Random Matrices with Complex Gaussian Entries

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Introduction

The eigenvalue distribution of a selfadjoint random $n \times n$ matrix A, for n large, was first studied by E. Wigner in 1955, and has since then been an active research area in mathematical physics (see Mehta's book [Meh] and references given there).

For a rectangular random $m \times n$ matrix B, the eigenvalue distribution of B^tB (or B^*B in the complex case), has in fact been studied for much longer by probabilists and statisticians, starting with the work of J. Wishart (1928) and P.L. Hsu (1939) (see [An], [Mu] and [Se]). However, the asymptotic eigenvalue distribution for random matrices of the form B^tB, when both m and n are large numbers, was first studied in the papers of Wachter, Grenander, Silverstein and Jonsson (cf. [Wa2], [GS] and [Jo]) from the period 1977-1982. A problem of particular interest has been the asymptotic behavior of the largest and the smallest eigenvalue of B^tB (cf. [Gem], [Si], [YBK] and [BY]).

Many of the results, in the papers cited above, on the asymptotic eigenvalue distribution of B^tB for large m and n, were obtained by very complicated combinatorial methods. These papers deal only with random matrices with real valued entries, but this is not an essential problem; the proofs can be generalized to the complex case without much extra effort.

In this paper, we give a new and entirely analytical treatment of some of the key results on asymptotic eigenvalue distributions, both for selfadjoint random matrices A (the *Wigner case*), and for matrices of the form B^*B (the *Wishart case*), under the extra assumption, that the entries of A and B are complex Gaussian random variables. By focusing on the complex Gaussian case, we have been able to obtain both simpler proofs and stronger results – particularly in the Wishart case – than one can obtain for more general random matrices. Our treatment is based on the derivation of explicit formulas for the mean values $\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big)$ in the Wigner case, and $\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big)$ in the Wishart case, as functions of a complex parameter s.

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The results and methods of this paper play a key role in our two papers (in preparation), [HT] and [Th], in which we apply random matrix techniques to obtain results in C^*-algebra theory.

As indicated above, we shall study two classes of complex Gaussian random matrices:

The first class, denoted $\mathrm{SGRM}(n,\sigma^2)$, is a class of selfadjoint $n\times n$ random matrices $A=(a_{jk})$, satisfying that the entries a_{jk}, $1\le j\le k\le n$, form a set of $\frac{1}{2}n(n+1)$ independent, Gaussian random variables, which are complex valued whenever $j<k$, and fulfill that

$$\mathbb{E}(a_{jk})=0, \quad \text{and} \quad \mathbb{E}(|a_{jk}|^2)=\sigma^2, \qquad \text{for all } j,k,$$

(cf. Definition 1.1 for details). The case $\sigma^2=\frac{1}{2}$ gives the normalization used by Wigner in [Wig3] and by Mehta in [Meh], whereas the case $\sigma^2=\frac{1}{n}$ yields the normalization used by Voiculescu in [Vo].

The second class, denoted $\mathrm{GRM}(m,n,\sigma^2)$, is a class of $m\times n$ random matrices $B=(b_{jk})$, for which the entries b_{jk}, $1\le j\le m$, $1\le k\le n$, form a set of mn independent, complex valued, Gaussian random variables, satisfying that

$$\mathbb{E}(b_{jk})=0, \quad \text{and} \quad \mathbb{E}(|b_{jk}|^2)=\sigma^2, \qquad \text{for all } j,k,$$

(see Definition 5.1 for details). For B in $\mathrm{GRM}(m,n,\sigma^2)$, the distribution of the random matrix B^*B is called the complex Wishart distribution (cf. [Go], [Ja] and [Kh]).

The class $\mathrm{SGRM}(n,\sigma^2)$ of selfadjoint, Gaussian random matrices is treated in Sections 1-4, whereas Sections 5-8 are devoted to the study of the rectangular, Gaussian random matrices in the class $\mathrm{GRM}(m,n,\sigma^2)$. We give next a short description of the contents of each of the sections 1-8.

In [Wig3], Wigner showed that for an element A of $\mathrm{SGRM}(n,\frac{1}{2})$, the "mean density" of the distribution of the eigenvalues of A is given by

$$\tfrac{1}{n}\textstyle\sum_{k=0}^{n-1}\varphi_k(x)^2, \tag{0.1}$$

where $\varphi_0,\varphi_1,\varphi_2,\dots$, is the sequence of Hermite functions. This result is the main objective of Section 1. In Section 2, we use (0.1) to show that for A in $\mathrm{SGRM}(n,\sigma^2)$ and s in $\mathbb{C}$, we have

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big)=n\cdot\exp(\tfrac{\sigma^2s^2}{2})\cdot\Phi(1-n,2;-\sigma^2s^2), \tag{0.2}$$

where Tr_n is the usual unnormalized trace on $M_n(\mathbb{C})$, and Φ is the confluent hypergeometric function (cf. formula (2.10) in Section 2). From (0.2), we obtain a simple proof of Wigner's Semi-circle Law in the sense of "convergence in moments", i.e., for a sequence (X_n) of random matrices, such that $X_n\in\mathrm{SGRM}(n,\frac{1}{n})$ for all n,

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[X_n^p]\big)=\tfrac{1}{2\pi}\int_{-2}^{2}x^p\sqrt{4-x^2}\,dx, \qquad (p\in\mathbb{N}), \tag{0.3}$$

where $\mathrm{tr}_n=\frac{1}{n}\mathrm{Tr}_n$ is the normalized trace on $M_n(\mathbb{C})$.

In Section 3, we apply (0.2) to show, that if (X_n) is a sequence of random matrices, defined on the same probability space, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n, then

$$\begin{aligned} \lim_{n\to\infty} \lambda_{\max}\big(X_n(\omega)\big) &= 2, \qquad \text{for almost all } \omega, & (0.4)\\ \lim_{n\to\infty} \lambda_{\min}\big(X_n(\omega)\big) &= -2, \qquad \text{for almost all } \omega, & (0.5) \end{aligned}$$

where $\lambda_{\max}\big(X_n(\omega)\big)$ and $\lambda_{\min}\big(X_n(\omega)\big)$ denote the largest and smallest eigenvalues of $X_n(\omega)$, for each point ω in the underlying probability space. This result is analogous to results of Geman (cf. [Gem]) and Silverstein (cf. [Si]) for the (real) Wishart case.

We conclude our studies of the class $\mathrm{SGRM}(n, \sigma^2)$ in Section 4, where we apply (0.2) together with the differential equation for the confluent hyper-geometric function, to obtain a recursion formula for the numbers:

$$C(p,n) = \mathbb{E}\big(\mathrm{Tr}_n[A^{2p}]\big), \qquad (A \in \mathrm{SGRM}(n,1),\ p \in \mathbb{N}),$$

namely

$$C(p+1,n) = n \cdot \tfrac{4p+2}{p+2} \cdot C(p,n) + \tfrac{p(4p^2-1)}{p+2} \cdot C(p-1,n), \tag{0.6}$$

(cf. Theorem 4.1). This gives a new proof of a recursion formula due to Harer and Zagier (cf. [HZ]).

In Section 5 we apply a results of Bronk, Goodman and James (cf. [Bro], [Go] and [Ja]) to show that for an element B of $\mathrm{GRM}(m,n,1)$, where $m \geq n$, the mean density of the distribution of the eigenvalues of B^*B is given by

$$\tfrac{1}{n} \textstyle\sum_{k=0}^{n-1} \varphi_k^{m-n}(x)^2, \tag{0.7}$$

where, for any non-negative α, $\varphi_0^\alpha, \varphi_1^\alpha, \varphi_2^\alpha, \ldots,$ is a particular orthonormal sequence in $L_2(\mathbb{R}_+)$. The functions $\varphi_0^\alpha, \varphi_1^\alpha, \varphi_2^\alpha, \ldots,$ can be expressed in terms of the generalized Laguerre polynomials, $L_0^\alpha, L_1^\alpha, L_2^\alpha, \ldots,$ of order α, as follows:

$$\varphi_k^\alpha(x) = \big[\tfrac{k!}{\Gamma(k+\alpha+1)} x^\alpha \exp(-x)\big]^{\frac{1}{2}} L_k^\alpha(x). \tag{0.8}$$

From (0.7) and (0.8) we derive in Section 6 the following two formulas:

If $m \geq n$, $B \in \mathrm{GRM}(m,n,1)$, and $s \in \mathbb{C}$ such that $\mathrm{Re}(s) < n$,

$$\begin{aligned} \mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big) &= \sum_{k=1}^{n} \frac{F(k-m, k-n, 1; s^2)}{(1-s)^{m+n+1-2k}}, & (0.9)\\ \mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big) &= mn\frac{F(1-m, 1-n, 2; s^2)}{(1-s)^{m+n}}, & (0.10) \end{aligned}$$

where $F(a,b,c;z)$ is the hyper-geometric function (cf. formula (6.8) in Section 6). From (0.10) we deduce the complex counterpart of results of Grenander and Silverstein (cf. [GS]), Jonsson (cf. [Jo]) and Wachter (cf. [Wa1]):

Let (Y_n) be a sequence of random matrices, such that for all n, $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$, where $m(n) \geq n$. Then, if $\lim_{n\to\infty} \frac{m(n)}{n} = c$, the mean distribution of the eigenvalues of $Y_n^* Y_n$ converges in moments to the probability measure μ_c on $[0, \infty[$ with density

$$\frac{d\mu_c(x)}{dx} = \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x),$$

where $a = \left(\sqrt{c} - 1\right)^2$ and $b = \left(\sqrt{c} + 1\right)^2$. Specificly,

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[(Y_n^* Y_n)^p]\big) = \int_a^b x^p \; d\mu_c(x), \qquad (p \in \mathbb{N}), \tag{0.11}$$

(cf. Theorem 6.7).

In Section 7, we use (0.9) to prove the complex versions of results of Geman (cf. [Gem]) and Silverstein (cf. [Si]): If (Y_n) is a sequence of random matrices, defined on the same probability space, and such that $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$ for all n, then, if $m(n) \geq n$ for all n, and $\lim_{n\to\infty} \frac{m(n)}{n} = c$, we have that

$$\lim_{n\to\infty} \lambda_{\max}(Y_n^* Y_n) = \left(\sqrt{c}+1\right)^2, \qquad \text{almost surely}, \tag{0.12}$$

$$\lim_{n\to\infty} \lambda_{\min}(Y_n^* Y_n) = \left(\sqrt{c}-1\right)^2, \qquad \text{almost surely}. \tag{0.13}$$

Finally, in Section 8, we use (0.10) combined with the differential equation for the hypergeometric function, to derive a recursion formula for the numbers:

$$D(p, m, n) = \mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big), \qquad (B \in \mathrm{GRM}(m, n, 1), \; p \in \mathbb{N}),$$

namely

$$D(p+1, m, n) = \tfrac{(2p+1)(m+n)}{p+2} \cdot D(p, m, n) + \tfrac{(p-1)(p^2-(m-n)^2)}{p+2} \cdot D(p-1, m, n). \tag{0.14}$$

The recursion formula for the moments of the measure μ_c, discovered by Oravecz and Petz in [OP], can be considered as a limit case of (0.14).

It would be interesting to know the counterparts of the explicit formulas (0.2), (0.6), (0.9), (0.10) and (0.14), for random matrices with real or quaternionic Gaussian entries. The counterpart of the Wigner density (cf. (0.1)) for real and quaternionic, selfadjoint, Gaussian random matrices, can be found in Mehta's book [Meh], but to our knowledge, the counterparts of Bronk's density (cf. (0.7)) for real and quaternionic Wishart matrices B^*B, have not been computed. In [HSS], Hanlon, Stanley and Stembridge compute explicitly the moments $\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big)$, $p \leq 4$, in the real, complex and quaternionic cases.

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1 Selfadjoint Gaussian Random Matrices

Recall first, that for ξ in $\mathbb{R}$ and σ^2 in $]0,\infty[$, $N(\xi,\sigma^2)$ denotes the Gaussian distribution with mean ξ and variance σ^2.

1.1 Definition. Let $(\Omega,\mathcal{F},P)$ be a (classical) probability space, let n be a positive integer and let $A\colon \Omega \to M_n(\mathbb{C})$ be a complex random $n\times n$-matrix defined on Ω. For i,j in $\{1,2,\dots,2p\}$, let $a(i,j)$ denote the entry at position (i,j) of A. We say that A is a (standard) selfadjoint Gaussian random $n\times n$-matrix with entries of variance σ^2, if the following conditions are satisfied:

(i) The entries $a(k,l), 1\le k\le l\le n$, form a set of $\frac{1}{2}n(n+1)$ independent, complex valued random variables.

(ii) For each k in $\{1,2,\dots,n\}$, $a(k,k)$ is a real valued random variable with distribution $N(0,\sigma^2)$.

(iii) When $k<l$, the real and imaginary parts $\mathrm{Re}(a(k,l))$ and $\mathrm{Im}(a(k,l))$ of $a(k,l)$ are independent, identically distributed random variables with distribution $N(0,\frac{1}{2}\sigma^2)$.

(iv) When $k>l$, $a(k,l)=\overline{a(l,k)}$.

We denote by $\mathrm{SGRM}(n,\sigma^2)$ the set of all such random matrices (defined on Ω). □

Note that if $A=(a(k,l))_{1\le k,l\le n}\in \mathrm{SGRM}(n,\sigma^2)$, then

$$\mathbb{E}(|a(k,l)|^2)=\sigma^2, \qquad (k,l\in\{1,2,\dots,n\}), \tag{1.1}$$

where $\mathbb{E}$ denotes expectation w.r.t. P. Note also, that the distribution of the real valued random variable $a(k,k)$ has density

$$x\mapsto \tfrac{1}{\sqrt{2\pi\sigma^2}}\exp(-\tfrac{x^2}{2\sigma^2}), \qquad (x\in\mathbb{R}), \tag{1.2}$$

w.r.t. Lebesgue measure on $\mathbb{R}$, whereas, if $k<l$, the distribution of the complex valued random variable $a(k,l)$ has density

$$z\mapsto \tfrac{1}{\sigma^2\pi}\exp(-\tfrac{|z|^2}{\sigma^2}), \qquad (z\in\mathbb{C}), \tag{1.3}$$

w.r.t. the Lebesgue measure on $\mathbb{C}$.

For any positive integer n, we denote by $\mathbf{1}_n$ the unit of $M_n(\mathbb{C})$. By tr_n we denote the trace on $M_n(\mathbb{C})$ satisfying that $\mathrm{tr}_n(\mathbf{1}_n)=1$, and we put $\mathrm{Tr}_n=n\cdot\mathrm{tr}_n$. Note that for any $H=(h_{kl})_{1\le k,l\le n}$ in $M_n(\mathbb{C})_{\mathrm{sa}}$, we have that

$$\mathrm{Tr}_n(H^2)=\sum_{k=1}^n h_{kk}+2\sum_{k<l}|h_{kl}|^2. \tag{1.4}$$

It follows thus from (1.2) and (1.3), that the distribution of an element A of $\mathrm{SGRM}(n,\sigma^2)$, has density

$$H \mapsto c_1 \exp(-\tfrac{1}{2\sigma^2}\mathrm{Tr}_n(H^2)), \qquad (H \in M_n(\mathbb{C})_{\mathrm{sa}}), \tag{1.5}$$

w.r.t. the Lebesgue measure

$$dH = \prod_{k=1}^{n} dh_{kk} \prod_{1\le k<l\le n} d(\mathrm{Re}(h_{kl}))\ d(\mathrm{Im}(h_{kl})), \tag{1.6}$$

on $M_n(\mathbb{C})_{\mathrm{sa}}$. The normalization constant c_1 in (1.5) is given by $c_1 = \big(2^{k/2}(\pi\sigma^2)^{k^2/2}\big)^{-1}$.

The following lemma can be extracted from Wigner's paper [Wig3] (see also [Meh, Chapter 5]).

1.2 Lemma. ([Wig3]) *For an element H of $M_n(\mathbb{C})_{\mathrm{sa}}$, denote by $\lambda_1(H) \le \cdots \le \lambda_n(H)$, the ordered eigenvalues of H, and consider then the mapping*

$$\eta\colon H \mapsto (\lambda_1(H), \dots, \lambda_n(H))\colon M_n(\mathbb{C})_{\mathrm{sa}} \to \mathbb{R}^n.$$

Then η transforms the measure on $M_n(\mathbb{C})_{\mathrm{sa}}$ with density given in (1.6) *onto the measure on $\mathbb{R}^n$, which has the following density w.r.t. Lebesgue measure:*

$$(\lambda_1,\dots,\lambda_n) \mapsto c_2 \prod_{1\le j\le k\le n} (\lambda_j-\lambda_k)^2 \cdot 1_\Lambda(\lambda_1,\dots,\lambda_n), \qquad ((\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n), \tag{1.7}$$

where $\Lambda = \{(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n \mid \lambda_1\le\cdots\le\lambda_n\}$ and $c_2 = \pi^{\frac12 n(n-1)}\big(\prod_{j=1}^{n-1} j!\big)^{-1}$.

1.3 Remark. The proof of Lemma 1.2 given in [Wig3] and [Meh] is somewhat heuristic, but it is possible to give a precise mathematical proof along the same lines (cf. [HZ, Section 4]). The formula corresponding to (1.7) for real symmetric matrices was obtained already in 1939 by Hsu (cf. [Hs]). □

A key result of Wigner's paper [Wig3] is the following theorem, which is needed in the subsequent sections of this paper.

1.4 Theorem. ([Wig3]) *Let A be an element of $\mathrm{SGRM}(n,\sigma^2)$. Then the joint distribution of the ordered eigenvalues $\lambda_1(A)\le\lambda_2(A)\le\cdots\le\lambda_n(A)$ of A, has density w.r.t. Lebesgue measure on $\mathbb{R}^n$, given by*

$$(\lambda_1,\dots,\lambda_n) \mapsto c_3 \cdot \Big(\prod_{1\le j<k\le n}(\lambda_j-\lambda_k)^2\Big)\cdot \exp(-\tfrac{1}{2\sigma^2}\textstyle\sum_{j=1}^n \lambda_j^2)\cdot 1_\Lambda(\lambda_1,\dots,\lambda_n), \tag{1.8}$$

where the normalization constant c_3 is given by

$$c_3 = \Big((2\pi)^{n/2}\big(\textstyle\prod_{j=1}^{n-1} j!\big)\Big)^{-1}. \tag{1.9}$$

Let $g_{n,\sigma^2}\colon \mathbb{R}^n \to \mathbb{R}$ be the density function obtained by taking the average of the function in (1.8) over all permutations of $\lambda_1, \dots, \lambda_n$, i.e.,

$$g_{n,\sigma^2}(\lambda_1,\dots,\lambda_n) = \tfrac{c_3}{n!}\cdot\Big(\prod_{1\le j<k\le n}(\lambda_j-\lambda_k)^2\Big)\cdot\exp(-\tfrac{1}{2\sigma^2}\textstyle\sum_{j=1}^n\lambda_j^2), \quad ((\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n). \tag{1.10}$$

Then the function

$$h_{n,\sigma^2}(\lambda) = \int_{\mathbb{R}^{n-1}} g_{n,\sigma^2}(\lambda,\lambda_2,\dots,\lambda_n)\,d\lambda_2\cdots d\lambda_n, \qquad (\lambda\in\mathbb{R}), \tag{1.11}$$

is given by

$$h_{n,\sigma^2}(\lambda) = \tfrac{1}{n\sigma\sqrt{2}}\textstyle\sum_{k=0}^{n-1}\big[\varphi_k(\tfrac{\lambda}{\sigma\sqrt{2}})\big]^2, \qquad (\lambda\in\mathbb{R}), \tag{1.12}$$

where $\varphi_0, \varphi_1, \varphi_3, \dots,$ is the sequence of Hermite functions:

$$\varphi_k(x) = \tfrac{1}{(2^k k!\sqrt{\pi})^{1/2}}H_k(x)\exp(-\tfrac{x^2}{2}), \qquad (k\in\mathbb{N}_0), \tag{1.13}$$

and $H_0, H_1, H_2, \dots,$ are the Hermite polynomials:

$$H_k(x) = (-1)^k\exp(x^2)\cdot\Big(\frac{d^k}{dx^k}\exp(-x^2)\Big), \qquad (k\in\mathbb{N}_0), \tag{1.14}$$

(cf. [HTF, Vol. 2, p.193, formula (7)]).

