, eq.(15ii)):

(5) $$Z_k \geq Z(L_k) \geq Z(L_k + R) \geq Z(L_k - R)^d = Z_k^{\,d}$$

At this stage we define an ultra-violet cut-off N_{k+1} as (see §II-3, eq.(20)):

(6) $$N_{k+1} = 1/\delta_{k+1} \,.\, \mathrm{Ln}\{2\, C_0 \,.\, \mathrm{Ln}^D(1/\varepsilon_k) \,.\, \varepsilon_k^{\,1-c}\}$$

At last if $\| E_k \|_{r_k, \rho_k, L_k} \leq \varepsilon_k$ we get :

(7) $$\varepsilon_{k+1} = \varepsilon_k^{\,c} \quad \Rightarrow \quad \| E_{k+1} \|_{r_{k+1}, \rho_{k+1}, L_{k+1}} \leq \varepsilon_{k+1}$$

It follows in particular that if $\varepsilon_0 \leq e^{-1}$, there is $\chi_1 > 0$ such that :

(8) $$N_k \leq \chi_1 . \delta_k^{-1} . \mathrm{Ln}(1/\varepsilon_k)$$

From (4) and (7) it also follows that there is $b > 0$ such that $Z_k = \varepsilon_k^{\,b}$. Since we need $Z_k^{\,\lambda} \leq \varepsilon_k$ (prop.4), it follows that $\lambda \geq 1/b$. Given $h > 0$ and $0 < g < 2$, we introduce the following notation :

(9) $$\xi_k = (2+h)/g \,.\, \mathrm{Ln}(1/Z_k) = (2+h).b/g \,.\, \mathrm{Ln}(1/\varepsilon_k)$$

IV-2) CONSTRUCTION OF THE RÜSSMANN FUNCTION :

In §III-2 (eq.5) we introduced a function ψ similar to the functions entering in the Rüssmann work [49] in describing the non-resonant condition. We set :

(10) $$S(s) = -\mathrm{Ln}(\psi(s))$$

We remark that S is smooth, positive, decreasing and concave on $\mathbb{R}_+$. Moreover, if we normalize ψ to $\psi(0) = 1$, then $S(0) = 0$. Since ψ vanishes at infinity, it follows that S is a diffeomorphism of $\mathbb{R}_+$. Let now $\Phi^{(1)}$ be a positive decreasing function on $\mathbb{R}$ such that :

(11) $$\Phi^{(1)} < 1 \qquad \lim_{\eta \to \infty} \Phi^{(1)}(\eta).e^{q.\eta} = \infty \quad \forall q > 0$$
$$\text{and} \qquad \int_\xi^\infty d\eta .\, \Phi^{(1)}(\eta) < \infty .$$

We set and S will be defined through the following formula :

(12) $$S^{-1}(\xi) = S^{-1}(1) . \exp\{1/\mathrm{Ln}\tau \,.\, \int_1^\xi d\eta/\eta \;\mathrm{Ln}(1/\Phi^{(1)}(\eta))\}$$

where $\tau > 1$. Then we get :

Lemma 1 : S satisfies the following inequality:

(13) $$S^{-1}(\xi) \leq \Phi^{(1)}(\xi) \,.\, S^{-1}(\tau.\xi)$$

◇

Proof : From (12) it follows that :

$$S^{-1}(\tau.\xi) = S^{-1}(\xi) \,.\, \exp\{1/\mathrm{Ln}\tau \,.\, \int_\xi^{\tau\xi} d\eta/\eta \,.\, \mathrm{Ln}(1/\Phi^{(1)}(\eta))\}$$

Since $\mathrm{Ln}(1/\Phi^{(1)})$ is increasing, the integrand is bounded from below by its value at ξ. Performing the integral gives the result. ◇

Let now $\chi_2 > 1$ be chosen and χ be equal to $(2+h).b/g$. We set :

(14) $$\delta_k = \chi_1 \chi_2^{\,2}/\chi \,.\, \xi_k \,.\, \Phi^{(1)}(\xi_k)$$

and we have :

Lemma 2 : Let Φ be given by $\Phi(\xi) = \int_\xi^\infty d\eta .\, \Phi^{(1)}(\eta)$. If :

(15) $$\varepsilon_0 \leq \exp-\{g/b.(2+h) \,.\, \Phi^{-1}(K.\rho_0)\} \qquad K = (c-1).\chi/2.c.\chi_1 \chi_2^{\,2}$$

then $\sum_{k \geq 1} \delta_k < \rho_0/2$.

◇

Proof : From the definition we obtain :

$$\sum_{k \geq 1} \delta_k = \chi_1 \chi_2^{\,2}/\chi \,.\, \sum_{k \geq 1} \xi_k . \Phi^{(1)}(\xi_k)$$

By definition $\xi_k = \xi_0 . c^k$ and therefore, $\xi_k - \xi_{k-1} = (c-1)/c.\xi_k$. Therefore the previous sum can be dominated by the integral :

$$\sum_{k \geq 1} \delta_k \leq c.\chi_1 \chi_2^{\,2}/\chi.(c-1) \,.\, \int_{\xi_0}^\infty d\eta .\, \Phi^{(1)}(\eta) = \Phi(\xi_0)/2.K < \rho_0/2 .$$

◇

The choice of S determines the function ψ and therefore gives a constraint on the function w introduced in the §III-2. On the other hand, we have not yet given the behavior of the function Z, and the values of the L_k's are not yet specified. This can be done by imposing :

(16) $$w(L_k) = w_k = \chi_2 . S^{-1}(\xi_k)$$

Lemma 3 : The sequence w_k satisfies the inequalities :

(17) $$S^{-1}(\xi_k) < w_k < S^{-1}(\tau.\xi_k)/N_k$$

◇

Proof : since $\chi_2 > 1$ we have :

$$S^{-1}(\xi_k) < \chi_2 . S^{-1}(\xi_k) = w_k < \chi_2^2 . S^{-1}(\xi_k)$$

$$\leq \chi_2^2 . \Phi^{(1)}(\xi_k) . S^{-1}(\tau.\xi_k) = \delta_k . \chi . S^{-1}(\tau.\xi_k)/\xi_k . \chi_1$$

