Statistics of energy levels and eigenfunctions in disordered and chaotic systems: Supersymmetry approach

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1 Introduction

The supersymmetry method pioneered by Efetov [1] has proven to be a very powerful tool of study of the statistical properties of energy levels and eigenfunctions in disordered systems. The aim of these lectures is to present a tutorial introduction to the method, as well as an overview of the recent developments. One of the major achievements of Efetov [1] was a proof of applicability of the random matrix theory (RMT) to a disordered metallic sample. More recently, the focus of the research interest shifted to the study of system-specific deviations from the universal (RMT) behavior. This will be the central topic of the present lectures. Since these deviations are determined by the underlying classical dynamics, this issue is closely related to the subject of quantum chaos.

We begin by introducing the basic notions and ideas of the supermatrix σ-model method in Sec. 2, considering the level statistics in the Gaussian unitary ensemble (GUE) as the simplest example. In Sec. 3 we turn to the problem of level correlations in a diffusive sample. We outline a derivation of the diffusive σ-model and discuss deviations of the level statistics from RMT. Section 4 is devoted to the eigenfunction statistics. Finally, in Sec. 5 we discuss very recent ideas of application of the method to chaotic ballistic systems.

It is appropriate to give here references to a few recent review articles and books which are close in their topics to the present lectures. These are the Efetov's book [2], the reviews by Guhr, Müller-Groeling, and Weidenmüller [3] and by the author [4], and the proceedings volume of the NATO Advanced Study Institute [5] (in particular the review by Altland, Offer, and Simons [6] there). Other references to the relevant literature will be given in appropriate places below.

2 Introduction to the supersymmetry method and application to RMT

2.1 Green's function approach

We consider the Gaussian unitary ensemble (GUE) defined [7, 8] as an ensemble of $N \times N$ $(N \to \infty)$ Hermitian matrices $\hat{H} = \hat{H}^\dagger$ with probability density

$$\mathcal{P}(\hat{H}) = \mathcal{N} \exp\left\{ -\frac{N}{2} \mathrm{Tr}\, \hat{H}^2 \right\} , \tag{2.1}$$

where $\mathcal{N}$ is a normalization factor. According to (2.1), all the matrix elements of $\hat{H}$ have a Gaussian distribution with variance

$$\langle H_{ij} H^*_{i'j'} \rangle = \frac{1}{N} \delta_{ii'} \delta_{jj'} . \tag{2.2}$$

The factor N in the exponent of (2.1) allows one to keep the eigenvalues E_i of $\hat{H}$ finite in the limit $N \to \infty$, with $\langle E_i^2 \rangle = 1$. We demonstrate below how the well-known results [7, 8] for the average density of states and the two-level correlation function are derived within the supermatrix σ-model method.

The density of states (DOS) is defined as

$$\nu(E) = \frac{1}{N} \mathrm{Tr}\, \delta(E - \hat{H}). \tag{2.3}$$

The first step of the strategy is to express the quantity of interest in terms of the retarded and/or advanced Green's functions

$$\hat{G}_{\mathrm{R,A}} = (E - \hat{H} \pm i\eta)^{-1} \ . \tag{2.4}$$

We have

$$\nu(E) = -\frac{1}{\pi N} \, \mathrm{Im}\, \mathrm{Tr}\, (E - \hat{H} + i\eta)^{-1}|_{\eta \to +0} \ . \tag{2.5}$$

The second step is to write the Green's function (in a general case, a product of Green's functions) as a functional integral. More precisely, in the case of RMT this will be an integral over a vector field with the number of components proportional to N; for the problem of a particle in a random potential (Sec. 3) the discrete index will be replaced by a continuously changing spatial coordinate, which will result in a functional integral. The simplest way to do it is to introduce an N-component complex vector ϕ_i, $i = 1, 2, \ldots, N$, so that

$$(E - \hat{H} + i\eta)^{-1}_{kl} = -i \frac{\int [\mathrm{d}\phi^* \mathrm{d}\phi] \phi_k \phi_l^* \exp\{i \sum_{ij} \phi_i^* [(E + i\eta)\delta_{ij} - H_{ij}]\phi_j\}}{\int [\mathrm{d}\phi^* \mathrm{d}\phi] \exp\{i \sum_{ij} \phi_i^* [(E + i\eta)\delta_{ij} - H_{ij}]\phi_j\}} \ , \tag{2.6}$$

where $[\mathrm{d}\phi^* \mathrm{d}\phi] = \prod_{i=1}^N \mathrm{d}\phi_i^* \mathrm{d}\phi_i$. We have to put the imaginary unit i in front of the quadratic form in the exponent of Eq. (2.6) to get a convergent integral; the convergence being guaranteed by $\eta > 0$.