1.5 Remark. Note that (1.8) and (1.9) are simple consequences of (1.5) and Lemma 1.2. The proof of (1.12), in the case $\sigma^2 = \frac{1}{2}$, can be found [Wig3] and in Mehta's book [Meh, Chapter 5]. The general case follows from this, via the simple observation, that if $A \in \mathrm{SGRM}(n,\sigma^2)$, then $\frac{1}{\sqrt{2}\sigma}A \in \mathrm{SGRM}(n,\frac{1}{2})$. The mentioned proofs in [Wig3] and [Meh] both rely on the Vandermonde determinant formula:

$$\prod_{1\le j<k\le n}(\lambda_k-\lambda_j) = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}\end{pmatrix}, \tag{1.15}$$

and on the orthogonality relation for the Hermite functions:

$$\int_{\mathbb{R}} \varphi_n(x)\varphi_m(x)\,dx = \delta_{n,m}, \qquad (n,m\in\mathbb{N}). \tag{1.16}$$

It can be recommended to read the proof of (1.12) from Wigner's original paper [Wig3], because the proof given in [Meh] is more complicated, since it is extracted as a special case of a much stronger result. □

1.6 Corollary. *Let $f\colon \mathbb{R}\to\mathbb{R}$ be a Borel function, and let $a\mapsto f(a)$ be the map from $M_n(\mathbb{C})_{\mathrm{sa}}$ into itself, obtained by the usual function calculus for selfadjoint operators on Hilbert space. Consider furthermore the function h_{n,σ^2} given by (1.12). Then for any element A of* $\mathrm{SGRM}(n,\sigma^2)$*, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[f(A)]\big) = n\int_{\mathbb{R}} f(\lambda)h_{n,\sigma^2}(\lambda)\;d\lambda, \tag{1.17}$$

provided that the integral on the right hand side of (1.17) *is well-defined (i.e., $f\ge 0$ or $\int_{\mathbb{R}}|f(\lambda)|h_{n,\sigma^2}(\lambda)\;d\lambda<\infty$).*

Proof. Assume first that $f\ge 0$. Since

$$\mathrm{Tr}_n[f(A)] = f(\lambda_1(A))+\cdots+f(\lambda_n(A)),$$

is a symmetric function of the eigenvalues $\lambda_1(A),\dots,\lambda_n(A)$, it follows from Theorem 1.4, that

$$\mathbb{E}\big(\mathrm{Tr}_n[f(A)]\big) = \int_{\mathbb{R}^n}\big(\textstyle\sum_{j=1}^n f(\lambda_j)\big)\cdot g_{n,\sigma^2}(\lambda_1,\dots,\lambda_n)\;d\lambda_1\cdots d\lambda_n.$$

Using then that g_{n,σ^2} is invariant under permutations of $\lambda_1,\dots,\lambda_n$, it follows that

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[f(A)]\big) &= n\cdot\int_{\mathbb{R}^n} f(\lambda_1)\cdot g_{n,\sigma^2}(\lambda_1,\dots,\lambda_n)\;d\lambda_1\cdots d\lambda_n\\ &= n\int_{\mathbb{R}} f(\lambda)h_{n,\sigma^2}(\lambda)\;d\lambda,\end{aligned}$$

which proves that (1.17) holds whenever $f\ge 0$. For general, complex valued Borel functions f, satisfying that $\int_{\mathbb{R}}|f(\lambda)|h_{n,\sigma^2}(\lambda)\;d\lambda<\infty$, (1.17) follows then from the positive case, and the standard decomposition:

$$f = (\mathrm{Re}f)^+ - (\mathrm{Re}f)^- + i\big((\mathrm{Im}f)^+ - (\mathrm{Im}f)^-\big). \qquad \blacksquare$$

2 Wigner's Semi-circle Law

A real valued random variable is said to be semi-circular distributed with mean μ and variance $\sigma^2>0$, if its distribution has the following density w.r.t. Lebesgue measure:

$$x\mapsto \tfrac{1}{2\pi\sigma^2}\sqrt{4\sigma^2-(x-\mu)^2}\cdot 1_{[-2\sigma,2\sigma]}(x), \qquad (x\in\mathbb{R}). \tag{2.1}$$

It is easily checked that the mean and variance of this distribution are in fact μ and σ^2 respectively. We shall mostly consider the case where $\mu=1$ and $\sigma^2=1$ (the standard semi-circular distribution), in which case the density (2.1) becomes

$$x\mapsto \tfrac{1}{2\pi}\sqrt{4-x^2}\cdot 1_{[-2,2]}(x), \qquad (x\in\mathbb{R}). \tag{2.2}$$

Wigner's semi-circle Law asserts, that for large n, the distribution of eigenvalues of an element A of $\mathrm{SGRM}(n,\sigma^2)$ is approximately a semi-circular distribution with mean 0 and variance $n\sigma^2$ (cf. [Wig1], [Wig3] and [Meh]). Wigner did not state his result in precise mathematical terms, but this has been done subsequently by several people; see for example [Gr, pp. 178-180], [Ar] and [Vo].

In [Vo], Voiculescu considered a sequence (X_n) of random matrices, such that for each n, $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$. Voiculescu proved in this case that

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n(X_n^p)\big) = \tfrac{1}{2\pi}\int_{-2}^{2} x^p\sqrt{4-x^2}\; dx, \qquad (p \in \mathbb{N}).$$

By Corollary 1.6, this means that the probability measures μ_n with densities

$$h_{n,\frac{1}{n}}(x) = \tfrac{1}{\sqrt{2n}} \textstyle\sum_{k=0}^{n-1} \varphi_k\big(\sqrt{\tfrac{n}{2}}x\big), \qquad (x \in \mathbb{R}), \tag{2.3}$$

obtained by putting $\sigma^2 = \frac{1}{n}$ in (1.12), converge "in moments" (see Definition 2.4 below) to the standard semi-circular distribution. Below we shall give another proof of this result, based on the study of special functions. We start by quoting a classical result from probability theory:

2.1 Proposition. *Let $\mu, \mu_1, \mu_2, \mu_3, \ldots,$ be probability measures on $\mathbb{R}$, and consider the corresponding distribution functions:*

$$F(x) = \mu\big(]-\infty, x]\big), \quad F_n(x) = \mu_n\big(]-\infty, x]\big), \qquad (x \in \mathbb{R},\ n \in \mathbb{N}).$$

Let $C_0(\mathbb{R})$ and $C_b(\mathbb{R})$ denote the set of continuous functions on $\mathbb{R}$ that vanish at $\pm\infty$, respectively the set of continuous, bounded functions on $\mathbb{R}$.

Then the following conditions are equivalent:

(i) $\lim_{n\to\infty} F_n(x) = F(x)$ *for all points x of $\mathbb{R}$ in which F is continuous.*

(ii) $\forall f \in C_0(\mathbb{R})\colon\ \lim_{n\to\infty} \int_{\mathbb{R}} f\ d\mu_n = \int_{\mathbb{R}} f\ d\mu.$

(iii) $\forall f \in C_b(\mathbb{R})\colon\ \lim_{n\to\infty} \int_{\mathbb{R}} f\ d\mu_n = \int_{\mathbb{R}} f\ d\mu.$

(iv) $\forall t \in \mathbb{R}\colon\ \lim_{n\to\infty} \int_{\mathbb{R}} \exp(itx)\ d\mu_n(x) = \int_{\mathbb{R}} \exp(itx)\ d\mu(x).$

Proof. Cf. [Fe, Chapter VIII: Criterion 1,Theorem 1,Theorem 2 and Chapter XV: Theorem 2]. ∎

2.2 Definition. Let $\mu, \mu_1, \mu_2, \mu_3, \ldots,$ be probabilities on $\mathbb{R}$. If (i) (and hence all of the conditions (i)-(iv)) in Proposition 2.1 is satisfied, then we say that μ_n converges weakly to μ. □

2.3 Remark. Condition (i) in Proposition 2.1 actually implies that $\mu_n(I) \to \mu(I)$, as $n \to \infty$, for any interval I in $\mathbb{R}$ for which F is continuous in both endpoints of I (here $\pm\infty$ should be considered as points of continuity for F). In particular, $\mu_n(I) \to \mu(I),\ n \to \infty$, for any interval I, if μ does not have any atoms, i.e., if $\mu(\{x\}) = 0$ for any x in $\mathbb{R}$. □

2.4 Definition. Let $\mu, \mu_1, \mu_2, \mu_3, \dots,$ be probabilities on $\mathbb{R}$, which have moments of all orders, i.e.,

$$\int_{\mathbb{R}} |x|^p \, d\mu(x) < \infty, \quad \text{and} \quad \int_{\mathbb{R}} |x|^p \, d\mu_n(x) < \infty, \qquad (p, n \in \mathbb{N}).$$

We say then that μ_n converges to μ in moments, if

$$\int_{\mathbb{R}} x^p \, d\mu_n(x) \to \int_{\mathbb{R}} x^p \, d\mu(x), \qquad (p \in \mathbb{N}). \qquad \square$$

2.5 Lemma. *Let* (φ_n) *denote the sequence of Hermite functions introduced in* (1.13) *of Theorem 1.4. We then have*

$$\varphi_0'(x) = -\tfrac{1}{\sqrt{2}}\varphi_1(x), \tag{2.4}$$

$$\varphi_n'(x) = \sqrt{\tfrac{n}{2}}\varphi_{n-1}(x) - \sqrt{\tfrac{n+1}{2}}\varphi_{n+1}(x), \qquad (n \in \mathbb{N}), \tag{2.5}$$

$$\frac{d}{dx}\Big(\sum_{k=0}^{n-1} \varphi_k(x)^2\Big) = -\sqrt{2n}\varphi_n(x)\varphi_{n-1}(x), \qquad (n \in \mathbb{N}). \tag{2.6}$$

Proof. The equations (2.4) and (2.5) follow from (1.13) and the elementary formulas

$$xH_n(x) = \tfrac{1}{2}H_{n+1} + nH_{n-1}(x), \tag{2.7}$$

$$H_n'(x) = 2nH_{n-1}(x), \tag{2.8}$$

(cf. [HTF, Vol. 2, p. 193, formulas (10) and (14)]). Moreover, (2.6) is easily derived from (2.4) and (2.5). ■

For any non-negative integer n, and any complex number w, we apply the notation

$$(w)_n = \begin{cases} 1, & \text{if } n = 0, \\ w(w+1)(w+2)\cdots(w+n-1), & \text{if } n \in \mathbb{N}. \end{cases} \tag{2.9}$$

Recall then, that the confluent hyper-geometric function $(a, c, x) \mapsto \Phi(a, c; x)$ is defined by the expression:

$$\Phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!} = 1 + \frac{a}{c}\frac{x}{1} + \frac{a(a+1)}{c(c+1)}\frac{x^2}{2} + \cdots, \tag{2.10}$$

for a, c, x in $\mathbb{C}$, such that $c \notin \mathbb{Z} \setminus \mathbb{N}$ (cf. [HTF, Vol. 1, p.248]). Note, in particular, that if $a \in \mathbb{Z} \setminus \mathbb{N}$, then $x \mapsto \Phi(a, c; x)$ is a polynomial in x of degree $-a$, for any permitted c.

2.6 Lemma. *For any* s *in* $\mathbb{C}$ *and* k *in* $\mathbb{N}_0$,

$$\begin{aligned} \int_{\mathbb{R}} \exp(sx)\varphi_k(x)^2 \, dx &= \exp(\tfrac{s^2}{4})\Phi(-k, 1; -\tfrac{s^2}{2}) \\ &= \exp(\tfrac{s^2}{4})\sum_{j=0}^{k} \frac{k(k-1)\cdots(k+1-j)}{(j!)^2}\Big(\frac{s^2}{2}\Big)^j, \end{aligned} \tag{2.11}$$

and for s in $\mathbb{C}$ and n in $\mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{R}} \exp(sx)\big(\textstyle\sum_{k=0}^{n-1}\varphi_k(x)^2\big)\ dx & \\ &= n\cdot\exp(\tfrac{s^2}{4})\Phi(1-n,2;-\tfrac{s^2}{2}) \\ &= n\cdot\exp(\tfrac{s^2}{4})\sum_{j=0}^{n-1}\frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!}\Big(\frac{s^2}{2}\Big)^j.\end{aligned} \tag{2.12}$$

Proof. For l,m in $\mathbb{N}_0$ and s in $\mathbb{R}$, we have that

$$\int_{\mathbb{R}} \exp(sx)\varphi_l(x)\varphi_m(x)\ dx = \frac{1}{(2^{l+m}l!m!\pi)^{1/2}}\int_{\mathbb{R}}\exp(sx-x^2)H_l(x)H_m(x)\ dx. \tag{2.13}$$

By the substitution $y=x-\frac{s}{2}$, the integral on the right hand side of (2.13) becomes

$$\exp(\tfrac{s^2}{4})\int_{\mathbb{R}}\exp(-y^2)H_l(y+\tfrac{s}{2})H_l(y+\tfrac{s}{2})\ dy. \tag{2.14}$$

Note here, that by (1.14) we have for a in $\mathbb{R}$ and k in $\mathbb{N}_0$,

$$\begin{aligned}H_k(x+a) &= (-1)^k\exp((x+a)^2)\cdot\Big(\frac{d^k}{dx^k}\exp(-(x+a)^2)\Big) \\ &= (-1)^k\exp(x^2+2ax)\sum_{j=0}^{k}\binom{k}{j}\Big(\frac{d^j}{dx^j}\exp(-x^2)\Big)\Big(\frac{d^{k-j}}{dx^{k-j}}\exp(-2ax)\Big),\end{aligned}$$

which can be reduced to

$$H_k(x+a)=\sum_{j=0}^{k}\binom{k}{j}(2a)^{k-j}H_j(x). \tag{2.15}$$

It follows thus that the quantity in (2.14) equals

$$\exp(\tfrac{s^2}{4})\int_{\mathbb{R}}\exp(-y^2)\Big(\sum_{j=0}^{l}\binom{l}{j}(s)^{l-j}H_j(y)\Big)\Big(\sum_{j=0}^{m}\binom{m}{j}(s)^{m-j}H_j(y)\Big)\ dy,$$

which by the orthogonality relations (1.16) can be reduced to

$$\exp(\tfrac{s^2}{4})\sum_{j=0}^{\min\{l,m\}}\binom{l}{j}\binom{m}{j}2^j j!\sqrt{\pi}\ s^{l+m-2j}.$$

Altogether, we have shown that for m,l in $\mathbb{N}_0$ and s in $\mathbb{R}$,

$$\int_{\mathbb{R}}\exp(sx)\varphi_l(x)\varphi_m(x)\ dx = \frac{\exp(\frac{s^2}{4})}{\sqrt{l!m!}}\sum_{j=0}^{\min\{l,m\}} j!\binom{l}{j}\binom{m}{j}\Big(\frac{s}{\sqrt{2}}\Big)^{l+m-2j}. \tag{2.16}$$

But since both sides of (2.16) are analytical functions of $s \in \mathbb{C}$, the formula (2.16) holds for all s in $\mathbb{C}$.

Putting now $l = m = k$, and substituting j by $k - j$, (2.16) becomes

$$\begin{aligned}\int_{\mathbb{R}} \exp(sx)\varphi_k(x)^2 \, dx &= \frac{\exp(\frac{s^2}{4})}{k!} \sum_{j=0}^{k} (k-j)! \binom{k}{j}^2 \Big(\frac{s}{\sqrt{2}}\Big)^{2j} \\ &= \exp(\tfrac{s^2}{4}) \sum_{j=0}^{k} \frac{k(k-1)\cdots(k+1-j)}{(j!)^2} \Big(\frac{s^2}{2}\Big)^j,\end{aligned}$$

and this proves (2.11).

The formula (2.12) is trivial in the case $s = 0$, because of the orthogonality relations (1.16). If $s \in \mathbb{C} \setminus \{0\}$, then by (2.6) and partial integration, we get that

$$\int_{\mathbb{R}} \exp(sx)\big(\textstyle\sum_{k=0}^{n-1} \varphi_k(x)^2\big) \, dx = \frac{\sqrt{2n}}{s} \displaystyle\int_{\mathbb{R}} \exp(sx)\varphi_n(x)\varphi_{n-1}(x) \, dx.$$

Using now (2.16) in the case $l = n, \ m = n - 1$, we get, after substituting j by $n - 1 - j$, that

$$\begin{aligned}\frac{\sqrt{2n}}{s} \int_{\mathbb{R}} \exp(sx)\varphi_n(x)\varphi_{n-1}(x) \, dx &= \frac{\sqrt{2}\exp(\frac{s^2}{4})}{s(n-1)!} \sum_{j=0}^{n-1} (n-1-j)! \binom{n}{j+1} \binom{n-1}{j} \Big(\frac{s}{\sqrt{2}}\Big)^{2j+1} \\ &= n \exp(\tfrac{s^2}{4}) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \Big(\frac{s^2}{2}\Big)^j,\end{aligned}$$

and (2.12) follows. ∎

2.7 Lemma. (i) *For any element A of* $\mathrm{SGRM}(n, \sigma^2)$ *and any s in $\mathbb{C}$, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = n \cdot \exp(\tfrac{\sigma^2 s^2}{2}) \cdot \Phi(1 - n, 2; -\sigma^2 s^2). \tag{2.17}$$

(ii) *Let (X_n) be a sequence of random matrices, such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n in $\mathbb{N}$. Then for any s in $\mathbb{C}$, we have that*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) = \frac{1}{2\pi} \int_{-2}^{2} \exp(sx)\sqrt{4 - x^2} \, dx, \tag{2.18}$$

and the convergence is uniform on compact subsets of $\mathbb{C}$.