On the other hand, from (8) & (9) we have $N_k \leq \chi_1 . \xi_k / \chi . \delta_k$ and therefore we get the result. ◇

Proposition 10 : If $h > 0$, $f > 0$, $b \geq 1/\lambda$, $0 < g < 2$ are chosen in order that :

(18) $$\tau = f.g/(2+h).b > 1$$

and if we define the sequence σ as :

(19) $$2\pi C_2^2 . \sigma_x = Z(|x|)^{2+h}$$

then with S defined by the equation (12) and $\psi = e^{-S}$, we get the following set of inequalities :

(20) $$\varepsilon_k^{(2+h)b} = Z_k^{2+h} \geq Z(L_k)^{2+h} = 2\pi C_2^2 . \sigma(L_k) \geq \psi(w_k)^g \geq \psi(N_k . w_k)^g \geq \varepsilon_k^{f.g}$$

◇

Proof : In lemma 3 we have shown that $S^{-1}(\xi_k) < w_k < N_k . w_k < S^{-1}(\tau\xi_k)$. Applying S on each sides, and exponentiating, gives the result. ◇

IV-3) THE NON RESONANT CONDITION :

Propositions 8 and 9 impose two conditions leading to the definition of the new radius of analyticity after each step. The first one in proposition 8 gives the radius r_k' as follows :

(21a) if $|\partial h_k / \partial A_x| \leq \eta_k \leq 1$

(21b) then $r_k^{(1)} \leq r_k . \gamma . \psi(N_{k+1} . w_{k+1}) / 8 N_{k+1}$

Since $\gamma < 1$, we get $r_k^{(1)} \leq r_k / 8$. The other one concerns the size of the interaction (prop.9 and eq.15 §III-3) :

(22a) $$\zeta_{k+1} = \zeta_k + \delta\zeta_k < 1/2$$

(22b) $$\varepsilon_k^{(1)} \leq 2/5 . Z(L_{k+1})^2 . (r_k - r_k^{(1)}) . r_k^{(1)} \quad \text{and} \quad r_{k+1} \leq r_k^{(1)}/5$$

In addition we know from the lemma (§II-3) that :

(23) $$\varepsilon_k^{(1)} \leq C_0 . \varepsilon_k . Ln^D(1/\varepsilon_k)$$

Thus a sufficient condition to verify (22b) is given by :

(24) $$\varepsilon_k . Ln^D(1/\varepsilon_k) \leq (\gamma/32 C_0) . r_k^2 . \psi(N_{k+1} w_{k+1}) . Z(L_{k+1})^2 / N_{k+1}$$

$$r_{k+1} = r_k . (\gamma/40) . \psi(N_{k+1} . w_{k+1}) / N_{k+1}$$

Proposition 11 : If :

(25) (i) $a.(c-1) > c.f$

(ii) $\Phi^{(1)}(\xi) \geq (40/\gamma) . e^{-\xi(a(c-1)-cf)/\chi c}$ for all $\xi > \xi_0$

then the hypothesis $r_k \geq \varepsilon_k^*$ implies $r_{k+1} \geq \varepsilon_{k+1}^*$.

◇

Proof : From (24) we have :

$r_{k+1}/\varepsilon_{k+1}^{a} = r_k(\gamma/40).\psi(N_{k+1}.w_{k+1})/N_{k+1}.\varepsilon_{k+1}^{a} \geq (\gamma/40).\varepsilon_k^{-a(c-1)+cf}/N_{k+1}$

from (8), (9) & (14) it leads to :

$$r_{k+1}/\varepsilon_{k+1}^{a} \geq (\gamma/40).\varepsilon_{k+1}^{-a(1-1/c)+f}.\chi_2^{2}.\Phi^{(1)}(\xi_{k+1})$$

from (9), (25 ii) and $\chi_2^{2} > 1$, the left hand side is bounded from below by 1.

◇

Proposition 12 : If :

(26) (i) $2bcd+2a+cf < 1$

(ii) $\Phi^{(1)}(\xi) \geq (32C_0/\gamma).(\xi/cd)^D.e^{-\xi(1-2a-2bcd-cf)/\chi c}$ for all $\xi > \xi_0$

(iii) $r_k \geq \varepsilon_k^{a}$

then (24) is true.

◇

Proof : It is sufficient to check :

$$\varepsilon_k.Ln^D(1/\varepsilon_k) \leq (\gamma/32C_0).\varepsilon_k^{2a+fc+2bcd}.\Phi^{(1)}(\xi_{k+1})$$

from which we see that (26i) is necessary. If we use (8) (14)and (9) again, the proof of this estimate follows the same line as the proof of proposition 11.

◇

IV-4) DIFFERENTIABILITY AND RECURSIVE CONSTRAINTS :

The final hamiltonian is a function of A defined as the sum of an infinite series :

(27) $H_\infty(A) = \Sigma_{x\in Z^D} A_x^2/2 + h_\infty(A)$ $\qquad h_\infty(A) = \Sigma_{k>0}\,\delta h_k$

where δh_k is the amount of change of the integrable part of the hamiltonian after the k^{th} step. Thanks to the proposition 4, we have :

(28) $|\delta h_k(A)| \leq \varepsilon_k^{(1)}$ $\qquad$ on $D_{\omega_k}(r_k)$

where $\omega_k(A) = A + \partial h_k/\partial A$.