The next step is to average over the ensemble of random matrices $\hat{H}$. However, direct averaging of Eq. (2.6) is complicated by the fact that $\hat{H}$ enters not only the numerator but also the denominator. If there were no denominator, the averaging over $\hat{H}$ with the probability density (2.1) would be straightforward (a Gaussian integral over $\hat{H}$). One possible way to get rid of the denominator is the replica trick first introduced by Edwards and Anderson [9] in the context of the spin glass theory. The idea of the method is to introduce n species of the field, $\phi_i^{(\alpha)}$, $\alpha = 1, 2, \ldots, n$. Then the denominator Z of Eq. (2.6) is transformed to Z^n and disappears in the limit $n \to 0$. However, the replica trick turns out to be ill-founded [10] for the σ-model approach to the problem of the level and eigenfunction statistics reviewed in these lectures.[1]. An alternative method, which uses a combination of commuting and anticommuting [11] variables instead of the $n \to 0$ replica limit, was proposed by McKane [12] and by Parisi and Sourlas [13]. The effective theories which are obtained in this way are invariant with respect to a transformation mixing commuting ("bosonic") and anticommuting ("fermionic") degrees of freedom, which is conventionally referred to as supersymmetry.[2]

A number of publications containing an introduction to the supersymmetry approach are available; we mention, in addition to the Efetov's review [1] and his more

[1] Note that if the σ-model is treated perturbatively (including the renormalization group treatment which is a resummation of the perturbative expansion), then the replica trick is completely equivalent to the supersymmetry method.

[2] Mathematicians also use the term "graded" (or, more specifically, $\mathbb{Z}_2$-graded) to characterize the arising algebraic structures.

recent book [2], the papers [14, 15, 16]. In particular, in Refs. [15, 16] a detailed exposition of the calculation of the GUE level statistics with the supersymmetry is given. In our presentation we will concentrate on "ideological" aspects of the method and skip some technical details of calculations, which can be found in the literature cited above.

2.2 Supermathematics

We describe now briefly the basic properties of anticommuting (Grassmannian) variables and introduce the notions of the supermathematics which we will use. For a detailed exposition of the Grassmannian mathematics and of the superanalysis the reader is referred to the book [17] (see also [2] for a physicist's summary of the most important properties). We introduce the Grassmannian variables χ_k, χ_k^*, $k = 1, \ldots, N$, which all anticommute to each other:

$$\chi_k \chi_l = -\chi_l \chi_k, \quad \chi_k^* \chi_l = -\chi_l \chi_k^*, \quad \chi_k^* \chi_l^* = -\chi_l^* \chi_k^* \,. \tag{2.7}$$

Note that χ_k and χ_k^* are to be considered as two independent variables. According to (2.7), the square of a Grassmannian variable is zero. As a consequence, any function of Grassmanians, when expanded in a power series, may contain only the terms of the zeroth and the first order in each Grassmannian variable. For this reason, integration over Grassmannians is uniquely defined by the following rules:

$$\int \chi_k \mathrm{d}\chi_k = \int \chi_k^* \mathrm{d}\chi_k^* = \frac{1}{\sqrt{2\pi}} \,; \qquad \int \mathrm{d}\chi_k = \int \mathrm{d}\chi_k^* = 0 \,, \tag{2.8}$$

the differentials $\mathrm{d}\chi_k$, $\mathrm{d}\chi_k^*$ anticommuting with each other and with the Grassmannian variables. Note that the Grassmannian integration is just a formally defined algebraic operation and the question "What is the domain of integration in Eqs. (2.8)?" does not make sense.

Using the rules (2.7), (2.8), one can calculate a Gaussian integral over the Grassmannians,

$$\int \mathrm{d}\chi_1^* \mathrm{d}\chi_1 \ldots \mathrm{d}\chi_N^* \mathrm{d}\chi_N \exp\{-\sum_{kl} \chi_k^* K_{kl} \chi_l\} = \det\left(\frac{K}{2\pi}\right) \,. \tag{2.9}$$

An analogous integral over commuting variables would give $\det^{-1}(K/2\pi)$. This property of the anticommuting variables is the reason for introducing them: it will allow us to replace the Gaussian integral over the commuting variables in the denominator of Eq. (2.6) by an analogous Grassmannian integral in the numerator, thus solving the denominator problem!

It is convenient to define the "complex conjugation" for the Grassmannians by the following rules[3]

$$(\chi_k)^* = \chi_k^*; \quad (\chi_k^*)^* = -\chi_k; \quad (\chi_k \chi_l)^* = \chi_k^* \chi_l^* \,. \tag{2.10}$$

[3]The notion of complex conjugation of Grassmannian variables is introduced for notational convenience only; it allows one to make the treatment of Grassmannians similar to that of ordinary (commuting) variables and thus to introduce compact supersymmetric notations. The fact that two Grassmannian variables have been declared complex conjugate to each other is, however, irrelevant for evaluation of integrals over them, which are simply defined by the rules (2.8).