Proof. From Corollary 1.6 and (2.12), we get by a simple substitution, that

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) &= \int_{\mathbb{R}} \exp(s\lambda) h_{n,\sigma^2}(\lambda) \, d\lambda \\ &= n \cdot \exp(\tfrac{\sigma^2 s^2}{2}) \cdot \Phi(1 - n, 2; -\sigma^2 s^2),\end{aligned}$$

for any s in $\mathbb{C}$, and this proves (i).

By application of (i), it follows then, that for X_n from $\mathrm{SGRM}(n, \frac{1}{n})$ and s in $\mathbb{C}$, we have that

$$\begin{aligned}\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) &= \exp(\tfrac{s^2}{2n}) \cdot \Phi(1-n, 2; -\tfrac{s^2}{n}) \\ &= \exp(\tfrac{s^2}{2n}) \sum_{j=0}^{n-1} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \Big(\frac{s^2}{n}\Big)^j.\end{aligned} \tag{2.19}$$

By Lebesgue's Theorem on Dominated Convergence (for series), it follows thus that

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) = \sum_{j=0}^{\infty} \frac{s^{2j}}{j!(j+1)!}.$$

The even moments of the standard semi-circular distribution are:

$$\tfrac{1}{2\pi} \int_{\mathbb{R}} x^{2p} \sqrt{4-x^2}\, dx = \tfrac{1}{p+1} \binom{2p}{p}, \qquad (p \in \mathbb{N}_0),$$

and the odd moments vanish. Hence, using the power series expansion of $\exp(sx)$, we find that

$$\tfrac{1}{2\pi} \int_{-2}^{2} \exp(sx) \sqrt{4-x^2}\, dx = \sum_{j=0}^{\infty} \frac{s^{2j}}{(2j)!(j+1)} \binom{2j}{j} = \sum_{j=0}^{\infty} \frac{s^{2j}}{j!(j+1)!}.$$

Therefore,

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big) = \tfrac{1}{2\pi} \int_{-2}^{2} \exp(sx) \sqrt{4-x^2}\, dx, \qquad (s \in \mathbb{C}). \tag{2.20}$$

Note next, that by (2.19), we have that

$$\big|\mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big)\big| \le \sum_{j=0}^{\infty} \frac{|s|^{2j}}{j!(j+1)!}, \qquad (s \in \mathbb{C}),$$

so the functions $s \mapsto \mathbb{E}\big(\mathrm{tr}_n[\exp(sX_n)]\big)$, $(n \in \mathbb{N})$, are uniformly bounded on any fixed bounded subset of $\mathbb{C}$. Hence by a standard application of Cauchy's Integral Formula and Lebesgue's theorem on Dominated Convergence, it follows that the convergence in (2.20) is uniform on compact subsets of $\mathbb{C}$. $\blacksquare$

Lemma 2.7(i), will also be needed in Sections 3 and 4. We conclude this section by using Lemma 2.7(ii) to give a short proof of Wigner's Semi-circle Law.

2.8 Theorem. (cf. [Wig3], [Gr], [Vo]) *Let (X_n) be a sequence of random matrices, such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n. We then have*

(i) *For any p in $\mathbb{N}$,*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[X_n^p]\big) = \tfrac{1}{2\pi} \int_{-2}^{2} x^p \sqrt{4-x^2}\, dx. \tag{2.21}$$

(ii) *For every continuous bounded function $f\colon \mathbb{R} \to \mathbb{C}$,*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big) = \tfrac{1}{2\pi} \int_{-2}^{2} f(x) \sqrt{4-x^2}\, dx.$$

Proof. By Corollary 1.6, Lemma 2.7(ii) and the Cauchy Integral Formulas, we have that

$$\lim_{n\to\infty} \frac{d^p}{ds^p}\Big(\int_{\mathbb{R}} \exp(sx)h_{n,\frac{1}{n}}(x)\ dx\Big) = \frac{d^p}{ds^p}\Big(\tfrac{1}{2\pi}\int_{-2}^{2} \exp(sx)\sqrt{4-x^2}\ dx\Big),$$

for all s in $\mathbb{C}$. Putting $s = 0$, it follows that

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[X_n^p]\big) = \lim_{n\to\infty} \Big(\int_{\mathbb{R}} x^p h_{n,\frac{1}{n}}(x)\ dx\Big) = \tfrac{1}{2\pi}\int_{-2}^{2} x^p\sqrt{4-x^2}\ dx,$$

which proves (i). Putting $s = it$ in Lemma 2.7(ii), it follows, that for any t in $\mathbb{R}$,

$$\lim_{n\to\infty} \int_{\mathbb{R}} \exp(itx)h_{n,\frac{1}{n}}(x)\ dx = \int_{\mathbb{R}} \exp(itx)\ d\gamma(x), \tag{2.22}$$

where $d\gamma = \frac{1}{2\pi}\sqrt{4-x^2}\cdot 1_{[-2,2]}(x)\ dx$. Hence by Proposition 2.1,

$$\mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big) = \lim_{n\to\infty} \int_{\mathbb{R}} f(x)h_{n,\frac{1}{n}}(x)\ dx = \int_{\mathbb{R}} f(x)\ d\gamma(x),$$

for any continuous bounded function f on $\mathbb{R}$, and this proves (ii). ■

2.9 Remark. Arnold's strengthening of Wigner's Semi-circle Law to a result about almost sure convergence of the empirical distributions of the eigenvalues (cf. [Ar]), will be taken up in Section 3 (see Proposition 3.6). A good survey of the history of Wigner's Semi-circle Law is given by Olson and Uppuluri in [OU]. □

3 Almost Sure Convergence of the Largest and Smallest Eigenvalues of Selfadjoint, Gaussian Random Matrices

The main result of this section is contained in Theorem 3.1 below. Due to the results of Geman ([Gem]) and Silverstein ([Si]) for the Wishart case (see also Section 7 of this paper), the result is not unexpected, but to our knowledge, a proof of it has not previously been published.

3.1 Theorem. *Let (X_n) be a sequence of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$, for each n in $\mathbb{N}$. For each ω in Ω and n in $\mathbb{N}$, let $\lambda_{\max}\big(X_n(\omega)\big)$ and $\lambda_{\min}\big(X_n(\omega)\big)$ denote the largest respectively the smallest eigenvalue of $X_n(\omega)$. We then have*

$$\lim_{n\to\infty} \lambda_{\max}(X_n) = 2, \qquad \textit{almost surely,} \tag{3.1}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(X_n) = -2, \qquad \textit{almost surely.} \tag{3.2}$$

For the proof of Theorem 3.1, we need some lemmas:

3.2 Lemma. (Borel-Cantelli) *Let $F_1, F_2, F_3, \ldots,$ be a sequence of measurable subsets of Ω, and assume that $\sum_{n=1}^{\infty} P(\Omega \setminus F_n) < \infty$. Then $P(F_n \text{ eventually}) = 1$, where*

$$(F_n \text{ eventually}) = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} F_n,$$

i.e., for almost all ω in Ω, $\omega \in F_n$ eventually as $n \to \infty$.

Proof. Cf. [Bre, Lemma 3.14]. ∎

3.3 Lemma. *Let (X_n) be a sequence of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n in $\mathbb{N}$. We then have,*

$$\limsup_{n\to\infty} \lambda_{\max}(X_n) \leq 2, \qquad \textit{almost surely}, \tag{3.3}$$

and

$$\liminf_{n\to\infty} \lambda_{\min}(X_n) \geq -2, \qquad \textit{almost surely}. \tag{3.4}$$

Proof. By (2.19), we have for any n in $\mathbb{N}$, that

$$\begin{aligned}
\mathbb{E}\big(\mathrm{Tr}_n[\exp(tX_n)]\big) &= n \cdot \exp(\tfrac{t^2}{2n}) \sum_{j=0}^{\infty} \frac{(n-1)(n-2)\cdots(n-j)}{j!(j+1)!} \Big(\frac{t^2}{n}\Big)^j \\
&\leq n \cdot \exp(\tfrac{t^2}{2n}) \sum_{j=0}^{\infty} \frac{t^{2j}}{j!(j+1)!} \\
&\leq n \cdot \exp(\tfrac{t^2}{2n}) \Big[\sum_{j=0}^{\infty} \frac{t^j}{j!}\Big]^2.
\end{aligned}$$

It follows thus, that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tX_n)]\big) \leq n \cdot \exp(\tfrac{t^2}{2n} + 2t), \qquad (t \in \mathbb{R}_+). \tag{3.5}$$

Note here, that since all eigenvalues of $\exp(tX_n)$ are positive, we have that

$$\mathrm{Tr}_n[\exp(tX_n)] \geq \lambda_{\max}(\exp(tX_n)) = \exp(t\lambda_{\max}(X_n)),$$

and hence by integration,

$$\mathbb{E}\big(\exp(t\lambda_{\max}(X_n))\big) \leq n \cdot \exp(\tfrac{t^2}{2n} + 2t), \qquad (t \in \mathbb{R}_+).$$

It follows thus, that for any ϵ in $]0, \infty[$,

$$\begin{aligned}
P(\lambda_{\max}(X_n) \geq 2 + \epsilon) &= P\big(\exp(t\lambda_{\max}(X_n) - t(2+\epsilon)) \geq 1\big) \\
&\leq \mathbb{E}\big(\exp(t\lambda_{\max}(X_n) - t(2+\epsilon))\big) \\
&\leq \exp(-t(2+\epsilon))\mathbb{E}\big(\exp(t\lambda_{\max}(X_n))\big),
\end{aligned}$$

and hence by (3.5),

$$P(\lambda_{\max}(X_n) \geq 2+\epsilon) \leq n \cdot \exp(\tfrac{t^2}{2n} - \epsilon t), \qquad (t \in \mathbb{R}_+). \tag{3.6}$$

As a function of $t \in \mathbb{R}_+$, the right hand side of (3.6) attains its minimum when $t = n\epsilon$. For this value of t, (3.6) becomes,

$$P(\lambda_{\max}(X_n) \geq 2+\epsilon) \leq n \cdot \exp(\tfrac{-n\epsilon^2}{2}).$$

Hence by the Borel-Cantelli Lemma (Lemma 3.2),

$$\limsup_{n\to\infty} \lambda_{\max}(X_n) \leq 2+\epsilon, \qquad \text{almost surely.}$$

Since this holds for arbitrary positive ϵ, we have proved (3.3). We note finally that (3.4) follows from (3.3), since the sequence $(-X_n)$ of random matrices also satisfies that $-X_n \in \mathrm{SGRM}(n, \frac{1}{n})$ for all n. ■

To complete the proof of Theorem 3.1, we shall need an "almost sure convergence version" of Wigner's semi-circle law. This strengthened version of the semi-circle law was proved by Arnold in [Ar]. Arnold's result is proved for real symmetric random matrices, with rather general conditions on the entries. His proof is combinatorial and can be generalized to the complex case. For convenience of the reader, we include below a short proof in the case of complex Gaussian random matrices (cf. Proposition 3.6 below). The proof relies on the following lemma, due to Pisier (cf. [Pi, Theorem 4.7]), which is related to the "concentration of measure phenomenon" (cf. [Mi]).

3.4 Lemma. ([Pi]) *Let $G_{N,\sigma}$ denote the Gaussian distribution on $\mathbb{R}^N$ with density*

$$\frac{dG_{N,\sigma}(x)}{dx} = (2\pi\sigma^2)^{-N/2} \exp(-\tfrac{\|x\|^2}{2\sigma^2}), \tag{3.7}$$

where $\|x\|$ is the Euclidean norm of x. Furthermore, let $F\colon \mathbb{R}^N \to \mathbb{R}$ be a function that satisfies the Lipschitz condition

$$|F(x) - F(y)| \leq c\|x-y\|, \qquad (x, y \in \mathbb{R}^N), \tag{3.8}$$

for some positive constant c. Then for any positive number t, we have that

$$G_{N,\sigma}\big(\{x \in \mathbb{R}^N \mid |F(x) - \mathbb{E}(F)| > t\}\big) \leq 2\exp(-\tfrac{Kt^2}{c^2\sigma^2}),$$

where $\mathbb{E}(F) = \int_{\mathbb{R}^N} F(x)\, dG_{N,\sigma}(x)$, and $K = \frac{2}{\pi^2}$.

Proof. For $\sigma = 1$, this is proved in [Pi, Theorem 4.7], and the general case follows easily from this case, by using that $G_{N,\sigma}$ is the range measure of $G_{N,1}$ under the mapping $x \mapsto \sigma x\colon \mathbb{R}^N \to \mathbb{R}^N$, and that the composed function $x \mapsto F(\sigma x)$, satisfies a Lipschitz condition with constant $c\sigma$. ■

The following result is also well-known:

3.5 Lemma. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that satisfies the Lipschitz condition*

$$|f(s) - f(t)| \le c|s-t|, \qquad (s, t \in \mathbb{R}). \tag{3.9}$$

Then for any n in $\mathbb{N}$, and all matrices A, B in $M_n(\mathbb{C})_{\mathrm{sa}}$, we have that

$$\|f(A) - f(B)\|_{HS} = c\|A - B\|_{HS},$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm, i.e., $\|C\|_{HS} = \mathrm{Tr}_n(C^*C)^{1/2}$, for all C in $M_n(\mathbb{C})_{\mathrm{sa}}$.

Proof. The proof can be extracted from the proof of [Co, Proposition 1.1]: Note first that we may write,

$$A = \sum_{i=1}^n \lambda_i E_i, \quad B = \sum_{i=1}^n \mu_i F_i,$$

where $\lambda_1, \dots, \lambda_n$ and $\mu_1, \dots, \mu_n$ are the eigenvalues of A and B respectively, and where $E_1 \dots, E_n$ and $F_1, \dots, F_n$ are two families of mutually orthogonal one-dimensional projections (adding up to $\mathbf{1}_n$). Using then that $\mathrm{Tr}_n(E_iF_j) \ge 0$ for all i, j, we find that

$$\begin{aligned}
\|f(A) - f(B)\|_{HS}^2 &= \mathrm{Tr}_n(f(A)^2) + \mathrm{Tr}_n(f(B)^2) - 2\mathrm{Tr}_n(f(A)f(B)) \\
&= \sum_{i,j=1}^n (f(\lambda_i) - f(\mu_j))^2 \cdot \mathrm{Tr}_n(E_iF_j) \\
&\le c^2 \cdot \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 \cdot \mathrm{Tr}_n(E_iF_j) \\
&= c^2\|A - B\|_{HS}^2. \qquad \blacksquare
\end{aligned}$$

3.6 Proposition. (cf. [Ar]) *Let (X_n) be a sequence of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, P)$, and such that $X_n \in \mathrm{SGRM}(n, \frac{1}{n})$, for each n in $\mathbb{N}$. For each ω in Ω, let $\mu_{n,\omega}$ denote the empirical distribution of the ordered eigenvalues $\lambda_1(X_n(\omega)) \le \lambda_2(X_n(\omega)) \le \dots \le \lambda_n(X_n(\omega))$, of $X_n(\omega)$, i.e., with the usual Dirac measure notation,*

$$\mu_{n,\omega} = \tfrac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(X_n(\omega))}. \tag{3.10}$$

Then for almost all ω in Ω, $\mu_{n,\omega}$ converges weakly to the standard semi-circular distribution γ, with density $x \mapsto \frac{1}{2\pi}\sqrt{4 - x^2} \cdot 1_{[-2,2]}(x)$.

Hence, for any interval I in $\mathbb{R}$, and almost all ω in Ω, we have that

$$\lim_{n\to\infty} \Big(\tfrac{1}{n} \cdot \mathrm{card}\big(\mathrm{sp}[X_n(\omega)] \cap I\big)\Big) = \gamma(I).$$

Proof. Note first that for any f in $C_0(\mathbb{R})$, we have that

$$\int_{\mathbb{R}} f(x)\ d\mu_{n,\omega}(x) = \mathrm{tr}_n\big[f(X_n(\omega))\big],$$

for all ω in Ω. Hence by Proposition 2.1, it suffices to show, that for almost all ω in Ω, we have that

$$\lim_{n\to\infty} \mathrm{tr}_n\big[f(X_n(\omega))\big] = \int_{\mathbb{R}} f\ d\mu, \qquad \text{for all } f \text{ in } C_0(\mathbb{R}). \tag{3.11}$$

By separability of the Banach space $C_0(\mathbb{R})$, it is enough to check that (3.11) holds almost surely for each fixed f in $C_0(\mathbb{R})$ or for each fixed f in some dense subset of $C_0(\mathbb{R})$. In the following we shall use, as such a dense subset, $C_c^1(\mathbb{R})$, i.e., the set of continuous differentiable functions on $\mathbb{R}$ with compact support. So consider a function f from $C_c^1(\mathbb{R})$, and put

$$F(A) = \mathrm{tr}_n\big[f(A)\big], \qquad (X \in M_n(\mathbb{C})_{\mathrm{sa}}).$$

Then for any A, B in $M_n(\mathbb{C})_{\mathrm{sa}}$, we have that

$$|F(A) - F(B)| \le \tfrac{1}{n}|\mathrm{Tr}_n\big[f(A)\big] - \mathrm{Tr}_n\big[f(B)\big]| \le \tfrac{1}{\sqrt{n}}\|f(A) - f(B)\|_{HS},$$

and since f is Lipschitz with constant $c = \sup_{x\in\mathbb{R}}|f'(x)| < \infty$, it follows then by Lemma 3.5, that

$$|F(A) - F(B)| \le \tfrac{c}{\sqrt{n}}\|A - B\|_{HS}, \qquad (A, B \in M_n(\mathbb{C})_{\mathrm{sa}}). \tag{3.12}$$

The linear bijection $\Phi\colon M_n(\mathbb{C})_{\mathrm{sa}} \to \mathbb{R}^{n^2}$, given by

$$\Phi(A) = \big((a_{ii})_{1\le i\le n}, (\sqrt{2}\mathrm{Re}(a_{ij}))_{1\le i<j\le n}, (\sqrt{2}\mathrm{Im}(a_{ij}))_{1\le i<j\le n}\big), \qquad (A = (a_{ij}) \in M_n(\mathbb{C})_{\mathrm{sa}}),$$

maps the distribution on $M_n(\mathbb{C})_{\mathrm{sa}}$ of an element of $\mathrm{SGRM}(n, \frac{1}{n})$ (cf. (1.5)) onto the joint distribution of n^2 independent, identically distributed random variables with distribution $N(0,1)$, i.e., the distribution $G_{n^2,n^{-1/2}}$ on $\mathbb{R}^{n^2}$ with density

$$\frac{dG_{n^2,n^{-1/2}}(x)}{dx} = \Big(\frac{n}{2\pi}\Big)^{-n^2/2}\exp(-\tfrac{n\|x\|^2}{2}), \qquad (x \in \mathbb{R}^{n^2}).$$