In particular, if ν is an integer, δh_k is ν times differentiable and using the Cauchy formula, it satisfies :

(29) $\mathrm{Max}_{|\underline{\nu}|\leq\nu}\,|\partial^{\underline{\nu}}\delta h_k/\partial A^{\underline{\nu}}| \leq \varepsilon_k^{(1)}(r_k.Z(L_{k+1}))^{-\nu}$

Since δh_k depends upon the sites with $|x|_1 \leq L_{k+1}$ (§II-3,eq.18 & 20b), the sum of these derivatives converges if :

(30) $\Sigma_{k\geq 0}\,\varepsilon_k^{(1)}(r_k.Z(L_{k+1}))^{-\nu} < \infty$

Thanks to the previous section, this is implied by the condition :

(31) $a+dbc < 1/\nu$

The same kind of estimates hold for the transformed $f_\infty = \Pi_{k\geq 1}\exp(\mathbf{L}_{G_k})\,f(A,\theta)$ of a function f in some $\mathcal{H}(D_\omega(r),\rho,L)$. In particular, f_∞ is the sum of an infinite series $\Sigma_{k\geq 0} f_k$ with :

(32) $f_k = [\exp(\mathbf{L}_{G_k})-1]\Pi_{j=0}^{k-1}\exp(\mathbf{L}_{G_j})f$

In particular, we get, when $L_k > L$:

(33) $\|f_k\|_{r_k,\rho_k,L_k} \leq \|f\|_{r,\rho,L}.\varepsilon_{k-1}^{c-1}/4.\ \Pi_{j\geq 0}(1+\varepsilon_j^{c-1}/4)$

The same argument shows that :

Proposition 13 : If :

(34) $bd+a < (c-1)/c\nu$

then the canonical transformation $\Pi_{k\geq 1}\exp(\mathbf{L}_{G_k})$ transforming $H(A,\theta)$ into $H_\infty(A)$ transforms every element of $\mathcal{H}(D_\omega(r),\rho,L)$ into a C^ν function. In particular, the corresponding invariant torii are ν-1 times differentiable.

◇

We must also check that the bounds on the different derivatives of h_∞

are compatible : one of this bounds is given in proposition 7 (eq.8), the others in propositions 8 (eq.14) and 9 (eq.16). It is not difficult to convince oneself that the first of this three implies the others. Actually, using the Cauchy formula, and considering that δh_k depends upon the sites within the distance L_{k+1} and is analytic in a polydisc of radius r_k, the first of these bounds is implied by the following ones :

$$(35)\quad \beta_0 \leq \Sigma_{k\geq 0}\, \beta_k < 1/2 \qquad \text{with}$$

$$\beta_k \leq \sup_{x\in Z^D} \Sigma_{y\in Z^D} |\partial^2 \delta h_k/\partial A_x \partial A_y| / Z(L_{k+1})^{2+h} \leq \varepsilon^{(1)} L_{k+1}/r_k^{\,2}\, Z(L_{k+1})^{4+h}$$

To get the convergence of this series it is necessary that ε_0 be small enough and that:

$$(36)\qquad 2a + (4+h)bcd < 1$$

◇

There is one more constraint to impose, in order to allow us to define the canonical transformation $\exp(L_{G_k})$. It is given in proposition 4, by (21i) namely :

$$(37)\quad \varepsilon_k^{\,2-c}.Ln^{2D}(1/\varepsilon_k) \leq (\gamma/16eC_0)\; r_{k+1}.\psi(N_{k+1}.w_{k+1}).Z(L_{k+1}).\delta_{k+1}$$

In much the same way we obtain :

Proposition 14 : If :

(38) (i) $bd+a+f < 2/c-1$

(ii) $\Phi^{(1)}(\xi) \geq (16eC_0/\gamma\chi_1 c).(\xi/c\chi)^{2D-1}.e^{-\xi(2-c-ac-bcd-fc)/\chi c}$ for all $\xi > \xi_0$

(iii) $r_k \geq \varepsilon_k^{\,a}$

then (37) is true.

◇

IV-5) SOLVING THE CONSTRAINTS ON THE EXPONENTS :

To summarize, we have obtained the following relation between the various exponents that we introduced along the previous sections :

(39)	(i)	$b < fg/(2+h)$	(prop. 10)
	(ii)	$f < a(c-1)/c$	(prop. 11)
	(iii)	$a+bd < (c-1)/cv$	(prop. 13)
	(iv)	$2a+2bcd+cf < 1$	(prop. 12)
	(v)	$a+bd+f < 2/c-1$	(prop. 14)
	(vi)	$2a+(4+h)bcd < 1$	(eq.36)

As can be seen from the proofs of the various results, these constraints are almost optimal. Therefore the best results will be obtained by replacing the inequalities by equalities.

Proposition 15 : For $b > 0$, $1 < c < 2$, and $v \geq 2$ given, there exists $a > 0$, $d > 1$, $f > 0$, $0 < g < 2$, $h > 0$ such that (39) is true if and only if :

$$(40)\quad b < b(v,c) = \text{Min}\{(2-c)(c-1)/(3c-2)c,\,(c-1)^2/vc(2c-1)\}$$

◇

Proof : Using (i) & (ii) we get a necessary condition in the form :

(i)	$f > b$
(ii)	$a > c/(c-1)b$
(iii)	$b(2c-1)/(c-1) < (c-1)/cv$
(iv)	$b(3c-2)/(c-1) < (2-c)/c$
(v)	$bc(3c-1)/(c-1) < 1$
(vi)	$b2c(2c-1)/(c-1) < 1$

We remark that (v)&(vi) are consequences of (iii) if $v \geq 2$, which implies (40). Conversely, if (40) is satisfied, it is certainly possible to find $d > 1$ such that

(iii)	$bc/(c-1)+bd < (c-1)/cv$
(iv)	$2bc/(c-1)+2bcd+cb < 1$
(v)	$bc/(c-1)+bd+b < 2/c-1$
(vi)	$2bc/(c-1)+4bcd < 1$

Proceeding in the same way, we easily check that it is possible to choose $a > 0$, then $f > 0$ and finally, $h > 0$ and $0 < g < 2$ such that (40) be satisfied.