Moreover, the Euclidean norm on $\mathbb{R}^{n^2}$ corresponds, via the mapping Φ, to the Hilbert-Schmidt norm on $M_n(\mathbb{C})_{\mathrm{sa}}$. Hence by (3.12) and Lemma 3.4, we get for any positive t, that

$$P\big(\{\omega \in \Omega \mid |F(X_n(\omega)) - \mathbb{E}(F(X_n))| > t\}\big) \le \exp(-\tfrac{n^2Kt^2}{c^2}),$$

where $K = \frac{2}{\pi^2}$. Hence by the Borel-Cantelli Lemma, it follows that

$$\big|\mathrm{tr}_n\big[f(X_n(\omega))\big] - \mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big)\big| \le t, \qquad \text{eventually},$$

for almost all ω. Since $t > 0$ was arbitrary, we get by Theorem 2.8, that

$$\lim_{n\to\infty} \mathrm{tr}_n\big[f(X_n(\omega))\big] = \lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[f(X_n)]\big) = \tfrac{1}{2\pi}\int_{-2}^{2} f(x)\sqrt{4 - x^2}\ dx,$$

for almost all ω. The last assertion in the proposition follows by application of Remark 2.3. This completes the proof. ■

Proof of Theorem 3.1. By Lemma 3.3, we have that

$$\limsup_{n\to\infty} \lambda_{\max}(X_n(\omega)) \le 2, \qquad \text{for almost all } \omega \text{ in } \Omega.$$

On the other hand, given any positive ϵ, it follows from Proposition 3.6, that

$$\operatorname{card}\big(\operatorname{sp}[X_n(\omega)] \cap [2-\epsilon, \infty[\big) \to \infty, \quad \text{as } n \to \infty, \qquad \text{for almost all } \omega \text{ in } \Omega,$$

and hence that

$$\liminf_{n\to\infty} \lambda_{\max}(X_n(\omega)) \ge 2-\epsilon, \qquad \text{for almost all } \omega \text{ in } \Omega.$$

Since this is true for any positive ϵ, it follows that (3.1) holds, and (3.2) follows from (3.1) by considering the sequence $(-X_n)$. ■

4 The Harer-Zagier Recursion Formula

In [HZ, Section 4, Proposition 1], Harer and Zagier considered the numbers:

$$C(p,n) = 2^{-k/2}\pi^{-k^2/2}\int_{M_n(\mathbb{C})_{\mathrm{sa}}} \operatorname{Tr}_n(A^{2p}) \exp(-\tfrac{1}{2}\operatorname{Tr}_n(A^2))\, dA, \qquad (n \in \mathbb{N}, p \in \mathbb{N}_0),$$

where $dA = \prod_{i=1}^n da_{ii} \prod_{i<j} d(\operatorname{Re}(a_{i,j}))\, d(\operatorname{Im}(a_{i,j}))$.
Comparing with Section 1, it follows, that if $A \in \mathrm{SGRM}(n,1)$, then for all p in $\mathbb{N}_0$,

$$C(p,n) = \mathbb{E}\big(\operatorname{Tr}_n[A^{2p}]\big).$$

Harer and Zagier proved that

$$C(p,n) = \sum_{j=1}^{[\frac{p}{2}]} \varepsilon_j(p) n^{p+1-2j}, \qquad (n, p \in \mathbb{N}),$$

where the coefficients $\varepsilon_j(p)$ satisfy the following recursion formula:

$$(p+2)\varepsilon_j(p+1) = (4p+2)\varepsilon_j(p) + p(4p^2-1)\varepsilon_{j-1}(p-1),$$

(cf. [HZ, p. 460, line 3], with (n,g) substituted by $(p+1,j)$), and using this formula, they could quickly generate tables of the coefficients $\varepsilon_j(p)$, $1 \le p \le 12$.

Below we give a new proof of the above recursion formula, based on Lemma 2.7 and the differential equation for the confluent hyper-geometric function $x \mapsto \Phi(a,c;x)$. A different treatment of this result of Harer and Zagier can be found in [Meh, pp. 117-120].

4.1 Theorem. *Let A be an element of* $\mathrm{SGRM}(n,1)$, *and define*

$$C(p,n) = \mathbb{E}\big(\operatorname{Tr}_n[A^{2p}]\big), \qquad (p \in \mathbb{N}_0). \tag{4.1}$$

Then $C(0,n) = n$, $C(1,n) = n^2$, and for fixed n in $\mathbb{N}$, the numbers $C(p,n)$ satisfy the recursion formula:

$$C(p+1,n) = n \cdot \tfrac{4p+2}{p+2} \cdot C(p,n) + \tfrac{p(4p^2-1)}{p+2} \cdot C(p-1,n), \qquad (p \ge 1). \tag{4.2}$$

Proof. Let a, c be complex numbers, such that $c \notin \mathbb{Z} \setminus \mathbb{N}$. Then the confluent hypergeometric function

$$x \mapsto \Phi(a,c;x) = 1 + \tfrac{a}{c}\tfrac{x}{1} + \tfrac{a(a+1)}{c(c+1)}\tfrac{x^2}{2} + \tfrac{a(a+1)(a+2)}{c(c+1)(c+2)}\tfrac{x^3}{3!} + \cdots, \qquad (x \in \mathbb{C}),$$

is an entire function, and $y = \Phi(a,c;x)$ satisfies the differential equation

$$x\frac{d^2y}{dx^2} + (c-x)\frac{dy}{dx} - ay = 0, \tag{4.3}$$

(cf. [HTF, Vol. 1, p.248, formula (2)]). By Corollary 1.6, we have for any A in $\mathrm{SGRM}(n,1)$, that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = n\int_{\mathbb{R}} \exp(sx)h_n(x)\ dx, \qquad (s \in \mathbb{C}),$$

where $h_n(x) = \frac{1}{n\sqrt{2}}\sum_{k=0}^{n-1}\varphi_k(\frac{x}{\sqrt{2}})^2$. Hence by (2.17) in Lemma 2.7, we get that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = n \cdot \exp(\tfrac{s^2}{2}) \cdot \Phi(1-n, 2; -s^2), \qquad (s \in \mathbb{C}).$$

Since h_n is an even function, $\mathbb{E}\big(\mathrm{Tr}_n[A^{2q-1}]\big) = 0$, for any q in $\mathbb{N}$, and consequently

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(sA)]\big) = \sum_{p=0}^{\infty}\frac{s^{2p}}{(2p)!}\mathbb{E}\big(\mathrm{Tr}_n[A^{2p}]\big).$$

It follows thus, that $\frac{C(p,n)}{(2p)!}$ is the coefficient to x^p in the power series expansion of the function

$$\sigma_n(x) = n \cdot \exp(\tfrac{x}{2}) \cdot \Phi(1-n, 2; -x).$$

By (4.3) the function $\rho_n(x) = \Phi(1-n, 2; -x)$, satisfies the differential equation

$$x\rho_n''(x) + (2+x)\rho_n'(x) - (n-1)\rho_n(x) = 0,$$

which implies that $\sigma_n(x) = n \cdot \exp(\frac{x}{2}) \cdot \rho_n(x)$, satisfies the differential equation

$$x\sigma_n''(x) + 2\sigma_n'(x) - (\tfrac{x}{4} + n)\sigma_n(x) = 0. \tag{4.4}$$

We know that σ_n has the power series expansion:

$$\sigma_n(x) = \sum_{p=0}^{\infty}\alpha_p x^p, \qquad \text{where} \quad \alpha_p = \frac{C(p,n)}{(2p)!}, \quad (p \in \mathbb{N}). \tag{4.5}$$

Inserting (4.5) in (4.4), we find that

$$(p+1)(p+2)\alpha_{p+1} - n\alpha_p - \tfrac{1}{4}\alpha_{p-1} = 0, \qquad (p \geq 1), \tag{4.6}$$

and that

$$2\alpha_1 - n\alpha_0 = 0. \tag{4.7}$$

Inserting then $C(p,n) = \frac{C(p,n)}{(2p)!}$, in (4.6), we obtain (4.2). Moreover, it is clear that $C(0,n) = \mathrm{Tr}_n(\mathbf{1}_n) = n$, and thus, by (4.7), $C(1,n) = 2\alpha_1 = n\alpha_0 = n^2$. ∎

4.2 Corollary. ([HZ]) *With $C(p,n)$ as introduced in (4.1), we have that*

$$C(p,n) = \sum_{j=0}^{[\frac{p}{2}]} \varepsilon_j(p) n^{p+1-2j}, \qquad (p \in \mathbb{N}_0, n \in \mathbb{N}), \tag{4.8}$$

where the coefficients $\varepsilon_j(p)$, $j, p \in \mathbb{N}_0$, are determined by the conditions

$$\begin{aligned} \varepsilon_j(p) &= 0, \quad \text{whenever } j \geq [\tfrac{p}{2}] + 1, && (4.9)\\ \varepsilon_0(p) &= \tfrac{1}{p+1}\tbinom{2p}{p}, \qquad (p \in \mathbb{N}_0), && (4.10)\\ \varepsilon_j(p+1) &= \tfrac{4p+2}{p+2} \cdot \varepsilon_j(p) + \tfrac{p(4p^2-1)}{p+2} \cdot \varepsilon_{j-1}(p-1), \qquad (p, j \in \mathbb{N}). && (4.11) \end{aligned}$$

Proof. It is immediate from (4.2) of Theorem 4.1, that for fixed p, $C(p,n)$ is a polynomial in n of degree $p+1$ and without constant term. Moreover, it follows from (4.2), that only $n^{p+1}, n^{p-1}, n^{p-3}$, etc., have non-zero coefficients in this polynomial. Therefore $C(p,n)$ is of the form set out in (4.8) for suitable coefficients

$$\varepsilon_j(p), \qquad p \geq 0, \quad 0 \leq j \leq [\tfrac{p}{2}].$$

Inserting (4.8) in (4.2), and applying the convention (4.9), we obtain (4.11), and also that

$$\varepsilon_0(p+1) = \tfrac{4p+2}{p+2} \cdot \varepsilon_0(p), \qquad (p \geq 1). \tag{4.12}$$

Clearly, $\varepsilon_0(0) = \varepsilon_0(1) = 1$, and thus by induction on (4.12), we obtain (4.10). ■

From Theorem 4.1 or Corollary 4.2, one gets, that for any A in SGRM$(n,1)$,

$$\begin{aligned} \mathbb{E}\big(\mathrm{Tr}_n[A^2]\big) &= n^2,\\ \mathbb{E}\big(\mathrm{Tr}_n[A^4]\big) &= 2n^3 + n,\\ \mathbb{E}\big(\mathrm{Tr}_n[A^6]\big) &= 5n^4 + 10n^2,\\ \mathbb{E}\big(\mathrm{Tr}_n[A^8]\big) &= 14n^5 + 70n^3 + 21n,\\ \mathbb{E}\big(\mathrm{Tr}_n[A^{10}]\big) &= 42n^6 + 420n^4 + 483n^2, \end{aligned}$$

etc. (see [HZ, p. 459] for a list of the numbers $\varepsilon_j(p)$, $p \leq 12$). If, as in Sections 2 and 3, we replace the A above by an element X of SGRM$(n, \frac{1}{n})$, and Tr_n by tr_n, then we have to divide the above numbers by n^{p+1}. Hence for X in SGRM$(n, \frac{1}{n})$, we have

$$\begin{aligned} \mathbb{E}\big(\mathrm{tr}_n[X^2]\big) &= 1,\\ \mathbb{E}\big(\mathrm{tr}_n[X^4]\big) &= 2 + \tfrac{1}{n^2},\\ \mathbb{E}\big(\mathrm{tr}_n[X^6]\big) &= 5 + \tfrac{10}{n^2},\\ \mathbb{E}\big(\mathrm{tr}_n[X^8]\big) &= 14 + \tfrac{70}{n^2} + \tfrac{21}{n^4},\\ \mathbb{E}\big(\mathrm{tr}_n[X^{10}]\big) &= 42 + \tfrac{420}{n^2} + \tfrac{483}{n^4}, \end{aligned}$$

etc. Note that the constant term in $\mathbb{E}\big(\mathrm{tr}_n[X^{2p}]\big)$ is

$$\varepsilon_0(p) = \tfrac{1}{p+1}\tbinom{2p}{p} = \tfrac{1}{2\pi}\int_{-2}^{2} x^{2p}\sqrt{4-x^2}\; dx,$$

in concordance with Wigner's semi-circle law.

5 Rectangular Gaussian Random Matrices and the Complex Wishart Distribution

In 1928, Wishart proved that if B is a real, random $m \times n$ matrix ($m \geq n$), such that the entries are independent, identically distributed random variables with distribution $N(0,1)$, then the distribution of the random matrix $S = B^t B$ has density w.r.t. the Lebesgue measure $dS = \prod_{1 \leq i \leq j \leq n} ds_{ij}$ on the set $M_n(\mathbb{R})_s$ of symmetric matrices, given by

$$S \mapsto c_4 \cdot (\det S)^{(m-n-1)/2} \exp(-\tfrac{1}{2}\mathrm{Tr}_n(S)) \cdot 1_{M_n(\mathbb{R})_+}(S), \qquad (S \in M_n(\mathbb{R})_s), \tag{5.1}$$

where $M_n(\mathbb{R})_+$ denotes the set of positive semi-definite matrices, and c_4 is a normalization constant,

$$c_4 = \Big[2^{(mn)/2} \pi^{n(n-1)/4} \prod_{j=1}^{n} \Gamma(\tfrac{m+1-j}{2})\Big]^{-1},$$

(cf. [Wis] or [An, Chapter 7]). The distribution with density given in (5.1) is called *the Wishart distribution.*

In 1939, Hsu proved that under the same conditions, the joint distribution of the set of ordered eigenvalues $\lambda_1(S) \leq \lambda_2(S) \leq \cdots \leq \lambda_n(S)$ of S, has density w.r.t. Lebesgue measure on $\mathbb{R}^n$, given by

$$(\lambda_1, \ldots, \lambda_n) \mapsto c_5 \Big[\prod_{1 \leq j < k \leq n} (\lambda_k - \lambda_j)\Big]\Big[\prod_{j=1}^{n} \lambda_j\Big]^{\frac{m-n-1}{2}} \exp(\tfrac{-1}{2}\textstyle\sum_{j=1}^{n} \lambda_j) \cdot 1_{\Lambda_+}(\lambda_1, \ldots, \lambda_n), \tag{5.2}$$

where $\Lambda_+ = \{(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \mid 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n\}$, and where c_5 is yet a normalization constant,

$$c_5 = \Big[2^{(mn)/2} \pi^{-n/2} \prod_{j=1}^{n} \Gamma(\tfrac{m+1-j}{2}) \Gamma(\tfrac{n+1-j}{2})\Big]^{-1},$$

(cf. [Hs, pp. 256-267]). The corresponding results to (5.1) and (5.2) for complex rectangular Gaussian random matrices were obtained by Goodman (cf. [Go]) and James (cf. [Ja]) in 1963-64 (see Theorem 5.2 below).

5.1 Definition. Let m, n be positive integers, and let

$$B = (b(i,j))_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} : \Omega \to M_{m,n}(\mathbb{C}),$$

be a complex random $m \times n$ matrix defined on Ω. We say then that B is a (standard) Gaussian random $m \times n$ matrix with entries of variance σ^2, if the real valued random variables $\mathrm{Re}(b(i,j))$, $\mathrm{Im}(b(i,j))$, $1 \leq i \leq m$, $1 \leq j \leq n$, form a family of $2mn$ independent, identically distributed random variables, with distribution $N(0, \frac{\sigma^2}{2})$. We denote by $\mathrm{GRM}(m, n, \sigma^2)$ the set of such random matrices defined on Ω. $\square$

We note, that if $B = (b(i,j))_{\substack{1\le i\le m\\ 1\le j\le n}} \in \mathrm{GRM}(m,n,\sigma^2)$, then for any i in $\{1,2,\dots,m\}$ and j in $\{1,2,\dots,n\}$, the distribution of the complex valued random variable $b(i,j)$ has distribution with density

$$z \mapsto \tfrac{1}{\pi\sigma^2}\exp(\tfrac{-|z|^2}{\sigma^2}), \qquad (z\in\mathbb{C}),$$

w.r.t. Lebesgue measure on $\mathbb{C}$, and moreover

$$\mathbb{E}\big(|b(i,j)|^2\big) = \sigma^2.$$

In this and the following three sections, we shall consider exclusively the two cases where $\sigma^2 = 1$ or $\sigma^2 = \frac{1}{n}$.

5.2 Theorem. ([Go],[Ja]) *Let m,n be elements of $\mathbb{N}$, such that $m \ge n$, and let B be an element of* $\mathrm{GRM}(m,n,1)$. *Then the distribution of the random matrix $S = B^*B$, has density w.r.t. the Lebesgue measure*

$$dS = \prod_{i=1}^{n} ds_{ii} \prod_{1\le i<j\le n} d(\mathrm{Re}(s_{ij}))\ d(\mathrm{Im}(s_{ij})), \tag{5.3}$$

on $M_n(\mathbb{C})_{\mathrm{sa}}$, given by

$$S \mapsto c_6\cdot(\det S)^{m-n}\cdot\exp(-\mathrm{Tr}_n(S))\cdot 1_{M_n(\mathbb{C})_+}(S), \qquad (S\in M_n(\mathbb{C})_{\mathrm{sa}}), \tag{5.4}$$

where

$$c_6 = \Big[\pi^{n(n-1)/2}\prod_{j=1}^{n}(m-j)!\Big]^{-1}.$$

The joint distribution of the ordered eigenvalues $\lambda_1(S),\lambda_2(S),\dots,\lambda_n(S)$ of S, has density w.r.t. Lebesgue measure on $\mathbb{R}^n$, given by

$$(\lambda_1,\dots,\lambda_n) \mapsto c_7\Big[\prod_{1\le i<j\le n}(\lambda_i-\lambda_j)^2\Big]\Big[\prod_{j=1}^{n}\lambda_j\Big]^{m-n}\exp(-\textstyle\sum_{j=1}^{n}\lambda_j)\cdot 1_{\Lambda_+}(\lambda_1,\dots,\lambda_n), \tag{5.5}$$

where $\Lambda_+ = \{(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n \mid 0\le\lambda_1\le\cdots\le\lambda_n\}$, and

$$c_7 = \Big[\prod_{j=1}^{n}(n-j)!(m-j)!\Big]^{-1}.$$

5.3 Remark. The computation of the density (5.4) is due to Goodman (cf. [Go]), and the distribution with density given by (5.4) is called *the complex Wishart distribution.* After Goodman's paper, the complex Wishart distribution and its quaternionic counterpart has been studied by several authors (see [Ja], [Kh], [ABJ], [HSS] and [LM]). James gives in [Ja] a survey of both the real and the complex Wishart distribution and states the formula (5.5) (cf. [Ja, pp. 487-489]). Unfortunately, James does not give a proof of (5.5) or a reference to a proof of it. However, it is not hard to derive (5.5) from (5.4), by application of the method of Wigner, that we described in Section 1. Indeed, by Lemma 1.2, the range-measure (on $\mathbb{R}^n$) of the measure dS given in (5.3) under the mapping

$$S \mapsto (\lambda_1(S), \lambda_2(S), \dots, \lambda_n(S)), \qquad (S \in M_n(\mathbb{C})_{\mathrm{sa}}),$$

(where $\lambda_1(S) \le \dots \le \lambda_n(S)$ are the ordered eigenvalues of S), has density w.r.t. Lebesgue measure on $\mathbb{R}^n$, given by

$$(\lambda_1, \dots, \lambda_n) \mapsto c_2 \prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2 \cdot 1_\Lambda(\lambda_1, \dots, \lambda_n), \tag{5.6}$$

where $\Lambda = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \lambda_1 \le \lambda_2 \le \dots \le \lambda_n\}$, and where

$$c_2 = \pi^{n(n-1)/2} \Big[\prod_{j=1}^{n-1} j! \Big]^{-1}.$$

Since $\prod_{j=1}^{n-1} j! = \prod_{j=1}^{n} (n-j)!$, (5.5) now follows from (5.4). □

The following theorem was proved by Bronk (cf. [Bro]) in 1965.

5.4 Theorem. ([Bro]) *Assume that $\alpha \in]-1, \infty[$, and let $S = (s_{ij})_{1 \le i,j \le n}$ be a selfadjoint random $n \times n$ matrix, for which the joint distribution of the entries s_{ij}, $1 \le i, j \le n$, is the measure*

$$c_8 \cdot (\det S)^\alpha \cdot \exp(-\mathrm{Tr}_n(S)) \cdot 1_{M_n(\mathbb{C})_+}(S)\, dS, \tag{5.7}$$

where dS is Lebesgue measure on $M_n(\mathbb{C})_{\mathrm{sa}}$ (cf. (5.3)), and where

$$c_8 = \Big[\int_{M_n(\mathbb{C})_+} (\det S)^\alpha \cdot \exp(-\mathrm{Tr}_n(S))\, dS \Big]^{-1}.$$

Then the joint distribution of the ordered eigenvalues $\lambda_1(S), \dots, \lambda_n(S)$ of S has density w.r.t. Lebesgue measure on $\mathbb{R}^n$, given by

$$(\lambda_1, \dots, \lambda_n) \mapsto c_9 \Big[\prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2 \Big] \Big[\prod_{j=1}^{n} \lambda_j \Big]^\alpha \exp(-\textstyle\sum_{j=1}^n \lambda_j) \cdot 1_{\Lambda_+}(\lambda_1, \dots, \lambda_n), \tag{5.8}$$

for some positive normalization constant c_9.

Let $g_n^\alpha \colon \mathbb{R}^n \to \mathbb{R}$ denote the density function obtained by taking the average of the function in (5.8) over all permutations of the eigenvalues, i.e.,

$$g_n^\alpha(\lambda_1, \dots, \lambda_n) = \frac{c_9}{n!} \Big[\prod_{1 \le j < k \le n} (\lambda_j - \lambda_k)^2 \Big] \Big[\prod_{j=1}^{n} \lambda_j \Big]^\alpha \exp(-\textstyle\sum_{j=1}^n \lambda_j). \tag{5.9}$$

Then the function

$$h_n(\lambda_1) = \int_{\mathbb{R}^{n-1}} g_n^\alpha(\lambda_1, \lambda_2, \dots, \lambda_n)\, d\lambda_2 \cdots d\lambda_n, \qquad (\lambda_1 \in \mathbb{R}), \tag{5.10}$$

can be expressed as

$$h_n^\alpha(x) = \tfrac{1}{n}\textstyle\sum_{j=0}^{n-1}\varphi_k^\alpha(x)^2, \qquad (x\in\mathbb{R}), \tag{5.11}$$

where

$$\varphi_k^\alpha(x) = \left[\tfrac{k!}{\Gamma(k+\alpha+1)}x^\alpha\exp(-x)\right]^{1/2}\cdot L_k^\alpha(x), \qquad (k\in\mathbb{N}_0), \tag{5.12}$$

and $(L_k^\alpha)_{k\in\mathbb{N}_0}$ *is the sequence of generalized Laguerre polynomials of order* α*, i.e.,*

$$L_k^\alpha(x) = (k!)^{-1}x^{-\alpha}\exp(x)\cdot\frac{d^k}{dx^k}\big(x^{k+\alpha}\exp(-x)\big), \qquad (k\in\mathbb{N}_0). \tag{5.13}$$

5.5 Remark. The exact values of c_8 and c_9 can be read out of Bronk's proof (cf. [Bro]), namely

$$c_8 = \Big[\pi^{n(n-1)/2}\prod_{j=1}^{n}\Gamma(\alpha+j)\Big]^{-1},$$

$$c_9 = \Big[\prod_{j=1}^{n}\Gamma(j)\Gamma(\alpha+j)\Big]^{-1}.$$

If $\alpha\in\mathbb{N}_0$ and $m,n\in\mathbb{N}$, such that $m-n=\alpha$, then the constants c_8 and c_9 coincide with the constants c_6 respectively c_7 given in Theorem 5.2. $\square$

5.6 Corollary. *Let B be an element of* $\mathrm{GRM}(m,n,1)$*, let* φ_k^α*,* $\alpha\in]0,\infty[$*,* $k\in\mathbb{N}_0$*, be the functions introduced in* (5.12)*, and let* $f\colon[0,\infty[\to\mathbb{R}$ *be a Borel function.*

(i) *If* $m\geq n$*, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[f(B^*B)]\big) = \int_0^\infty f(x)\Big[\textstyle\sum_{j=0}^{n-1}\varphi_k^{m-n}(x)^2\Big]\,dx,$$

whenever the integral on the right hand side is well-defined.

(ii) *If* $m<n$*, we have that*

$$\mathbb{E}\big(\mathrm{Tr}_n[f(B^*B)]\big) = (n-m)f(0) + \int_0^\infty f(x)\Big[\textstyle\sum_{j=0}^{m-1}\varphi_k^{n-m}(x)^2\Big]\,dx,$$

whenever the integral on the right hand side is well-defined.

Proof. (i) The proof of (i) can be copied from the proof of Corollary 1.6, using Theorem 5.2 and Theorem 5.4 instead of Theorem 1.4.

(ii) Assume that $m<n$, and note that $B^*\in\mathrm{GRM}(n,m,1)$. If $T\in M_{m,n}(\mathbb{C})$, then T^*T and TT^* have the same list of non-zero eigenvalues counted with multiplicity, and hence T^*T must have $n-m$ more zeroes in its list of eigenvalues than TT^* has. Combining these facts with (i), we obtain (ii). $\blacksquare$

5.7 Remark. Bronk's proof of (5.11) in Theorem 5.4 is a fairly simple generalization of Wigner's method from [Wig3] (see also Theorem 1.4 in this paper). It is based on the orthogonality relation for the generalized Laguerre polynomials:

$$\int_0^\infty L_j^\alpha(x) L_k^\alpha(x) \cdot x^\alpha \exp(-x)\, dx = \begin{cases} \frac{\Gamma(k+\alpha+1)}{k!}, & \text{if } j = k, \\ 0, & \text{if } j \neq k, \end{cases} \tag{5.14}$$

(cf. [HTF, Vol. 2, p.188, formula (2)]), which implies that the sequence of functions $(\varphi_k^\alpha)_{k\in\mathbb{N}_0}$ introduced in formula (5.12), is an orthonormal sequence in the Hilbert space $L_2([0,\infty[, dx)$, i.e., that

$$\int_0^\infty \varphi_j^\alpha(x)\varphi_k^\alpha(x)\, dx = \delta_{j,k}, \qquad (j,k \in \mathbb{N}_0).$$

Apparently Bronk did not know about the results of Goodman and James, quoted in Theorem 5.2 above, and he included in his paper an independent proof of Theorem 5.2, in the case where $m = n$. However, Bronk seems to have been unaware of the connection between his work and the complex Wishart distribution for $m > n$. $\square$

6 The Asymptotic Eigenvalue Distribution in the Complex Wishart Case

In the paper [GS] from 1977, Grenander and Silverstein considered random $m \times n$ matrices, $T = (t_{jk})$, satisfying that the entries t_{jk}, $1 \le j \le m, 1 \le k \le n$, form a family of mn independent, identically distributed random variables, such that $\mathbb{E}(t_{jk}^2) = 1$ and $\mathbb{E}(|t_{jk}|^p) < \infty$ for all p in $\mathbb{N}$. Letting n tend to ∞, under the assumptions that $m = cn$ for some fixed, positive integer c, and that the distribution of the entries t_{jk} is independent of n, Grenander and Silverstein proved that the empirical distribution of the eigenvalues of $\frac{1}{n}T^tT$ converges in probability to the distribution μ_c on $[0,\infty[$, given by the density

$$\frac{d\mu_c(x)}{dx} = \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x), \qquad (x \in [0,\infty[), \tag{6.1}$$

where $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. More precisely, if F_n denotes the (random) distribution function for the empirical distribution of the eigenvalues of $\frac{1}{n}T^tT$, then Grenander and Silverstein showed that for all x in $\mathbb{R}$, $F_n(x) \to F_c(x)$ in probability, where F_c is the distribution function for μ_c.

In [Wa2] from 1978 (but written independently of [GS]), Wachter strengthened Grenander and Silverstein's result considerably by showing that in fact (with notation as above), $F_n(x) \to F_c(x)$, almost surely, for all x in $\mathbb{R}$, even under the more general assumption that $\frac{m}{n} \to c \in [1,\infty[$, as $n \to \infty$. The actual form of the limit measure μ_c is not exposed in [Wa2], but it can be found in Jonsson's paper [Jo, Theorem 2.1], and, according to Geman (cf. [Gem]), also in Wachter's thesis [Wa1, Theorem 7.7]. The papers [GS], [Wa2] and [Jo] deal exclusively with random matrices with real valued entries, but it is not hard to generalize their results to the complex case. In [OP], Oravecz and Petz give a new treatment of some of the results of Grenander, Silverstein, Wachter and Jonsson, and the paper contains also a few comments about the complex case.

The proofs given in [GS],[Wa2],[Jo] and [OP] are mainly combinatorial. The primary aim of this section is to use Corollary 5.6 to give an entirely analytical proof of convergence (in the sense considered in Theorem 2.8), of the eigenvalue distribution of $\frac{1}{n}B_n^*B_n$ to the measure μ_c, under the assumptions that $B_n \in \mathrm{GRM}(m(n), n, c)$ for all n, and $\frac{m(n)}{n} \to c$, as $n \to \infty$ (cf. Theorem 6.7). The analog of Wachter's result on almost sure convergence of the empirical eigenvalue distribution for the complex Gaussian case, is discussed in Section 7 (cf. Proposition 7.4).

As in Section 5, for any α in $]-1, \infty[$, we denote by $(L_k^\alpha)_{k\in\mathbb{N}_0}$ the sequence of generalized Laguerre polynomials of order α, i.e.,

$$L_k^\alpha(x) = (k!)^{-1}x^{-\alpha}\exp(x)\frac{d^k}{dx^k}\big(x^{k+\alpha}\exp(-x)\big), \qquad (k \in \mathbb{N}_0,\ x > 0), \tag{6.2}$$

and by $(\varphi_k^\alpha)_{k\in\mathbb{N}_0}$ the sequence of functions given by

$$\varphi_k^\alpha(x) = \big(\tfrac{k!}{\Gamma(k+\alpha+1)}x^\alpha \exp(-x)\big)^{1/2}L_k^\alpha(x), \qquad (x > 0). \tag{6.3}$$

6.1 Lemma. *For any n in $\mathbb{N}_0$, we have that*

$$\frac{d}{dx}\Big(x\sum_{j=0}^{n-1}\varphi_j^\alpha(x)^2\Big) = \sqrt{n(n+\alpha)}\cdot\varphi_{n-1}^\alpha(x)\varphi_n^\alpha(x), \qquad (x > 0). \tag{6.4}$$

Proof. For each n in $\mathbb{N}$, we define

$$\rho_n(x) = \sum_{j=0}^{n-1}\tfrac{j!}{\Gamma(j+\alpha+1)}L_j^\alpha(x)^2, \qquad (x > 0). \tag{6.5}$$

Using [HTF, Volume 2, p.188, formula (7)], we have here that

$$\begin{aligned}\rho_n(x) &= \lim_{y\to x}\sum_{j=0}^{n-1}\tfrac{j!}{\Gamma(j+\alpha+1)}L_j^\alpha(y)L_j^\alpha(x)\\ &= \lim_{y\to x}\frac{n!\big(L_{n-1}^\alpha(y)L_n^\alpha(x) - L_n^\alpha(y)L_{n-1}^\alpha(x)\big)}{(y-x)\cdot\Gamma(n+\alpha)}.\end{aligned}$$

Therefore, it follows that

$$\rho_n(x) = \tfrac{n!}{\Gamma(n+\alpha)}\big((L_{n-1}^\alpha)'(x)L_n^\alpha(x) - (L_n^\alpha)'(x)L_{n-1}^\alpha(x)\big), \tag{6.6}$$

and hence that

$$\rho_n'(x) = \tfrac{n!}{\Gamma(n+\alpha)}\big((L_{n-1}^\alpha)''(x)L_n^\alpha(x) - (L_n^\alpha)''(x)L_{n-1}^\alpha(x)\big). \tag{6.7}$$

By [HTF, Volume 2, p.188, formula (10)], we have that

$$x(L_{n-1}^{\alpha})''(x) + (\alpha+1-x)(L_{n-1}^{\alpha})'(x) = -(n-1)L_{n-1}^{\alpha}(x),$$
$$x(L_n^{\alpha})''(x) + (\alpha+1-x)(L_n^{\alpha})'(x) = -nL_n^{\alpha}(x).$$

Combining these two formulas with (6.6) and (6.7), we find that

$$\begin{aligned} x\rho_n'(x) + (\alpha+1-x)\rho_n(x) &= \tfrac{n!}{\Gamma(n+\alpha)}\big(-(n-1)L_{n-1}^{\alpha}(x)L_n^{\alpha}(x) + nL_n^{\alpha}(x)L_{n-1}^{\alpha}(x)\big) \\ &= \tfrac{n!}{\Gamma(n+\alpha)}L_{n-1}^{\alpha}(x)L_n^{\alpha}(x). \end{aligned}$$

It follows now, that

$$\begin{aligned} \frac{d}{dx}\Big(x\sum_{j=0}^{n-1}\varphi_j^{\alpha}(x)^2\Big) &= \frac{d}{dx}\big(\rho_n(x)x^{\alpha+1}\exp(-x)\big) \\ &= \big(x\rho_n'(x) + (\alpha+1-x)\rho_n(x)\big)x^{\alpha}\exp(-x) \\ &= \tfrac{n!}{\Gamma(n+\alpha)}L_{n-1}^{\alpha}(x)L_n^{\alpha}(x)x^{\alpha}\exp(-x) \\ &= \tfrac{n!}{\Gamma(n+\alpha)}\Big(\tfrac{\Gamma(n+\alpha)}{(n-1)!}\tfrac{\Gamma(n+\alpha+1)}{n!}\Big)^{1/2}\varphi_{n-1}^{\alpha}(x)\varphi_n^{\alpha}(x) \\ &= \sqrt{n(n+\alpha)}\cdot\varphi_{n-1}^{\alpha}(x)\varphi_n^{\alpha}(x), \end{aligned}$$

which is the desired formula. ■

In order to state the next lemma, we need to introduce the hyper-geometric function F, which is given by the equation (cf. [HTF, Vol. 1, p.56, formula (2)]),

$$F(a,b,c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n n!}z^n, \tag{6.8}$$

with the notation introduced in (2.9). We note that $F(a,b,c;z)$ is well-defined whenever $c \notin \mathbb{Z}\setminus\mathbb{N}$ and $|z| < 1$. If either $-a \in \mathbb{N}_0$ or $-b \in \mathbb{N}_0$, then $F(a,b,c;z)$ becomes a polynomial in z, and is thus well-defined for all z in $\mathbb{C}$.

6.2 Lemma. *Consider α in $]-1,\infty[$ and j,k in $\mathbb{N}_0$. Then for any complex number s, such that $s \neq 0$ and $\mathrm{Re}(s) < 1$, we have that*

$$\int_0^{\infty}\varphi_j^{\alpha}(x)\varphi_k^{\alpha}(x)\exp(sx)\,dx = \gamma(\alpha,j,k)\cdot\frac{s^{j+k}}{(1-s)^{\alpha+j+k+1}}\cdot F(-j,-k,\alpha+1;s^{-2}), \tag{6.9}$$

where

$$\gamma(\alpha,j,k) = \frac{(-1)^{j+k}}{\Gamma(\alpha+1)}\Big(\frac{\Gamma(\alpha+j+1)\Gamma(\alpha+k+1)}{j!k!}\Big)^{1/2}. \tag{6.10}$$

Proof. The formula (6.9) can be extracted from the paper [Ma] by Mayr, but for the readers convenience, we include an elementary proof. Both sides of the equality (6.9) are analytical functions of $s \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1\}$, so it suffices to check (6.9) for all s in $]-\infty, 1[\setminus\{0\}$. By (6.3), we have that

$$\begin{aligned}
&\int_0^\infty \varphi_j^\alpha(x)\varphi_k^\alpha(x)\exp(sx)\ dx \\
&\quad = \left(\tfrac{j!k!}{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}\right)^{1/2}\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)x^\alpha\exp((s-1)x)\ dx \\
&\quad = \left(\tfrac{j!k!}{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}\right)^{1/2}\tfrac{1}{(1-s)^{\alpha+1}}\int_0^\infty L_j^\alpha(\tfrac{y}{1-s})L_k^\alpha(\tfrac{y}{1-s})y^\alpha\exp(-y)\ dy,
\end{aligned} \tag{6.11}$$

where, in the last equality, we applied the substitution $y = \frac{x}{1-s}$. We note here, that by [HTF, Volume 2, p.192, formula (40)], we have for any positive number λ, that

$$\begin{aligned}
L_k^\alpha(\lambda x) &= \sum_{r=0}^k \binom{k+\alpha}{r}\lambda^{k-r}(1-\lambda)^r L_{k-r}^\alpha(x) \\
&= \sum_{r=0}^k \binom{k+\alpha}{k-r}\lambda^r(1-\lambda)^{k-r}L_r^\alpha(x).
\end{aligned} \tag{6.12}$$

By application of this formula and the orthogonality relation (5.14) for the Laguerre polynomials, we obtain that

$$\begin{aligned}
&\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)x^\alpha\exp((s-1)x)\ dx \\
&\quad = \frac{1}{(1-s)^{\alpha+1}}\sum_{r=0}^{\min\{j,k\}}\binom{j+\alpha}{j-r}\binom{k+\alpha}{k-r}\Big(\frac{1}{1-s}\Big)^{2r}\Big(1-\frac{1}{1-s}\Big)^{j+k-2r}\frac{\Gamma(\alpha+r+1)}{r!} \\
&\quad = \frac{(-s)^{j+k}}{(1-s)^{\alpha+j+k+1}}\sum_{r=0}^{\min\{j,k\}}\binom{j+\alpha}{j-r}\binom{k+\alpha}{k-r}\frac{\Gamma(\alpha+r+1)}{r!}(-s)^{-2r} \\
&\quad = \frac{(-s)^{j+k}}{(1-s)^{\alpha+j+k+1}}\sum_{r=0}^{\min\{j,k\}}\frac{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)}{(j-r)!(k-r)!r!\Gamma(\alpha+r+1)}s^{-2r}.
\end{aligned} \tag{6.13}$$