◇

<u>Proposition 16</u> : The maximal value of b compatible with (40) is a monotone decreasing function of ν on the interval $[2,\infty)$ and satisfies :

(41a) $$b(\nu) = \sup_{1<c<2} b(\nu,c) = 1/6\nu \,.\, (1+10/9\nu + O(\nu^{-2}))^{-1}$$

as $\nu \to \infty$, and in addition $b(\nu) \leq b(2) = 1/19.18\ldots$. Consequently, if we choose $\lambda = 1/b$, then $\lambda \geq 20$, for $\nu=2$, and

(41b) $$\lambda \geq 6\nu + 20/3 + O(1/\nu) \quad \text{as } \nu \to \infty.$$

◇

<u>Proof</u> : let us set :

$$f_1(c) = (2-c)(c-1)/c(3c-2) \qquad f_2(c) = (c-1)^2/c(2c-1)$$

Clearly f_1 has a maximum c_0 in the interval (1,2), and $c_0 < 3/2$ as can be checked from the sign of the logarithmic derivative. On the other hand f_2/ν is increasing on (1,2), and cuts f_1 at the point $c(\nu)$ given by :

(42) $$c(\nu) = \frac{5(1+\nu) + \{25(1+\nu)^2 - 8(1+\nu)(3+2\nu)\}^{1/2}}{2(3+2\nu)}$$

The supremum of b is reached at $c = \mathrm{Max}(c_0, c(\nu))$. We have $\nu = f_2(c(\nu))/f_1(c(\nu))$ and the right hand side is increasing in $c(\nu) \in (1,2)$; for $c(\nu)=3/2 > c_0$ we get $\nu=5/4$ which is smaller than 2. Thus, for $\nu \geq 2$, $c(\nu)$ gives the supremum of b. As $\nu \to \infty$ we get $c(\nu) \approx 2 - 4/3\nu + O(1/\nu^2)$ and this gives the asymptotic value of $b(\nu)$. On the other hand, $b(\nu)$ is decreasing in ν and therefore, $b(\nu) \geq b(2)$. For $\nu=2$, we get from (42) :

$$c(\nu=2) = (15+\sqrt{57})/14 = 1.610\ldots$$

and this gives the numerical value of b(2) : $1/b(\nu=2) = 19.18\ldots$

We recall that the exponent λ giving how the interaction decreases toward zero as the action variable decreases, is limited by $\lambda \geq 1/b$. Therefore the minimal value of λ is given by $\lambda > 1/b(\nu)$, which is expressed by the proposition15. ◇

IV-6) <u>THE RÜSSMANN FUNCTION</u> :

It remains now to find the conditions on the functions ψ, w and Z that we introduced to measure the resonance condition and the rate of decreasing of the configurations. In §III-2 (eq. 5) ψ is defined via its Laplace transform, namely :

(43) $$\psi(s)^{(1-g/2)} = \int_0^\infty dt\; e^{-\varphi(t) - ts}$$

We will choose the following expression for φ :

(44) $$\varphi(t) = \omega_D \,.\, \mathrm{Ln}(1/t) \,.\, \int_0^\infty dL \,.\, L^{D-1} e^{-t.w(L)}$$

where ω_D is a positive constant to be defined below, and w is bigger than one, increasing, and such that for all $t>0$ the right hand side of (44) converges. Then we get :

<u>Proposition 17</u> : If ψ is given by the equation (43) above, and ω_D is big enough, then :

(45) $$\sum_{\Omega \in Z^{(Z^D)}\setminus\{0\}} \psi(|\Omega|_w)^{(1-g/2)} = C_1 < \infty$$

◇

<u>Proof</u> : It is a simple calculation to show that :

(46) $$C_1 = \int_{t>0} dt \,.\, e^{-\varphi(t)} \{\prod_{x \in Z^D} \coth(t.w(|x|_1)/2) - 1\}$$

We decompose this integral into the sum $\int_0^1 + \int_1^\infty$. For $t>1$ we get

$$\{\prod_{x \in Z^D} \coth(t.w(|x|_1)/2) - 1\} \leq$$
$$\leq \text{const.} \sum_{x \in Z^D} e^{-t w(|x|_1)} \,.\, \prod_{x \in Z^D} \coth(t.w(|x|_1)/2)$$
$$\leq \text{const.} e^{-t} \sum_{x \in Z^D} e^{-w(|x|_1)+1} \prod_{x \in Z^D} \coth(w(|x|_1)/2) = \text{const } e^{-t}$$

For $t<1$ we obtain :

$$\sum_{x \in Z^D} \mathrm{Ln}[\coth(t.w(|x|_1)/2)] \leq \text{const.} \mathrm{Ln}(1/t) \sum_{L \in \mathbf{N}} L^{D-1} e^{-t\, w(L)}$$

But it is easy to show that $D.L^{D-1} \le \{L^D - (L-1)^D\}/(1-e^{-1})$ if $L \ge D$, and since w is increasing we can estimate the series by the following integral :

$$\mathrm{Ln}(1/t).\sum_{L\in\mathbb{N}} L^{D-1}.e^{-t\,w(L)} \le \mathrm{const.}\,\mathrm{Ln}(1/t).\int_{L>0} dL.L^{D-1}.e^{-t\,w(L)} \le \mathrm{const.}\ \varphi(t)$$

Therefore chosing the constant ω_D big enough, allows us to bound the integrand in (46) by an integrable function.

◇

We will impose the following conditions on w :

<u>D-concavity</u> : the function $v(\xi) = w^{-1}(\xi)^D$ is increasing and concave on $[1,\infty)$, and $v(1) = 0$.

<u>κ-scaling</u> : there is $0 < \kappa < 1$ such that $\liminf_{\xi\to\infty} \ \xi.v'(\xi)/v(\xi)^{1-\kappa} > 0$.

The previous condition implies that v and w are continuous, and one-to-one. In the appendix we will prove the following result :

<u>Proposition 18</u> : If $S(s) = -\mathrm{Ln}(\psi(s))$, then there are two positive constants C_+, C_- and $L_0 > 0$, such that if $L > L_0$ we have :

(47) (i) $w(L) \le S^{-1}(C_+.L^{D(1+\kappa)})/L^D$

(ii) $w(L).\mathrm{Ln}(w(L)) \ge S^{-1}(C_-L^D)/L^D$

◇

Let us now estimate the speed of variation of the sequence L_k defined by the equations (16) and (17) of §IV-2.