We note here that

$$\frac{j!k!\Gamma(\alpha+1)}{(j-r)!(k-r)!r!\Gamma(\alpha+r+1)} = \frac{(-j)_r(-k)_r}{(\alpha+1)_r r!},$$

and hence it follows that

$$\begin{aligned}
&\int_0^\infty L_j^\alpha(x)L_k^\alpha(x)x^\alpha\exp((s-1)x)\ dx \\
&\quad = \frac{\Gamma(j+\alpha+1)\Gamma(k+\alpha+1)\cdot(-s)^{j+k}}{j!k!\Gamma(\alpha+1)\cdot(1-s)^{\alpha+j+k+1}}\cdot F(-j,-k,\alpha+1;s^{-2}).
\end{aligned} \tag{6.14}$$

Combining (6.11) and (6.14), we obtain (6.9). ∎

6.3 Lemma. *Assume that* $\alpha \in]-1,\infty[$, *and that* $n,k \in \mathbb{N}_0$. *Then for any complex number* s *such that* $\mathrm{Re}(s) < 1$, *we have that*

$$\int_0^\infty \varphi_k^\alpha(x)^2 \exp(sx)\; dx \;=\; \frac{F(-k-\alpha,-k,1;s^2)}{(1-s)^{\alpha+2k+1}} \tag{6.15}$$

$$\int_0^\infty \Big(\sum_{j=0}^{n-1} \varphi_j^\alpha(x)^2\Big) x \exp(sx)\; dx \;=\; n(n+\alpha)\frac{F(1-n-\alpha,1-n,2;s^2)}{(1-s)^{\alpha+2n}}. \tag{6.16}$$

Proof. By continuity, it suffices to prove (6.15) and (6.16) for all s in $\mathbb{C} \setminus \{0\}$. Before doing so, we observe that for j,k in $\mathbb{N}_0$ such that $j \le k$, we have that

$$\begin{aligned} F(-j,-k,\alpha+1;s^{-2}) &= \sum_{r=0}^{j} \frac{(-j)_r(-k)_r}{(\alpha+1)_r r!} s^{-2r} \\ &= \sum_{r=0}^{j} \frac{j!k!\Gamma(\alpha+1)}{(j-r)!(k-r)!r!\Gamma(\alpha+r+1)} s^{-2r}. \end{aligned}$$

Replacing now r by $j-r$ in the summation, it follows that

$$\begin{aligned} F(-j,-k,\alpha+1;s^{-2}) &= \sum_{r=0}^{j} \frac{j!k!\Gamma(\alpha+1)}{r!(k-j+r)!(j-r)!\Gamma(\alpha+j-r+1)} s^{2r-2j} \\ &= \frac{k!\Gamma(\alpha+1)}{(k-j)!\Gamma(\alpha+j+1)} \sum_{r=0}^{j} \frac{(-j)_r(-\alpha-j)_r}{r!(1+k-j)_r} s^{2r-2j}. \end{aligned}$$

Hence for j,k in $\mathbb{N}_0$ such that $j \le k$, we have that

$$F(-j,-k,\alpha+1;s^{-2}) = \frac{k!\Gamma(\alpha+1)}{(k-j)!\Gamma(\alpha+j+1)} \frac{F(-j-\alpha,-j,1+k-j;s^2)}{s^{2j}}. \tag{6.17}$$

Returning now to the proof of (6.15) and (6.16), we note that by Lemma 6.1 and (6.17), we have that

$$\begin{aligned} \int_0^\infty \varphi_k^\alpha(x)^2 \exp(sx)\; dx &= \frac{\Gamma(\alpha+k+1)\cdot s^{2k}}{k!\Gamma(\alpha+1)\cdot(1-s)^{\alpha+2k+1}} \cdot F(-k,-k,\alpha+1;s^{-2}) \\ &= \frac{F(-k-\alpha,-k,1;-s^2)}{(1-s)^{\alpha+2k+1}}, \end{aligned}$$

which proves (6.15).

Regarding (6.16), we get by partial integration, Lemma 6.1, Lemma 6.2 and (6.17), that

$$
\begin{aligned}
&\int_0^\infty \Big(\sum_{j=0}^{n-1}\varphi_j^\alpha(x)^2\Big)x\exp(sx)\;dx\\
&\quad=\frac{-1}{s}\int_0^\infty \frac{d}{dx}\Big(x\sum_{j=0}^{n-1}\varphi_j^\alpha(x)^2\Big)\exp(sx)\;dx\\
&\quad=\frac{-\sqrt{n(n+\alpha)}}{s}\int_0^\infty \varphi_{n-1}^\alpha(x)\varphi_n^\alpha(x)\exp(sx)\;dx\\
&\quad=\frac{-\sqrt{n(n+\alpha)}\cdot\gamma(\alpha,n-1,n)\cdot s^{2n-1}}{s(1-s)^{\alpha+2n}}\cdot F(-n+1,-n,\alpha+1;s^{-2})\\
&\quad=\frac{-\sqrt{n(n+\alpha)}\cdot\gamma(\alpha,n-1,n)\cdot s^{2n-1}\cdot n!\cdot\Gamma(\alpha+1)}{s(1-s)^{\alpha+2n}\cdot\Gamma(\alpha+n)}\cdot\frac{F(-n-\alpha+1,-n+1,2;s^2)}{s^{2n-2}}\\
&\quad=\frac{-\sqrt{n(n+\alpha)}\cdot\gamma(\alpha,n-1,n)\cdot n!\cdot\Gamma(\alpha+1)}{(1-s)^{\alpha+2n}\cdot\Gamma(\alpha+n)}\cdot F(1-n-\alpha,1-n,2;s^2).
\end{aligned}
\tag{6.18}
$$

Recall here from (6.10), that

$$
\gamma(\alpha,n-1,n)=\frac{-1}{\Gamma(\alpha+1)}\Big(\frac{\Gamma(\alpha+n)\Gamma(\alpha+n+1)}{(n-1)!n!}\Big)^{1/2}=\frac{-\Gamma(\alpha+n)}{\Gamma(\alpha+1)n!}\sqrt{n(n+\alpha)},
$$

and inserting this in (6.18), we obtain (6.16). ■

6.4 Theorem. *Assume that $m,n\in\mathbb{N}$ and that $B\in\mathrm{GRM}(m,n,1)$. Then for any complex number s, such that $\mathrm{Re}(s)<1$, we have that*

$$
\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big)=m\cdot n\cdot\frac{F(1-m,1-n,2;s^2)}{(1-s)^{m+n}},\tag{6.19}
$$

and that

$$
\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big)=\sum_{k=1}^{n}\frac{F(k-m,k-n,1;s^2)}{(1-s)^{m+n+1-2k}},\quad \text{if } m\ge n,\tag{6.20}
$$

$$
\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big)=(n-m)+\sum_{k=1}^{m}\frac{F(k-m,k-n,1;s^2)}{(1-s)^{m+n+1-2k}},\quad \text{if } m< n.\tag{6.21}
$$

Proof. To prove (6.19), assume first that $m\ge n$. Then by Corollary 5.6(i), we have that

$$
\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big)=\int_0^\infty\Big(\sum_{k=0}^{n-1}\varphi_k^{m-n}(x)^2\Big)x\exp(sx)\;dx,
$$

and hence (6.19) follows from (6.16) in Lemma 6.3. The case $m<n$ is proved similarly by application of Corollary 5.6(ii) instead of Corollary 5.6(i).

To prove (6.20), assume that $m \geq n$, and note then that by Corollary 5.6(i) and (6.15) in Lemma 6.3,

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[\exp(sB^*B)]\big) &= \int_0^\infty \Big(\sum_{k=0}^{n-1} \varphi_k^{m-n}(x)^2\Big) x\ dx \\ &= \sum_{k=0}^{n-1} \frac{F(-k-m+n,-k,1;s^2)}{(1-s)^{m-n+2k+1}}.\end{aligned}$$

Replacing then k by $n-k$ in this summation, we obtain (6.20).

We note finally that (6.21) is proved the same way as (6.20), by application Corollary 5.6(ii) instead of Corollary 5.6(i). ∎

6.5 Definition. For c in $]0,\infty[$, we denote by μ_c, the measure on $[0,\infty[$ given by the equation

$$\mu_c = \max\{1-c,0\}\delta_0 + \frac{\sqrt{(x-a)(b-x)}}{2\pi x} \cdot 1_{[a,b]}(x) \cdot dx,$$

where $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. It is not hard to check that

$$\int_a^b \frac{\sqrt{(x-a)(b-x)}}{2\pi x}\, dx = \begin{cases} 1, & \text{if } c \geq 1, \\ c, & \text{if } c < 1, \end{cases}$$

and this implies that μ_c is a probability measure for all c in $]0,\infty[$.

In [OP], the measure μ_c is called the *Machenko-Pastur distribution* (cf. [MP]). It is also known as the *free analog of the Poisson distribution* with parameter c (cf. [VDN]). □

6.6 Lemma. *Assume that $c \in]0,\infty[$, and let $(m(n))_n$ be a sequence of positive integers, such that $\lim_{n\to\infty} \frac{m(n)}{n} = c$. Consider furthermore a sequence (Y_n) of random matrices, such that for all n in $\mathbb{N}$, $Y_n \in \mathrm{GRM}(m(n),n,\frac{1}{n})$. We then have*

(i) *For any s in $\mathbb{C}$ and n in $\mathbb{N}$, such that $n > \mathrm{Re}(s)$, we have that*

$$\mathbb{E}\Big(\big|\mathrm{tr}_n[Y_n^*Y_n\exp(sY_n^*Y_n)]\big|\Big) < \infty.$$

(ii) *For any complex number s, we have that*

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[Y_n^*Y_n\exp(sY_n^*Y_n)]\big) = \int_0^\infty x\exp(sx)\ d\mu_c(x), \tag{6.22}$$

and the convergence is uniform on compact subsets of $\mathbb{C}$.

Proof. For each n in $\mathbb{N}$, put $B_n = \sqrt{n}Y_n$, and note that $B_n \in \mathrm{GRM}(m(n),n,1)$. If $s \in \mathbb{C}$ and $n \in \mathbb{N}$ such that $n > \mathrm{Re}(s)$, then by Theorem 6.4, we have that

$$\begin{aligned}\mathbb{E}\Big(\big|\mathrm{tr}_n[Y_n^*Y_n\exp(sY_n^*Y_n)]\big|\Big) &\leq \mathbb{E}\Big(\mathrm{tr}_n\big[Y_n^*Y_n\exp(\mathrm{Re}(s)Y_n^*Y_n)\big]\Big) \\ &\leq \tfrac{1}{n^2}\mathbb{E}\Big(\mathrm{Tr}_n\big[B_n^*B_n\exp(\tfrac{\mathrm{Re}(s)}{n}B_n^*B_n)\big]\Big) < \infty,\end{aligned}$$

which proves (i). Regarding (ii), Theorem 6.4 yields furthermore (still under the assumption that $n > \mathrm{Re}(s)$), that

$$\begin{aligned}\mathbb{E}\big(\mathrm{tr}_n[Y_n^*Y_n\exp(sY_n^*Y_n)]\big) &= \tfrac{1}{n^2}\mathbb{E}\big(\mathrm{Tr}_n[B_n^*B_n\exp(\tfrac{s}{n}B_n^*B_n)]\big)\\ &= \frac{m(n)\cdot F(1-m(n),1-n,2;\frac{s^2}{n^2})}{n\cdot(1-\frac{s}{n})^{m(n)+n}}.\end{aligned}$$

Here,

$$F(1-m(n),1-n,2;\tfrac{s^2}{n^2}) = \sum_{j=0}^{\infty}\frac{1}{j+1}\binom{m(n)-1}{j}\binom{n-1}{j}\frac{s^{2j}}{n^{2j}},$$

with the convention that $\binom{k}{j}=0$, whenever $j>k$, $(j,k\in\mathbb{N}_0)$. Since $\lim_{n\to\infty}\frac{m(n)}{n}=c$, it follows that for each fixed j in $\mathbb{N}$,

$$\lim_{n\to\infty}\frac{1}{j+1}\binom{m(n)-1}{j}\binom{n-1}{j}\frac{s^{2j}}{n^{2j}} = \frac{(cs^2)^j}{j!(j+1)!}.$$

Moreover, with $\gamma := \sup_{n\in\mathbb{N}}\frac{m(n)}{n} < \infty$, we have that

$$\Big|\frac{1}{j+1}\binom{m(n)-1}{j}\binom{n-1}{j}\frac{s^{2j}}{n^{2j}}\Big| \le \frac{(\gamma s^2)^j}{j!(j+1)!},$$

for all j,n. Hence by Lebesgue's Theorem on Dominated Convergence (for series), it follows that

$$\lim_{n\to\infty}F(1-m(n),1-n,2;\tfrac{s^2}{n^2}) = \sum_{j=0}^{\infty}\frac{(cs^2)^j}{j!(j+1)!}, \qquad (s\in\mathbb{C}), \tag{6.23}$$

and moreover

$$|F(1-m(n),1-n,2;\tfrac{s^2}{n^2})| \le \sum_{j=0}^{\infty}\frac{(c|s|^2)^j}{j!(j+1)!} \le \exp(\gamma|s|^2), \qquad (s\in\mathbb{C}). \tag{6.24}$$

A standard application of Cauchy's Integral Formula and Lebesgue's Theorem on Dominated Convergence (using (6.24)) now shows that the convergence in (6.23) actually holds uniformly on compact subsets of $\mathbb{C}$.

Recalling next that $\lim_{n\to\infty}(1-\frac{s}{n})^n = \exp(-s)$ for any complex number s, it follows that

$$\lim_{n\to\infty}(1-\tfrac{s}{n})^{m(n)+n} = \exp(-(c+1)s), \qquad (s\in\mathbb{C}). \tag{6.25}$$

Using then that

$$\big|(1-\tfrac{s}{n})^{m(n)+n}\big| \le (1-\tfrac{|s|}{n})^{(\gamma+1)n} \le \exp((\gamma+1)|s|), \qquad (s\in\mathbb{C}),$$

it follows as before, that (6.25) holds uniformly on compact subsets of $\mathbb{C}$.

Taken together, we have verified that,

$$\lim_{n\to\infty} \mathbb{E}\big(\mathrm{tr}_n[Y_n^*Y_n \exp(sY_n^*Y_n)]\big) = c\exp((c+1)s)\sum_{j=0}^{\infty} \frac{(cs^2)^j}{j!(j+1)!}, \qquad (s \in \mathbb{C}), \tag{6.26}$$

and that the convergence is uniform on compact subsets of $\mathbb{C}$.

It remains thus to show that

$$\int_0^\infty x\exp(sx)\ d\mu_c(x) = c\exp((c+1)s)\sum_{j=0}^{\infty} \frac{(cs^2)^j}{j!(j+1)!}, \qquad (s \in \mathbb{C}). \tag{6.27}$$

Note for this, that for any c in $]0,\infty[$,

$$\int_0^\infty x\exp(sx)\ d\mu_c(x) = \tfrac{1}{2\pi}\int_{c+1-2\sqrt{c}}^{c+1+2\sqrt{c}} \exp(sx)\sqrt{4c-(x-c-1)^2}\ dx,$$

since, in the case where $c<1$, the mass at 0 for μ_c does not contribute to the integral. Applying then the substitution $x = c+1+\sqrt{c}y$, we get that

$$\int_0^\infty x\exp(sx)\ d\mu_c(x) = \frac{c\exp((c+1)s)}{2\pi}\int_{-2}^{2}\sqrt{4-y^2}\exp(s\sqrt{c}y)\ dy, \tag{6.28}$$

and here, as we saw in the proof of Theorem 2.8,

$$\tfrac{1}{2\pi}\int_{-2}^{2}\sqrt{4-y^2}\exp(ty)\ dy = \sum_{j=0}^{\infty}\frac{t^{2j}}{j!(j+1)!}, \qquad (t\in\mathbb{C}). \tag{6.29}$$

Combining (6.28) and (6.29), we obtain (6.27). ■

6.7 Theorem. *Assume that $c \in]0,\infty[$ and let $(m(n))_n$ be a sequence of positive integers, such that $\lim_{n\to\infty}\frac{m(n)}{n} = c$. Consider furthermore a sequence (Y_n) of random matrices, satisfying that $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$ for all n. We then have*

(i) *For any s in $\mathbb{C}$ and n in $\mathbb{N}$, such that $n > \mathrm{Re}(s)$,*

$$\mathbb{E}\Big(\big|\mathrm{tr}_n[\exp(sY_n^*Y_n)]\big|\Big) < \infty,$$

and moreover

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[\exp(sY_n^*Y_n)]\big) = \int_0^\infty \exp(sx)\ d\mu_c(x), \qquad (s\in\mathbb{C}), \tag{6.30}$$

with uniform convergence on compact subsets of $\mathbb{C}$.

(ii) *For any positive integer p,*

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[(Y_n^*Y_n)^p]\big) = \int_0^\infty x^p\ d\mu_c(x). \tag{6.31}$$

(iii) *For any bounded continuous function $f\colon [0,\infty[\to\mathbb{C}$,*

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[f(Y_n^*Y_n)]\big) = \int_0^\infty f(x)\ d\mu_c(x). \tag{6.32}$$

Proof. Since $\exp(u) \le 1 + u\exp(u)$, for any u in $[0,\infty[$, the first statement of (i) follows immediately from Lemma 6.6.

Consider next an element Y of $\mathrm{GRM}(m,n,\frac{1}{n})$, and put $B = \sqrt{n}Y \in \mathrm{GRM}(m,n,1)$. Then by Corollary 5.6, we have that

$$\begin{aligned}
&\mathbb{E}\big(\mathrm{tr}_n[f(Y^*Y)]\big)\\
&\quad= \tfrac{1}{n}\mathbb{E}\big(\mathrm{Tr}_n[f(\tfrac{1}{n}B^*B)]\big)\\
&\quad= \begin{cases} \displaystyle\int_0^\infty \big(\textstyle\sum_{k=0}^{n-1}\varphi_k^{m-n}(nx)^2\big)f(x)\,dx, & \text{if } m \ge n,\\ (1-\frac{m}{n})f(0) + \displaystyle\int_0^\infty \big(\textstyle\sum_{k=0}^{m-1}\varphi_k^{n-m}(nx)^2\big)f(x)\,dx, & \text{if } m < n,\end{cases}
\end{aligned} \tag{6.33}$$

for any Borel function $f\colon [0,\infty[\to \mathbb{C}$, for which the integrals on the right hand side make sense.

From this formula, it follows easily that $s \mapsto \mathbb{E}\big(\mathrm{tr}_n[\exp(sY^*Y)]\big)$, is an analytic function in the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) < n\}$, and that

$$\frac{d}{ds}\mathbb{E}\big(\mathrm{tr}_n[\exp(sY^*Y)]\big) = \mathbb{E}\big(\mathrm{tr}_n[Y^*Y\exp(sY^*Y)]\big), \qquad (\mathrm{Re}(s) < n). \tag{6.34}$$

Now for each n in $\mathbb{N}$, define

$$f_n(s) = \mathbb{E}\big(\mathrm{tr}_n[\exp(sY_n^*Y_n)]\big), \qquad (\mathrm{Re}(s) < n),$$

where (Y_n) is as set out in the theorem. Define furthermore,

$$f(s) = \int_0^\infty \exp(sx)\,d\mu_c(x), \qquad (s \in \mathbb{C}).$$

Since μ_c has compact support, f is an entire function, and moreover

$$f'(s) = \int_0^\infty x\exp(sx)\,d\mu_c(x), \qquad (s \in \mathbb{C}).$$

It follows thus by (6.34) and Lemma 6.6, that

$$f_n'(s) \to f'(s), \quad \text{as } n \to \infty, \qquad (s \in \mathbb{C}), \tag{6.35}$$

with uniform convergence on compact subsets of $\mathbb{C}$. Now for fixed s in $\mathbb{C}$, we may choose a smooth path $\gamma\colon [0,1] \to \mathbb{C}$, such that $\gamma(0) = 0$ and $\gamma(1) = s$. Then since $f_n(0) = 1 = f(0)$ for all n, it follows that

$$f_n(s) - f(s) = \int_\gamma \big(f_n'(z) - f'(z)\big)\,dz,$$

whenever $n > \mathrm{Re}(s)$. Combining this fact with (6.35), it follows readily that $f_n(s) \to f(s)$ for all s in $\mathbb{C}$, and that the convergence is uniform on compact subsets of $\mathbb{C}$. This completes the proof of (i). We note next that (ii) follows from (i), by repeating the argument given in the proof of Theorem 2.8. Finally, to prove (iii), for any m, n in $\mathbb{N}$, let $\sigma_{m,n}$ denote the probability measure on $[0, \infty[$, given by

$$\sigma_{m,n} = \begin{cases} \Big(\sum_{k=0}^{n-1} \varphi_k^{m-n}(nx)^2\Big) \cdot dx, & \text{if } m \geq n \\ (1 - \frac{m}{n})\delta_0 + \Big(\sum_{k=0}^{m-1} \varphi_k^{n-m}(mx)^2\Big) \cdot dx, & \text{if } m < n. \end{cases}$$

Applying then (6.33) and (6.30) in the case $s = it$, $t \in \mathbb{R}$, it follows from the implication (iv) $\Rightarrow$ (iii) in Proposition 2.1, that (6.32) holds. ∎

6.8 Remark. In [OP, Proposition 1.1], Oravecz and Petz showed that

$$\int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{p}\sum_{k=1}^{p} \binom{p}{k}\binom{p}{k-1} c^k, \qquad (c > 0,\ p \in \mathbb{N}), \tag{6.36}$$

by solving a recursion formula for the moments of μ_c. It is also possible to derive this formula directly: For p in $\mathbb{N}$, the point-mass at 0 for μ_c (if $c < 1$), does not contribute to the integral on the left hand side of (6.35), and hence

$$\int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{2\pi}\int_{c+1-2\sqrt{c}}^{c+1+2\sqrt{c}} \sqrt{4c - (x-c-1)^2}\, x^{p-1} \, dx.$$

Applying now the substitution $x = c + 1 + 2\sqrt{c}\cos\theta$, $\theta \in [0, \pi]$, we get that

$$\begin{aligned} \int_0^\infty x^p \, d\mu_c(x) &= \tfrac{2c}{\pi}\int_0^\pi \sin^2\theta \cdot (c + 1 + 2\sqrt{c}\cos\theta)^{p-1} \, d\theta \\ &= \tfrac{c}{\pi}\int_{-\pi}^\pi \sin^2\theta \cdot (c + 1 + 2\sqrt{c}\cos\theta)^{p-1} \, d\theta. \end{aligned}$$

Consider next the functions,

$$g_p(\theta) = (1 + \sqrt{c}e^{i\theta})^{p-1}, \quad h(\theta) = e^{i\theta} g_p(\theta), \quad \text{and} \quad k(\theta) = e^{-i\theta} g_p(\theta), \qquad (\theta \in [0, \pi]).$$

Using then the formula: $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$, we find that

$$\begin{aligned} \int_0^\infty x^p \, d\mu_c(x) &= \tfrac{c}{2\pi}\int_{-\pi}^\pi \mathrm{Re}(1 - e^{i2\theta}) \cdot |g_p(\theta)|^2 \, d\theta \\ &= \tfrac{c}{2\pi}\Big(\int_{-\pi}^\pi |g(\theta)|^2 \, d\theta - \mathrm{Re}\Big(\int_{-\pi}^\pi h_p(\theta)\overline{k_p(\theta)} \, d\theta\Big)\Big). \end{aligned} \tag{6.37}$$

By the binomial formula and Parseval's formula, we have here that

$$\tfrac{c}{2\pi}\int_{-\pi}^\pi |g(\theta)|^2 \, d\theta = \sum_{j=0}^{p-1} \binom{p-1}{j}^2 c^j,$$

and that

$$\tfrac{c}{2\pi}\int_{-\pi}^\pi h_p(\theta)\overline{k_p(\theta)} \, d\theta = \sum_{j=0}^{p-1} \binom{p-1}{j-1}\binom{p-1}{j+1} c^j,$$

where we have put $\binom{p-1}{-1} = \binom{p-1}{p} = 0$. A simple computation shows that

$$\binom{p-1}{j}^2 - \binom{p-1}{j-1}\binom{p-1}{j+1} = \frac{1}{p+1}\binom{p}{j+1}\binom{p}{j}, \qquad (0 \le j \le p-1). \tag{6.38}$$

Now (6.36) follows by combining (6.37)-(6.38), and substituting j by $j-1$. $\square$

7 Almost Sure Convergence of the Largest and Smallest Eigenvalues of $Y_n^* Y_n$

In the paper [Gem] from 1981, Geman studied a sequence (T_n) of random matrices, such that for all n in $\mathbb{N}$, T_n is an $m(n) \times n$ random matrix, satisfying that the entries $t_{jk}^{(n)}$, $1 \le j \le m(n)$, $1 \le k \le n$, are independent, identically distributed, real valued random variables, with mean 0 and variance 1. Under the assumption that $\lim_{n\to\infty} \frac{m(n)}{n} = c$, and some extra conditions on the growth of the higher order moments of the entries $t_{jk}^{(n)}$, Geman proved that

$$\lim_{n\to\infty} \lambda_{\max}(\tfrac{1}{n} T_n^t T_n) = (\sqrt{c}+1)^2, \qquad \text{almost surely,} \tag{7.1}$$

where $\lambda_{\max}(\frac{1}{n} T_n^t T_n)$ denotes the largest eigenvalue of $\frac{1}{n} T_n^t T_n$. Under the additional assumptions that T_n is Gaussian for all n, (i.e., that $t_{jk}^{(n)} \sim N(0,1)$ for all j,k,n), and that $m(n) \ge n$ for all n, Silverstein proved in 1985, that

$$\lim_{n\to\infty} \lambda_{\min}(\tfrac{1}{n} T_n^t T_n) = (\sqrt{c}-1)^2, \qquad \text{almost surely,} \tag{7.2}$$

where $\lambda_{\min}(\frac{1}{n} T_n^t T_n)$ denotes the smallest eigenvalue of $\frac{1}{n} T_n^t T_n$ (cf. [Si] and [Sz, pp. 929-934]). Both Geman's condition and Silverstein's condition have later been relaxed to the condition that the entries of T_n have finite fourth moment, i.e., $\mathbb{E}\big(|t_{jk}^{(n)}|^4\big) < \infty$ (cf. [YBK] and [BY]). This condition is also necessary for (7.1) (cf. [BSY]).

The above quoted papers consider only real random matrices, but it is not hard to generalize the proofs to the complex case. In this section we prove (7.1) and (7.2) in the complex Gaussian case, by taking a different route, namely by applying the explicit formula for $\mathbb{E}\big(\mathrm{Tr}_n[\exp(B_n^* B_n)]\big)$, $B \in \mathrm{GRM}(m,n,1)$, that we obtained in Section 6. This route is similar to the one we took in Section 3.

7.1 Theorem. *Let c be a strictly positive number, and let $(m(n))_n$ be a sequence of positive integers, such that $\lim_{n\to\infty} \frac{m(n)}{n} = c$. Consider furthermore a sequence (Y_n) of random matrices, defined on the same probability space $(\Omega, \mathcal{F}, p)$, and such that $Y_n \in \mathrm{GRM}(m(n), n, \frac{1}{n})$, for all n. We then have*

$$\lim_{n\to\infty} \lambda_{\max}(Y_n^* Y_n) = (\sqrt{c}+1)^2, \qquad \textit{almost surely,} \tag{7.3}$$

and

$$\lim_{n\to\infty} \lambda_{\min}(Y_n^* Y_n) = \begin{cases} (\sqrt{c}-1)^2, & \text{if } c > 1, \\ 0, & \text{if } c \le 1, \end{cases} \qquad \textit{almost surely.} \tag{7.4}$$

We start by proving two lemmas:

7.2 Lemma. *Consider an element B of* $\mathrm{GRM}(m,n,1)$. *We then have*

(i) *For any t in $[0,\frac{1}{2}]$,*

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) \le n\exp\big((\sqrt{m}+\sqrt{n})^2t+(m+n)t^2\big), \tag{7.5}$$

(ii) *If $m \ge n$ and $t \ge 0$, then*

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tB^*B)]\big) \le n\exp\big(-(\sqrt{m}-\sqrt{n})^2t+(m+n)t^2\big). \tag{7.6}$$

Proof. (i) Assume first that $m \ge n$. Then by (6.20) in Theorem 6.4, we have that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) = \sum_{k=1}^{n}\frac{F(m-k,n-k,1;t^2)}{(1-t)^{n+m+1-2k}}, \qquad (t \in]-\infty,1[). \tag{7.7}$$

For k in $\{1,2,\dots,n\}$, we have here that

$$\begin{aligned} F(m-k,n-k,1;t^2) &= \sum_{j=0}^{\infty}\binom{m-k}{j}\binom{n-k}{j}t^{2j} \\ &\le \sum_{j=0}^{\infty}\frac{(m-k)^j(n-k)^j}{(j!)^2}t^{2j} \\ &\le \Big(\sum_{j=0}^{\infty}\frac{(\sqrt{(m-k)(n-k)}|t|)^j}{j!}\Big)^2, \end{aligned}$$

and thus we obtain the estimate

$$F(m-k,n-k,1;t^2) \le \exp(2\sqrt{(m-k)(n-k)}|t|), \qquad (k \in \{1,2,\dots,n\}). \tag{7.8}$$

For t in $[0,1[$ and k in $\mathbb{N}$, we have also that $(1-t)^{2k-1} \le 1$, and hence by (7.7) and (7.8), we get the estimate

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) \le \sum_{k=1}^{n}\frac{\exp(2\sqrt{mn}t)}{(1-t)^{m+n}} = \frac{n\exp(2\sqrt{mn}t)}{(1-t)^{m+n}}, \qquad (t \in [0,1[). \tag{7.9}$$

Regarding the denominator of the fraction on the right hand side of (7.9), note that for t in $[0,\frac{1}{2}]$, we have that

$$-\log(1-t) = \sum_{n=1}^{\infty}\frac{t^n}{n} \le 2+\tfrac{1}{2}(t^2+t^3+t^4+\cdots) \le t+t^2,$$

and hence that $(1-t)^{-1} \le \exp(t+t^2)$. Inserting this inequality in (7.9), we obtain (7.5), in the case where $m \ge n$.

If, conversely, $m < n$, then by application of (6.21) in Theorem 6.4, we get as above, that for t in $[0,1[$,

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tB^*B)]\big) \le (n-m) + \frac{m\exp(2\sqrt{mn}t)}{(1-t)^{m+n}} \le \frac{n\exp(2\sqrt{mn}t)}{(1-t)^{m+n}}.$$

Estimating then the denominator as above, it follows that (7.5) holds for all t in $[0,\frac{1}{2}]$.

(ii) Assume that $m \ge n$, and note then that for k in $\{1,2,\dots,n\}$, we have that

$$\sqrt{(m-k)(n-k)} \le \sqrt{mn} - k.$$

Combining this inequality with (7.7) and (7.8), we get for t in $[0,\infty[$, that

$$\begin{aligned}\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tB^*B)]\big) &= \sum_{k=1}^{n} \frac{F(m-k,n-k,1;t^2)}{(1+t)^{m+n+1-2k}} \\ &\le \frac{1}{(1+t)^{m+n+1}}\Big(\sum_{k=1}^{n} \frac{\exp(2(\sqrt{mn}-k)t)}{(1+t)^{-2k}}\Big) \\ &\le \frac{\exp(2\sqrt{mn}t)}{(1+t)^{m+n}}\Big(\sum_{k=1}^{n} \big((1+t)\exp(-t)\big)^{2k}\Big).\end{aligned}$$

Here, $(1+t)\exp(-t) \le 1$ for all t in $[0,\infty[$, and hence we see that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tB^*B)]\big) \le \frac{n\exp(2\sqrt{mn}t)}{(1+t)^{m+n}}, \qquad (t \in [0,\infty[). \tag{7.10}$$

Regarding the denominator of the fraction on the right hand side of (7.10), we note that for any t in $[0,\infty[$, we have by Taylor's formula with remainder term,

$$\log(1+t) = t - \tfrac{t^2}{2} + \tfrac{t^3}{3(1+\xi(t))^3},$$

for some number $\xi(t)$ in $[0,t[$. It follows thus that $\log(1+t) \ge t - \frac{t^2}{2}$, and hence that $(1+t)^{-1} \le \exp(-t+t^2)$, for any t in $[0,\infty[$. Combining this fact with (7.10), we obtain (7.6). ■

7.3 Lemma. *Let c, $(m(n))_n$ and (Y_n) be as set out in Theorem 7.1. We then have*

(i) *For almost all ω in Ω,*

$$\limsup_{n\to\infty} \lambda_{\max}\big(Y_n^*(\omega)Y_n(\omega)\big) \le (\sqrt{c}+1)^2. \tag{7.11}$$

(ii) *If $c > 1$, then for almost all ω in Ω,*

$$\liminf_{n\to\infty} \lambda_{\min}\big(Y_n^*(\omega)Y_n(\omega)\big) \ge (\sqrt{c}-1)^2. \tag{7.12}$$

Proof. For each n in $\mathbb{N}$, we put $c_n = \frac{m(n)}{n}$, and $B_n = \sqrt{n}Y_n \in \mathrm{GRM}(m(n), n, 1)$. By Lemma 7.2, we have then that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(tY_n^*Y_n)]\big) \le n\exp\big((\sqrt{c_n}+1)^2 t + \tfrac{1}{n}(c_n+1)t^2\big), \qquad (t \in [0, \tfrac{n}{2}]), \tag{7.13}$$

and that

$$\mathbb{E}\big(\mathrm{Tr}_n[\exp(-tY_n^*Y_n)]\big) \le n\exp\big(-(\sqrt{c_n}-1)^2 t + \tfrac{1}{n}(c_n+1)t^2\big), \qquad (t \in [0, \infty[), \tag{7.14}$$

Since all the eigenvalues of $\exp(\pm tY_n^*Y_n)$ are positive, we have here for any t in $[0, \infty[$, that

$$\begin{aligned}\mathrm{Tr}_n[\exp(tY_n^*(\omega)Y_n(\omega))] &\ge \lambda_{\max}\big(\exp(tY_n^*(\omega)Y_n(\omega))\big)\\ &= \exp\big(t\lambda_{\max}(Y_n^*(\omega)Y_n(\omega))\big), \qquad (\omega \in \Omega),\end{aligned} \tag{7.15}$$

and that

$$\begin{aligned}\mathrm{Tr}_n[\exp(-tY_n^*(\omega)Y_n(\omega))] &\ge \lambda_{\max}\big(\exp(-tY_n^*(\omega)Y_n(\omega))\big)\\ &= \exp\big(-t\lambda_{\min}(Y_n^*(\omega)Y_n(\omega))\big), \qquad (\omega \in \Omega).\end{aligned} \tag{7.16}$$

For fixed n in $\mathbb{N}$, t in $[0, \frac{n}{2}]$ and ϵ in $]0, 1[$, we get now by the Chebychev Inequality, (7.15) and (7.13),

$$\begin{aligned}P\Big(\lambda_{\max}(Y_n^*Y_n) \ge (\sqrt{c_n}+1)^2 + \epsilon\Big) &= P\Big(\exp\big[t\lambda_{\max}(Y_n^*Y_n) - t(\sqrt{c_n}+1)^2 - t\epsilon\big] \ge 1\Big)\\ &\le \mathbb{E}\Big(\exp\big[t\lambda_{\max}(Y_n^*Y_n) - t(\sqrt{c_n}+1)^2 - t\epsilon\big]\Big)\\ &\le \exp[-t(\sqrt{c_n}+1)^2 - t\epsilon]\mathbb{E}\big(\mathrm{Tr}_n[\exp(tY_n^*Y_n)]\big)\\ &\le n\exp\big(-t\epsilon + \tfrac{1}{n}(c_n+1)t^2\big).\end{aligned}$$

For fixed n in $\mathbb{N}$ and ϵ in $]0, \infty[$, the function $t \mapsto -t\epsilon + \frac{1}{n}(c_n+1)t^2$, attains its minimum at $t_0 = \frac{n\epsilon}{2(c_n+1)} \in [0, \frac{n}{2}]$. With this value of t, the above inequality becomes

$$P\big(\lambda_{\max}(Y_n^*Y_n) \ge (\sqrt{c_n}+1)^2 + \epsilon\big) \le n\exp\big(-t_0\epsilon + \tfrac{1}{n}(c_n+1)t_0^2\big) = n\exp\big(\tfrac{-n\epsilon^2}{4(c_n+1)}\big).$$

Since $c_n \to c$ as $n \to \infty$, the sequence (c_n) is bounded, and thus it follows that

$$\sum_{n=1}^{\infty} P\big(\lambda_{\max}(Y_n^*Y_n) \ge (\sqrt{c_n}+1)^2 + \epsilon\big) \le \sum_{n=1}^{\infty} n\exp\big(\tfrac{-n\epsilon^2}{4(c_n+1)}\big) < \infty.$$

Hence the Borel-Cantelli lemma yields, that on a set with probability one, we have that

$$\lambda_{\max}(Y_n^*Y_n) \le (\sqrt{c_n}+1)^2 + \epsilon, \quad \text{eventually},$$

and consequently that

$$\limsup_{n\to\infty} \lambda_{\max}(Y_n^*Y_n) \le \limsup_{n\to\infty} \big[(\sqrt{c_n}+1)^2 + \epsilon\big] = (\sqrt{c}+1)^2 + \epsilon.$$

Taken together, we have verified that for any ϵ in $]0,\infty[$, we have that

$$P\Big(\limsup_{n\to\infty}\lambda_{\max}(Y_n^*Y_n)\le(\sqrt{c}+1)^2+\epsilon\Big)=1,$$

and this proves (7.11). The proof of (7.12) can be carried out in exactly the same way, using (7.16) and (7.14) instead of (7.15) respectively (7.13). We leave the details to the reader. ■

To conclude the proof of Theorem 7.1, we must, as in Geman's paper [Gem], rely on Wachter's result from [Wa2] on almost sure convergence of the empirical distribution of the eigenvalues to the measure μ_c. As mentioned in the beginning of Section 6, the random matrices considered by Wachter have real valued (but not necessarily Gaussian) entries. His method works also for random matrices with complex valued entries, but in the following we shall give a short proof for the case of complex Gaussian random matrices, based on the "concentration of measures phenomenon" in the form of Lemma 3.4.