<u>Proposition 19</u> : In order to satisfy the equation 16 & 17 of §IV-2 it is sufficient that :

(48) $Z(E) = e^{-b.S(w(L+R)/\chi_2)/\chi}$

In this case there are two positive constants a_+ and a_- such that, for L big enough :

(49) $e^{-a_-.L^{D(1+K)}} \le Z(L) \le e^{-a_+.L^D}$

◇

<u>Proof</u> : The equation (16) reads for any $k \in \mathbb{N}$

$$w(L_k) = \chi_2.S^{-1}(\xi_k) \qquad \text{and} \qquad \xi_k = \chi.\mathrm{Ln}(1/\varepsilon_k) = \chi/b\,.\,\mathrm{Ln}(1/Z(L_k-R))$$

which is a consequence of (48). Now from (47), if L is big enough,

$$S^{-1}(\xi) < w(L) \le S^{-1}(C_+.L^{D(1+K)})$$

which implies $\xi < C_+.L^{D(1+K)}$ and therefore

$$Z(L) \ge Z(L-R)^d = e^{-\chi.\xi.d/b} \ge e^{-\mathrm{const.}\,L^{D(1+\square)}}$$

On the other hand,

$$S^{-1}(C_-.L^D)/L^D \le w(L).\mathrm{Ln}(w(L)) = \chi_2 S^{-1}(\xi).\mathrm{Ln}(\chi_2 S^{-1}(\xi)) \le \mathrm{const.}\, S^{-1}(\xi)/\Phi^{(1)}(\xi)$$

for $\mathrm{Ln}(\chi_2 S^{-1}(\xi)) \le \mathrm{const.}\,\xi \le \mathrm{const.}/\Phi^{(1)}(\xi)$ thanks to (11). Using the lemma1, the right hand side is bounded by $\mathrm{const.}S^{-1}(\tau.\xi)$. In much the same way, the left hand side is bounded below by $\mathrm{const.}S^{-1}(C'.L^D)$, where C' is some constant. Therefore we get $S^{-1}(C'.L^D) \le \mathrm{const.}\,S^{-1}(\tau.\xi)$. Since S is the logarithm of the Laplace transform of some positive function, it satisfies (Hölder's inequality) $S(p.s) \le p.S(s)$ for any $p \ge 1$, which leads to $\xi \ge \mathrm{const.}\,L^D$ and to the result by the same reasoning. ◇

It remains to show now that the function $\Phi^{(1)}$ defined in §IV-2 (eq. 12) satisfies the hypothesis given in equation (11).

<u>Proposition 20</u> : (i) If w obeys to D-concavity and κ-scaling then $\Phi^{(1)}$ satisfies

(50) $\limsup_{\xi\to\infty} \mathrm{Ln}(1/\Phi^{(1)}) / \xi = 0$

(ii) If in addition, the function $v(\xi) = w^{-1}(\xi)^D$ satisfies for some $2/3 < \beta < 1$:

(51) $v'(\xi)/v(\xi) \le \mathrm{const.}\ \{\xi.\mathrm{Ln}^{\beta}(\xi)\}^{-1}$ for ξ big

then $\Phi^{(1)}$ is integrable on $(1,\infty)$. ◇

Proof : We recall that $\Phi^{(1)}$ is defined through the equation (12), namely :

$$\mathrm{Ln}(1/\Phi^{(1)}) = \mathrm{Ln}\tau\,.\,\xi\,.\,\partial S^{-1}(\xi)/\partial\xi\,.\,(1/S^{-1}(\xi))$$

If we set $s = S^{-1}(\xi)$ (50) is equivalent to :

$$\liminf_{s\to\infty} s.S'(s) = +\infty$$

ie to :

$$\lim_{s\to\infty} s.S'(s) = \lim_{s\to\infty} \frac{\int_0^\infty dt\,.ts\,.e^{-\varphi(t)-ts}}{\int_0^\infty dt\,.e^{-\varphi(t)-ts}} = +\infty$$

Using a gaussian approximation of the exponent of the integrals (see lemma A1 in appendix A) , it is easy to get :

$$s.S'(s) \geq \frac{\int_{t(s)}^\infty dt.ts\, e^{-t\varphi(s)-(t-T)^2\sigma/2}}{\int_{-\infty}^{t(s)} dt\,.e^{-t\varphi(s)-(t-T)^2\sigma/2}} \geq s.\,t(s)$$

with $\sigma = \varphi''(T)$ and $t(s)$ is the unique solution of $s = -\varphi'(t(s))$. The proof will be finished if we show that $\lim_{t\to 0} t|\varphi'(t)| = \infty$. This is a consequence of (see App. A) :

$$t.|\varphi'(t)| \geq \omega_D.(1/e^a(1+t))\;.\;v(a/t) \qquad \text{for all } a > t$$

The second part of the proposition is more involved. A sufficient condition for integrability is

$$\liminf_{\xi\to\infty} \mathrm{Ln}(1/\Phi^{(1)})/\mathrm{Ln}\xi > 1$$

or a fortiori :

$$\lim_{s\to\infty} s.S'(s)\,.\,\mathrm{Ln}S(s)\,/\,S(s) = 0$$

The proof of this estimate will be found in Appendix B. ◇

At last we remark that there is a large class of functions w satisfying the D-concavity, the κ-scaling condition and the eq.(51) as we will see in the next section. In this case, starting from the data of w, we have built φ then S and therefore $\Phi^{(1)}$.

IV-7) THE SIZE OF THE INTERACTION :

There are various places where we obtained bounds restricting the size of ε_0. In this chapter, the first bound was given in the lemma 2 (eq.5), and three others in proposition 11,12, and 14 are related. They are :

(52) (i) $\varepsilon_0 < \exp-\{\Phi^{-1}(K.\rho_0)/\chi\}$ $\qquad K < (c-1)\chi/c\chi_1$

(ii) $\Phi^{(1)}(\xi) \geq C_3/y\,.\,(\xi/\chi c)^{2D-1}.e^{-q(\xi/\chi c)}$ $\qquad \forall\ \xi \geq \chi.\,\mathrm{Ln}(1/\varepsilon_0)$

with

$$\chi = (2+h)b/g$$

$$q = \mathrm{Min}\{(a(c-1)-cf),(1-2a-2bcd-df),(2-c-ac-bcd-cf),(1-2a-(4+h)bcd)\}$$

$$C_3 = \mathrm{Max}\{40, 32C_0, 16eC_0/\chi_1 c\}$$

(we have added the last exponent (1-2a-(4+h)bcd) in view of the remainder of this section). As we saw in the previous section, given $q_0 > 0$ with $q_0 < q$, there is ξ_0 such that if $\xi \geq \xi_0$ then $\Phi^{(1)}(\xi) \geq e^{-q_0(\xi/\chi c)}$. In particular if

(53) $\qquad \varepsilon_0 \leq e^{-\xi_0/\chi}$

then (52 ii) is satisfied as long as :

$$e^{-q_0(\xi/\chi c)} \geq C_3/y\,.\,(\xi/\chi.c)^{2D-1}\,e^{-q(\xi/\chi c)} \qquad \text{for all } \xi \geq \xi_0$$

If $\xi_0/\chi c > (2D-1)/(q-q_0)$ i.e. if

(54) $\qquad \varepsilon_0 \leq e^{-c(2D-1)/(q-q_0)}$

then (52ii) is satisfied as long as :

(55) $\qquad \varepsilon_0^{(q-q_0)/c}.\mathrm{Ln}^{2D-1}(1/\varepsilon_0) < y.c^{(2D-1)}/C_3$

The other inequalitities come from the propositions 7,8 and 9. They are related to the total size of h_∞. In proposition 7 we introduced the constant β_0 which is dominated by (see also §IV-4 eq.35) :

(56) $\qquad \beta_0 \leq \sum_{k\geq 0} \beta_k$ $\qquad$ with

$$\beta_k = C_0.\varepsilon_k.\mathrm{Ln}^D(1/\varepsilon_k).K(D).L_{k+1}^D / r_k^2.Z(L_{k+1})^{4+h}$$

as can be seen easily from (§III-2 eq.8), using the Cauchy formula, the estimate on the size of δh_k (eq.28 §IV-4), and the fact that δh_k is analytic in a polydisc of radius r_k, and depends upon the sites within the distance L_{k+1} of the origin. Here K(D) is a geometrical constant defined by :

(57) $$K(D) = \sup_{L>0} \#\{x \in \mathbf{Z}^D ; |x|_1 \leq L\} / L^D$$

The constraint $\beta_0 < 1$ can be obtained by imposing $\beta < 1$ provided $\beta_0 \leq \beta$. In particular, β_k is bounded from above by :

(58) $$\beta_k \leq K(D).C_0.(bcd)^D/K_1.\varepsilon_k^{(1-2a-(4+h)bcd)}\mathrm{Ln}^{2D}(1/\varepsilon_k)$$

for we have used the estimate (5-iii) of §II-3 on Z, and the relations between Z and ε as described in the previous sections. If in addition to them we recall the hypothesis $\varepsilon_0 < e^{-1}$, we get the bound :

(59) $$\varepsilon_0^q.\mathrm{Ln}^{2D}(1/\varepsilon_0) < C_4(D,bcd,q,c)$$

where

$$C_4(D,bcd,q,c)^{-1} = K(D).C_0.(bcd)^D/K_1.\sum_{k\geq 0} e^{-q(c^k-1)}c^k$$

Let us remark that if we want λ very close to its minimal value, q will be fairly small, and ε_0 will vanish like $(1/q)^{-1/q}$ as $q \to 0$.

The estimates needed in propositions 8 & 9 are actually consequences of (59) as can be easily seen from (§III-2, eq.14 & 15).

Let us finish this section by remarking that there are a lot of functions w's satisfying the three conditions described in the previous section. The limiting cases are given by the following examples :

(60) (i) $w(L) \approx e^{A.\mathrm{Ln}^{\alpha}(L)}$ as $L \to \infty$, with $\alpha > 3$, $A > 0$

$\Phi^{-1}(K.\rho) \approx e^{A.\mathrm{Ln}(K\rho)^{1/(\alpha-1)}}$

and

(ii) $w(L) \approx e^{A.L^{\kappa.D}}$ as $L \to \infty$, with $\kappa < 1$, $A > 0$

$\Phi^{-1}(K.\rho) \approx \mathrm{Ln}^2(K.\rho)$

****

-APPENDIX A : a proof of Proposition 18-

In this appendix we will prove the proposition 18. We recall that ψ was introduced in §II-2 (eq. 5) and defined via its Laplace transform as :

(1) $$\psi(s)^{(1-g/2)} = \int_0^{\infty} dt.\, e^{-\varphi(t)-ts}$$

In contrast with §IV-6 (eq. 44) we will choose the following expression for φ (which does not change the conclusion) :

(2) $$\varphi(t) = \omega_D.\mathrm{Ln}(1+1/t).\int_0^{\infty} dL.\, L^{D-1}.e^{-t.w(L)}$$

where ω_D is a positive constant to be defined below, and w is bigger than one, increasing, and such that for all $t > 0$ the right hand side of (50) converges. We have imposed the following conditions on w :

<u>D-concavity</u> : the function $v(\xi) = w^{-1}(\xi)^D$ is increasing and concave on $[1,\infty)$, and $v(1) = 0$.

<u>κ-scaling</u> : there is $0 < \kappa < 1$ such that $\liminf_{\xi\to\infty} \xi.v'(\xi)/v(\xi)^{1-\kappa} > 0$.

The previous condition implies that v and w are continuous, and one-to-one. We will prove the following result :

<u>Proposition 18</u> : If $S(s) = -\mathrm{Ln}(\psi(s))$, then there are two positive constants C_+, C_- and $L_0 > 0$, such that if $L > L_0$ we have :

(3) (i) $w(L) \leq S^{-1}(C_+.L^{D(1+\kappa)})/L^D$

(ii) $w(L).\mathrm{Ln}(w(L)) \geq S^{-1}(C_-.L^D)/L^D$ ◇

1- The Legendre transform $L\varphi$ of φ is defined as :

(4) $$L\varphi(s) = \mathrm{Inf}_{t>0}\{\varphi(t) + t.s\}$$

From (2), it follows that φ is positive on $\mathbf{R}_+$, it vanishes at infinity and it diverges at $t \to 0$. Therefore $L\varphi$ is finite at all $s > 0$. On the other hand φ is completely monotone i.e. $(-)^n.\varphi^{(n)}(t) > 0$ for all $n \in \mathbf{N}$. Thus the infimum in (4) is reached at a unique point $t(s)$, solution of the equation :