7.4 Proposition. (cf. [Wa2]) *Let c, $(m(n))_n$ and (Y_n) be as in Theorem 7.1, and for all n in $\mathbb{N}$ and ω in Ω, let $\mu_{n,\omega}$ denote the empirical distribution of the eigenvalues of $Y_n^*(\omega)Y_n(\omega)$, i.e.,*

$$\mu_{n,\omega}=\tfrac{1}{n}\textstyle\sum_{j=1}^n\delta_{\lambda_k(Y_n^*(\omega)Y_n(\omega))},$$

where, as usual, $\lambda_1(Y_n^*(\omega)Y_n(\omega))\le\cdots\le\lambda_n(Y_n^*(\omega)Y_n(\omega))$ are the ordered eigenvalues of $Y_n^*(\omega)Y_n(\omega)$. We then have

(i) *For almost all ω in Ω, $\mu_{n,\omega}$ converges weakly to the measure μ_c introduced in Definition 6.5.*

(ii) *On a set with probability 1, we have for any interval I in $\mathbb{R}$, that*

$$\lim_{n\to\infty}\big(\tfrac{1}{n}\cdot\mathrm{card}\big[\mathrm{sp}(Y_n^*Y_n)\cap I\big]\big)=\mu_c(I).$$

Proof. Note first that (ii) follows from (i) and Remark 2.3.

To prove (i), it suffices, as in the proof of Proposition 3.6, to show that for every fixed function f from $C_c^1(\mathbb{R})$, we have that

$$\lim_{n\to\infty}\mathrm{tr}_n[f(Y_n^*Y_n)]=\int_0^\infty f\,d\mu_c,\qquad\text{almost surely.}$$

So let such an f be given, and define $g\colon\mathbb{R}\to\mathbb{C}$ by the equation: $g(x)=f(x^2)$, $(x\in\mathbb{R})$. Then $g\in C_c^1(\mathbb{R})$, so in particular g is Lipschitz with constant

$$c=\sup_{x\in\mathbb{R}}|g'(x)|<\infty.$$

Consider furthermore fixed m,n in $\mathbb{N}$, and for A,B in $M_{m,n}(\mathbb{C})$, define $\tilde{A}$ and $\tilde{B}$ in $M_{m+n}(\mathbb{C})$ by the equations

$$\tilde{A}=\begin{pmatrix}0&A^*\\A&0\end{pmatrix},\qquad\tilde{B}=\begin{pmatrix}0&B^*\\B&0\end{pmatrix}.$$

By Lemma 3.5 it follows then that

$$\|g(\tilde{A}) - g(\tilde{B})\|_{HS} \leq c\|\tilde{A} - \tilde{B}\|_{HS}. \tag{7.17}$$

Note here that

$$\tilde{A}^2 = \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}, \qquad \tilde{B}^2 = \begin{pmatrix} B^*B & 0 \\ 0 & BB^* \end{pmatrix}.$$

so that

$$g(\tilde{A}) = \begin{pmatrix} f(A^*A) & 0 \\ 0 & f(AA^*) \end{pmatrix}, \qquad g(\tilde{B}) = \begin{pmatrix} f(B^*B) & 0 \\ 0 & f(BB^*) \end{pmatrix}.$$

Hence, it follows from (7.17) that

$$\|f(A^*A) - f(B^*B)\|_{HS}^2 + \|f(AA^*) - f(BB^*)\|_{HS}^2 \leq c^2\big(\|A-B\|_{HS}^2 + \|A^* - B^*\|_{HS}^2\big).$$

Since $\|A^* - B^*\|_{HS}^2 = \|A-B\|_{HS}^2$, the above inequality implies that

$$\|f(A^*A) - f(B^*B)\|_{HS} \leq c\sqrt{2}\|A-B\|_{HS},$$

and hence, by the Cauchy-Schwarz inequality, that

$$\big|\mathrm{tr}_n[f(A^*A)] - \mathrm{tr}_n[f(B^*B)]\big| \leq c\sqrt{\tfrac{2}{n}}\|A-B\|_{HS}.$$

It follows thus, that the function $F\colon M_{m,n}(\mathbb{C}) \to \mathbb{R}$, given by

$$F(A) = \mathrm{tr}_n[f(A^*A)], \qquad (A \in M_{m,n}(\mathbb{C})), \tag{7.18}$$

satisfies the Lipschitz condition

$$|F(A) - F(B)| \leq c\sqrt{\tfrac{2}{n}}\|A-B\|_{HS}, \qquad (A, B \in M_{m,n}(\mathbb{C})). \tag{7.19}$$

The linear bijection $\Phi\colon M_{m,n}(\mathbb{C}) \to \mathbb{R}^{2mn}$, given by

$$\Phi(A) = \big(\mathrm{Re}(A_{jk}), \mathrm{Im}(A_{jk})\big)_{\substack{1\leq j\leq m,\\ 1\leq k\leq n}} \qquad (A \in M_{m,n}(\mathbb{C})),$$

transforms the distribution on $M_{m,n}(\mathbb{C})$ of an element of $\mathrm{GRM}(m, n, \frac{1}{n})$ onto the joint distribution of $2mn$ independent, identically $N(0, \frac{1}{2n})$-distributed random variables, i.e., the distribution $G_{2mn,(2n)^{-\frac{1}{2}}}$ on $\mathbb{R}^{2mn}$ with density

$$\frac{dG_{2mn,(2n)^{-\frac{1}{2}}}(x)}{dx} = \Big(\frac{n}{\pi}\Big)^{mn} \exp(-n\|x\|^2), \qquad (x \in \mathbb{R}^{2mn}),$$

w.r.t. Lebesque measure on $\mathbb{R}^{2mn}$. Moreover, the Hilbert-Schmidt norm on $M_{m,n}(\mathbb{C})$ corresponds to the Euclidean norm on $\mathbb{R}^{2mn}$ via the mapping Φ. Combining these observations with Lemma 3.4 (in the case $\sigma^2 = \frac{1}{\sqrt{2n}}$) and (7.19), it follows that with (Y_n) as set out in the proposition, we have for any n in $\mathbb{N}$ and t from $]0, \infty[$, that

$$P\big(|F(Y_n) - \mathbb{E}(F(Y_n))| > t\big) \leq 2\exp(-\tfrac{n^2Kt^2}{c^2}),$$

where $K = \frac{2}{\pi^2}$. It follows thus by application of the Borel-Cantelli lemma, that

$$\lim_{n\to\infty} |F(Y_n) - \mathbb{E}(F(Y_n))| = 0, \qquad \text{almost surely.}$$

Using then (7.18) and Theorem 6.7(iii), we get that

$$\lim_{n\to\infty} \operatorname{tr}_n[f(Y_n^*Y_n)] = \int_0^\infty f \, d\mu_c, \qquad \text{almost surely,}$$

as desired. ∎

Proof of Theorem 7.1. By Lemma 7.3, we only need to show, that for any c from $]0,\infty[$, we have that

$$\liminf_{n\to\infty} \lambda_{\max}(Y_n^*Y_n) \;\geq\; (\sqrt{c}+1)^2, \qquad \text{almost surely,} \tag{7.20}$$

$$\limsup_{n\to\infty} \lambda_{\min}(Y_n^*Y_n) \;\leq\; \begin{cases} (\sqrt{c}-1)^2, & \text{if } c > 1, \\ 0, & \text{if } c \leq 1, \end{cases} \qquad \text{almost surely.} \tag{7.21}$$

By Proposition 7.4, it follows, that for any strictly positive ϵ and almost all ω from Ω, the numbers of eigenvalues of $Y_n^*(\omega)Y_n(\omega)$ in the intervals $\big[(\sqrt{c}+1)^2-\epsilon, (\sqrt{c}+1)^2\big]$ and $\big[(\sqrt{c}-1)^2, (\sqrt{c}-1)^2+\epsilon\big]$, both tend to ∞, as $n \to \infty$. This proves (7.20) and, when $c \geq 1$, also (7.21). If $c < 1$, then $m(n) < n$ eventually, and this implies that eventually, 0 is an eigenvalue for $Y_n^*(\omega)Y_n(\omega)$, for any ω in Ω. Hence we conclude that (7.21) holds in this case too. ∎

8 A Recursion Formula for the Moments of the complex Wishart distribution

In [HSS], Hanlon, Stanley and Stembridge used representation theory of the Lie group $U(n)$ to compute the moments $\mathbb{E}\big(\operatorname{Tr}_n[(B^*B)^p]\big)$ of B^*B, when $B \in \mathrm{GRM}(m,n,1)$. They derived the following formula (cf. [HSS, Theorem 2.5]):

$$\mathbb{E}\big(\operatorname{Tr}_n[(B^*B)^p]\big) = \tfrac{1}{p}\sum_{j=1}^{p}(-1)^{j-1}\frac{[m+p-j]_p[n+p-j]_p}{(p-j)!(j-1)!}, \qquad (p \in \mathbb{N}), \tag{8.1}$$

where we apply the notation: $[a]_p = a(a-1)\cdots(a-p+1)$, $(a \in \mathbb{C}, p \in \mathbb{N}_0)$.

By application of the results of Section 6, we can derive another explicit formula for the moments of B^*B.

8.1 Proposition. *Let m, n be positive integers, and let B be an element of* $\mathrm{GRM}(m,n,1)$. *Then for any p in $\mathbb{N}$, we have that*

$$\mathbb{E}\big(\operatorname{Tr}_n[(B^*B)^p]\big) = mn(p-1)!\sum_{j=0}^{[\frac{p-1}{2}]}\frac{1}{j+1}\binom{m-1}{j}\binom{n-1}{j}\binom{m+n+p-2j-2}{p-2j-1}. \tag{8.2}$$

Proof. In Section 6, we saw that that for any complex number s, such that $\mathrm{Re}(s) < 1$, we have the formula

$$\mathbb{E}\big(\mathrm{Tr}_n[B^*B\exp(sB^*B)]\big) = \frac{m\cdot n\cdot F(1-m,1-n,2;s^2)}{(1-s)^{m+n}}, \tag{8.3}$$

(cf. formula (6.19)). Hence, by Taylor series expansion, for any s in $\mathbb{C}$, such that $|s| < 1$, we have that

$$\sum_{p=1}^{\infty}\frac{\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big)\cdot s^{p-1}}{(p-1)!} = \frac{m\cdot n\cdot F(1-m,1-n,2;s^2)}{(1-s)^{m+n}}, \tag{8.4}$$

Formula (8.2) now follows by multiplying the two power series

$$F(1-m,1-n,2;-s^2) = \sum_{j=0}^{\infty}\frac{1}{j+1}\binom{m-1}{j}\binom{n-1}{j}s^{2j},$$

and

$$(1-s)^{-(m+n)} = \sum_{k=0}^{\infty}\binom{m+n+k-1}{k}s^k,$$

and comparing terms in (8.4). ■

We prove next a recursion formula for the moments of B^*B, similar to the Harer-Zagier recursion formula, treated in Section 4.

8.2 Theorem. *Let m,n be positive integers, let B be an element of* $\mathrm{GRM}(m,n,1)$, *and for p in $\mathbb{N}_0$, define*

$$D(p,m,n) = \mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big). \tag{8.5}$$

Then $D(0,m,n) = n$, $D(1,m,n) = mn$, and for fixed m,n, the numbers $D(p,m,n)$ satisfy the recursion formula

$$D(p+1,m,n) = \tfrac{(2p+1)(m+n)}{p+2}\cdot D(p,m,n) + \tfrac{(p-1)(p^2-(m-n)^2)}{p+2}\cdot D(p-1,m,n), \qquad (p\in\mathbb{N}). \tag{8.6}$$

Proof. Recall from Section 6, that the hyper-geometric function F is defined by the formula

$$F(a,b,c;x) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k,$$

for a,b,c,x in $\mathbb{C}$, such that $c\notin\mathbb{Z}\setminus\mathbb{N}_0$, and $|x| < 1$. For fixed a,b,c, the function $u(x) = F(a,b,c;x)$, is a solution to the differential equation

$$x(1-x)\frac{d^2u}{dx^2} + (c-(a+b+1)x)\frac{du}{dx} - abu = 0,$$

(cf. [HTF, Vol. 1, p.56, formula (1)]). In particular, if $a = 1-n$, $b = 1-m$ and $c = 2$, then u satisfies the differential equation

$$x(1-x)\frac{d^2u}{dx^2} + (2 + (m+n-3)x)\frac{du}{dx} - (m-1)(n-1)u = 0. \tag{8.7}$$

Define now, for these a, b, c,

$$v(t) = u(t^2) = F(1-m, 1-n, 2; t^2), \qquad (|t| < 1).$$

Then (8.7) implies that v satisfies the differential equation

$$t(1-t)\frac{d^2v}{dt^2} + (3 + (2m+2n-5)t^2)\frac{dv}{dt} - 4(m-1)(n-1)tv = 0, \qquad (|t| < 1). \tag{8.8}$$

Define next

$$w(t) = \frac{v(t)}{(1-t)^{m+n}} = \frac{F(1-m, 1-n, 2; t^2)}{(1-t)^{m+n}}, \qquad (|t| < 1).$$

A tedious, but straightforward computation, then shows that w satisfies the differential equation

$$\begin{aligned} t(1-t^2)\frac{d^2w}{dt^2} &+ (3 - 2(m+n)t - 5t^2)\frac{dw}{dt} \\ &- (3(m+n) + 4t - (m-n)^2t)w = 0, \qquad (|t| < 1). \end{aligned} \tag{8.9}$$

Introduce now the power series expansion $w(t) = \sum_{p=0}^{\infty} \alpha_p t^p$, of $w(t)$. Inserting this expansion in (8.9), one finds (after some reductions), that the coefficients α_p satisfy the formulas

$$\alpha_0 = 1, \quad \text{and} \quad \alpha_1 = m+n, \tag{8.10}$$

$$p(p+2)\alpha_p - (2p+1)(m+n)\alpha_{p-1} - (p^2 - (m-n)^2)\alpha_{p-2} = 0, \qquad (p \geq 2). \tag{8.11}$$

On the other hand, inserting the power series expansion of $w(t)$ in (8.4), yields the formula

$$D(p, m, n) = \mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^p]\big) = mn(p-1)!\alpha_{p-1}, \qquad (p \in \mathbb{N}). \tag{8.12}$$

Combining this formula with (8.11), it follows that (8.6) holds, whenever $p \geq 2$. Regarding the case $p = 1$, it follows from (8.10) and (8.12), that $D(1, m, n) = mn$, $D(2, m, n) = mn(m+n)$, and hence (8.6) holds in this case too. It remains to note that $D(0, m, n) = \mathbb{E}\big(\mathrm{Tr}_n[\mathbf{1}_n]\big) = n$. ∎

The recursion formula (8.6) is much more efficient than (8.1) and (8.2) to generate tables of the moments of B^*B. For an element B of $\mathrm{GRM}(m, n, 1)$, we get

$$\begin{aligned}
\mathbb{E}\big(\mathrm{Tr}_n[B^*B]\big) &= mn \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^2]\big) &= m^2n + mn^2 \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^3]\big) &= (m^3n + 3m^2n^2 + mn^3) + mn \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^4]\big) &= (m^4n + 6m^3n^2 + 6m^2n^3 + mn^4) + (5m^2n + 5mn^2) \\
\mathbb{E}\big(\mathrm{Tr}_n[(B^*B)^5]\big) &= (m^5n + 10m^4n^2 + 20m^3n^3 + 10m^2n^4 + mn^5) \\
&\qquad + (15m^3n + 40m^2n^2 + 15mn^3) + 8mn.
\end{aligned}$$

For $p \leq 4$, these moments were also computed in [HSS, p.172] by application of (8.1). Note that only terms of homogeneous degree $p+1-2j$, $j \in \{0,1,2,\dots,[\frac{p-1}{2}]\}$, appear in the above formulas. This is a general fact, which can easily be proved by Theorem 8.2 and induction. If we replace the B from $\mathrm{GRM}(m,n,1)$ considered above by an element Y from $\mathrm{GRM}(m,n,\frac{1}{n})$, and Tr_n by tr_n, then we have to divide the right hand sides of the above formulas by n^{p+1}. Thus with $c = \frac{m}{n}$, we obtain the formulas

$$\begin{aligned}
\mathbb{E}\big(\mathrm{tr}_n[Y^*Y]\big) &= c \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^2]\big) &= c^2+c \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^3]\big) &= (c^3+3c^2+c)+cn^{-2} \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^4]\big) &= (c^4+6c^3+6c^2+c)+(5c^2+5c)n^{-2} \\
\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^5]\big) &= (c^5+10c^4+20c^3+10c^2+c) \\
&\qquad\qquad + (15c^3+40c^2+15c)n^{-2}+8cn^{-4}.
\end{aligned}$$

In general $\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^p]\big)$ is a polynomial of degree $[\frac{p-1}{2}]$ in n^{-2}, for fixed c. By Theorem 8.2, the constant term $\gamma(p,c)$ in this polynomial, satisfies the recursion formula

$$\gamma(p+1,c) = \tfrac{(2p+1)(c+1)}{p+2}\cdot\gamma(p,c) - \tfrac{(p-1)(c-1)^2}{p+2}\cdot\gamma(p-1,c), \qquad (p \in \mathbb{N}),$$

and moreover, $\gamma(0,c)=1$, $\gamma(1,c)=c$. As was proved in [OP], for any c in $]0,\infty[$, the solution to this difference equation is exactly the sequence of moments of the free Poisson distribution μ_c with parameter c, i.e.,

$$\gamma(p,c) = \int_0^\infty x^p \, d\mu_c(x) = \tfrac{1}{p}\sum_{k=1}^{p}\binom{p}{k}\binom{p}{k-1}c^k, \qquad (p \in \mathbb{N}),$$

(cf. [OP, Formula (1.2) and Proposition 1.1]). Thus, if $Y_n \in \mathrm{GRM}(m(n),n,\frac{1}{n})$, for all n in $\mathbb{N}$, and $\frac{m(n)}{n} \to c$, as $n\to\infty$, then we have that

$$\lim_{n\to\infty}\mathbb{E}\big(\mathrm{tr}_n[(Y^*Y)^p]\big) = \gamma(p,c) = \int_0^\infty x^p \, d\mu_c(x),$$

in concordance with Theorem 6.7(ii).

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