(5) $$s + \varphi'(t(s)) = 0$$

The first step gives a comparison between S and $L\varphi$:

Lemma A1 : Let $L\varphi$ be the Legendre transform of φ and $t(s)$ be defined by (5). Then ψ satisfies the following bounds :

(6) $$\psi(s) \geq \{\pi/2\varphi''(t(s))\}^{1/2}.\, e^{-L\varphi(s)}$$

$$\psi(s) \leq [\{\pi/2\varphi''(t(s))\}^{1/2} + 1/s + \varphi(t(s))/s].\, e^{-L\varphi(s)}$$

◇

Proof : From the fundamental formula of calculus we get:

$$\varphi(t) + ts = L\varphi(s) + (t-t(s))^2 . \int_0^1 d\sigma.(1-\sigma).\varphi''(\sigma.t+(1-\sigma).t(s))$$

We already know that φ'' is positive. Since it is also decreasing, we get :

$$\int_0^{t(s)} dt.e^{-\varphi(t)-ts} \leq e^{-L\varphi(s)} \int_0^\infty dt.e^{-(t-t(s))^2 \varphi''(t(s))/2}$$

and

$$\int_{t(s)}^\infty dt.e^{-\varphi(t)-ts} \geq e^{-L\varphi(s)} \int_0^\infty dt.e^{-(t-t(s))^2 \varphi''(t(s))/2}$$

This gives the lower bound and the first part of the upper bound in (6). For $t > t(s)$ we will estimate the integrand through:

$$\varphi(t) + ts \geq L\varphi(s) \quad \text{for} \quad t(s) < t < L\varphi(s)/s$$

and

$$\varphi(t) + ts \geq t.s \quad \text{for} \quad t \geq L\varphi(s)/s$$

Thus we get:

$$\int_{t(s)}^{L\varphi(s)/s} dt.e^{-\varphi(t)-ts} \leq e^{-L\varphi(s)}.\{\varphi(t(s))/s\}$$

and

$$\int_{L\varphi(s)/s}^\infty dt.e^{-\varphi(t)-ts} \leq e^{-L\varphi(s)}.1/s$$

which achieves the result. ◇

2- Using the function $v(\xi) = w^{-1}(\xi)^D$ we get immediatly:

(7) $$\varphi(t) = \omega_D . Ln(1+1/t). \int_1^\infty d\xi.v'(\xi)\, e^{-t.\xi}$$

Lemma A2 : For any $a > 0$ and $t < a$, φ satisfies the following bounds :

(8) $$\omega_D . Ln(1+1/t).v(a/t).e^{-a} \leq \varphi(t) \leq \omega_D . Ln(1+1/t).v(a/t).\{1+e^{-a}/(a-t)\}$$

◇

Proof : The lower bound is immediatly obtained from (7) by restricting the interval of integration to $[1,a/t]$. For the upper bound we get:

$$\varphi(t) = \omega_D . Ln(1+1/t).\{\int_1^{a/t} d\xi.v'(\xi) + v'(a/t).\int_{a/t}^\infty d\xi.e^{-t\xi}\}$$

for v' is decreasing (since v is concave). This gives the upper bound if we remark that the concavity of v implies:

$$v'(\xi) \leq v(\xi)/(\xi-1) \qquad \text{for all } \xi > 1$$

◇

In much the same way we get the bounds:

(9) $$0 \leq -\varphi'(t) \leq \text{const}.\, Ln(a/t).(a/t).v(a/t)$$
$$0 \leq \varphi''(t) \leq \text{const}.\, Ln(a/t).(a/t)^2.v(a/t) \qquad \text{for } t < a/2$$

On the other hand we have:

$$-\varphi'(t) = \omega_D . [Ln(1+1/t). \int_1^\infty d\xi.\xi v'(\xi)\, e^{-t.\xi} + 1/[t(1+t)]. \int_1^\infty d\xi.v'(\xi)\, e^{-t.\xi}]$$

from which it follows immediatly:

(10) $$-\varphi'(t) \geq \text{const}.\,(a/t).v(a/t) \qquad \text{for } t < a$$

Thus we get the following estimates (where $C_- < 1 < C_+$)

(11) $$C_-.Ln(a/t(s)).v(a/t(s)) \leq L\varphi(s) \leq C_+.Ln(a/t(s)).v(a/t(s))$$

$$C_-.(a/t(s)).v(a/t(s)) \le s \le C_+.Ln(a/t(s)).(a/t(s)).v(a/t(s))$$

for $s > s_a$ such that $t(s_a) \le a/2$.

From the lemma A2 and the previous formulæ it follows that $\varphi(t(s))/s$, $1/s$ and $1/\sqrt{\varphi''(t(s))}$ tend to zero as $s \to \infty$. Thus for s big enough, $L\varphi(s) \le S(s)$. On the other hand we also get from the same estimates

$$Ln[\varphi''(t(s))]/L\varphi(s) \to 0 \qquad \text{as} \qquad s \to \infty$$

Therefore given $C > 1$, there is s_0 big enough such that

$$(12) \qquad L\varphi(s) \le S(s) \le C.L\varphi(s) \qquad \text{for all } s \ge s_0.$$

3- We are now ready to prove the proposition 18. Let s be given in the form $s = L^D.w(L)$ with L big enough. If we set $\xi = w(L)$, we get from (11) with $t = t(s)$

$$\xi v(\xi) \ge C_-.(a/t).v(a/t) \ge C_-.(a/t).v(C_-.a/t)$$

for v is increasing. It follows that $\xi \ge C_-.a/t$. Therefore, if ξ is big enough, we obtain from (11):

$$L\varphi(\xi.v(\xi)) \le C_+.Ln(\xi/C_-).v(\xi/C_-) \le \text{const.}\, Ln(\xi).v(\xi)$$

where again we used the concavity of v in the form

$$(13) \qquad v(\xi_1) \le v(\xi_0).(\xi_1-1)/(\xi_0-1) \qquad \text{if } \xi_1 > \xi_0 > 1$$

We now remark that thanks to the κ-scaling condition, we have

$$Ln(\xi) \le \text{const.}\, v(\xi)^{\kappa}$$

Using (12) we get for L big enough

$$S(L^D.w(L)) \le \text{const.}\, L^{D(1+\kappa)}$$

which is the first inequality to be proved.

Let us choose s in the form $s = L^D.w(L).Ln(w(L))$ with L big enough and let ξ be $w(L)$. From (11) we get immediatly $\xi \le C_+a/t$. Thus using the same trick as before we get for big ξ's:

$$L\varphi(s) \ge C_-Ln(a/t).v(a/t) \ge \text{const.}\, Ln(\xi).v(\xi) \ge \text{const.}\, v(\xi) = \text{const.}\, L^D$$

This implies immediatly the other bound on w.

◇

-APPENDIX B : a proof of Proposition 20-

As announced in the section IV-6, proposition 20, it is necessary to prove the following result :

Lemma B1 : If w satifies the D-concavity and the κ-scaling conditions, together with (if $v(\xi) = w^{-1}(\xi)^D$):

$$(1) \qquad v'(\xi)/v(\xi) \le \text{const.}\, [\xi.Ln^{\beta}(\xi)]^{-1} \qquad \text{for } \xi \text{ big and some } 2/3 < \beta < 1$$

then we have :

$$(2) \qquad \lim_{s\to\infty} s.S'(s).(LnS(s))/S(s) = 0$$

◇

Proof : Using the proof of the proposition 20 we get

$$s.S'(s) = s.t(s) + R(s) \qquad R(s) = \frac{\int_0^{\infty} dt.s(t-t(s)).e^{\varphi(t)-ts}}{\int_0^{\infty} dt.e^{-\varphi(t)-ts}}$$

We claim that R(s) is negligible compare to s.t(s). For indeed cutting the numerator into a sum of two integrals on the intervals $[0,L\varphi(s)/s]$ and $[L\varphi(s)/s,\infty)$, we get :

$$\int_{L\varphi/s}^{\infty} dt.s(t-t(s)).e^{-\varphi(t)-ts} \le \int_{L\varphi/s}^{\infty} dt.s.(t-t(s)).e^{-ts} \le e^{-L\varphi(s)}(\varphi(s)+1)/s$$

The other integrals are treated through a gaussian approximation as in the Appendix A . This leads to :

$$R(s) \leq const.\ \varphi''(t(s))^{1/2}.\{ s/\varphi''(L\varphi/s) + \varphi/s \}$$

Using the estimates of Appendix A, we get with $\xi=1/t$ and provided $t<1$,

$$R(s)/s.t(s) \leq const.\ \frac{Ln^{7/2}\xi.v(\xi)^{1/2}}{v(\xi/Ln\xi)}$$

Using the concavity of v, we get for ξ big enough

$$v(\xi/Ln\xi) \geq const.\ v(\xi)/Ln\xi$$

On the other hand the κ-scaling property implies :

$$Ln\,\xi \leq const.\ v(\xi)^{\kappa}$$

which shows that indeed $R(s)/s.t(s)$ tends to zero as $s\to\infty$.

Thus it remains to show that replacing $s.S'(s)$ by $s.t(s)$ leads to the result. We know that $s = -\varphi'(t(s))$, and that $S(s)$ behaves like $L\varphi(s)$ (App.A Lemma A1) as $s\to\infty$; since $L\varphi = \varphi + s.t(s)$ it follows that to get the result φ must dominate the product $st(s)$ at infinity. Thus it is sufficient to show that

$$Ln\varphi(t).t.|\varphi'(t)|/\varphi(t) \to \infty \quad as \quad t\to 0$$

Going back to the expression of φ' (see App. A, proof of Lemma A2) we get two terms :

$$Ln(L\varphi).t.|\varphi'(t)|/\varphi(t) \leq const.Ln(L\varphi)/Ln(1/t) + t.Ln(L\varphi).N/D$$

where

$$N = \int_0^\infty d\xi.\xi.v'(\xi).e^{-t\xi} \qquad D = \int_0^\infty d\xi.v'(\xi).e^{-t\xi}$$

As in App. A we get

$$D \geq const.\ v(1/t)$$

whereas we get for N the following estimate :

if $\xi_0 = (t.Ln^\gamma v)^{-1}$ with $v = v(1/t)$

$$N \leq \int_0^{\xi_0} d\xi.\xi.v'(\xi).e^{-t\xi} + \int_{\xi_0}^\infty d\xi.\xi.v'(\xi).e^{-t\xi} \leq \xi_0.v(\xi_0) + v'(\xi_0)/t^2$$

Using the hypothesis on v, the second term is dominated by

$$v'(\xi_0) \leq const.v(\xi_0).\{\xi_0.Ln^\beta(\xi_0)\}^{-1}$$

If we set $\xi = 1/t$, and we remark that v is increasing we get $v(\xi_0) \leq v$ and

$$N \leq const.\xi.v/Ln^\gamma(v) + const.\xi.v.Ln^\gamma(v)/Ln^\beta(\xi)$$

On the other hand we have

$$L\varphi \leq const.(Ln\xi).v \leq const.\ v^{1+\kappa}$$

from the κ-scaling. Thus patching together all the estimates leads to :

$$t.Ln(L\varphi).N/D \leq const.\ Ln^{1-\gamma}(v) + const.\ Ln^{1+\gamma}(v)/Ln^\beta(\xi)$$

Now we remark that thanks to (1), we get by integration

$$Ln\,v \leq const.\ Ln^{1-\beta}(\xi) \qquad \text{for } \xi \text{ big enough}$$

Hence as $t\to 0$, the right hand side converges to zero provided $1<\gamma$ and $\beta/(\gamma+1) > 1-\beta$. This implies $\beta > 2/3$ and in this case it suffices to choose

$$1 < \gamma < (2\beta-1)/(\beta-1)$$

In much the same way,

$$Ln(L\varphi)/Ln(1/t) \leq const.\ Lnv/Ln\xi$$

which converges to zero provided $\beta < 1$. This achieves the result . ◇

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