Furthermore, we introduce the notion of a supervector

$$\Phi = \begin{pmatrix} S_1 \\ \vdots \\ S_n \\ \chi_1 \\ \vdots \\ \chi_n \end{pmatrix} \quad ; \qquad \Phi^\dagger = (S_1^*, \dots, S_n^*, \chi_1^*, \dots, \chi_n^*), \tag{2.11}$$

where S_i are the commuting and χ_i the anticommuting components, and n is an arbitrary positive integer. A supermatrix has the structure

$$F = \begin{pmatrix} a & \sigma \\ \rho & b \end{pmatrix} \ , \tag{2.12}$$

where the boson-boson (bb) block a and the fermion-fermion (ff) block b are ordinary $n \times n$ matrices, while the boson-fermion (bf) and fermion-boson (fb) blocks σ, ρ are $n \times n$ matrices with anticommuting entries. For example, for any two supervectors Φ, Ψ, the tensor product $\Phi \otimes \Psi^\dagger$ is a supermatrix. The supertrace and the superdeterminant of the supermatrix (2.12) are defined as follows

$$\mathrm{Str}\, F = \mathrm{Tr}\, a - \mathrm{Tr}\, b \ , \tag{2.13}$$

$$\mathrm{Sdet}\, F = \exp \mathrm{Str} \ln F = \det(a - \sigma b^{-1} \rho) \det{}^{-1} b. \tag{2.14}$$

Finally, hermitian conjugation of a supermatrix is defined by

$$F^\dagger = \begin{pmatrix} a^\dagger & \rho^\dagger \\ -\sigma^\dagger & b^\dagger \end{pmatrix} \ , \tag{2.15}$$

where for the anticommuting (bf and fb) blocks the usual definition holds, $\sigma^\dagger = (\sigma^*)^T$.

It can be shown that with the above set of definitions, the usual properties of the vector and matrix algebra are valid for supervectors and supermatrices as well, in particular:

- if F is a supermatrix and Φ a supervector, then $\Psi = F\Phi$ is a supervector;
- hermitian conjugation satisfies the usual properties

$$F = \Phi \otimes \Psi^\dagger \qquad \Longrightarrow \qquad F^\dagger = \Psi \otimes \Phi^\dagger \ , \tag{2.16}$$

$$(\Phi^\dagger F \Psi)^\dagger = \Psi^\dagger F^\dagger \Phi \ , \tag{2.17}$$

$$(F^\dagger)^\dagger = F \ , \quad \text{etc.} \tag{2.18}$$

- supertrace and superdeterminant of a product:

$$\mathrm{Str} F_1 F_2 = \mathrm{Str} F_2 F_1 \ , \tag{2.19}$$

$$\mathrm{Sdet} F_1 F_2 = \mathrm{Sdet} F_1 \cdot \mathrm{Sdet} F_2 \ . \tag{2.20}$$

- Gaussian integrals:

$$\int \mathrm{d}\Phi^\dagger \mathrm{d}\Phi \exp(-\Phi^\dagger K \Phi) = \mathrm{Sdet}\, K^{-1} \tag{2.21}$$

$$\int \mathrm{d}\Phi^\dagger \mathrm{d}\Phi\, \Phi_\alpha \Phi_\beta^\dagger \exp(-\Phi^\dagger K \Phi) = (K^{-1})_{\alpha\beta}\, \mathrm{Sdet}\, K^{-1} \ . \tag{2.22}$$

It is assumed here that the boson-boson block K_{bb} of the matrix K defines a quadratic form with a positively defined real part, $\mathrm{Re}\, \mathbf{S}^\dagger K_{\mathrm{bb}} \mathbf{S} > 0$, so that the integral over the bosonic components $\mathbf{S}$ of the supervector Φ converges. Integrals of the type (2.22) with a product of a larger number of Φ_α's in the preexponential factor can be evaluated via the Wick theorem (taking into account that interchanging two anticommuting variables produces a minus sign).

The notion of a supermanifold (including integration and change of coordinates on it) will become important for us in the course of calculation of the DOS-DOS correlation function (Sec. 2.4). We restrict ourselves to saying that the corresponding definitions are natural extensions of those for ordinary analytic manifolds and refer the reader to Berezin's book [17] (see also [18] for a summary). A discussion of the structure of supermanifolds relevant to the supersymmetry treatment of the random matrix ensembles and a more extended list of the related mathematical literature can be found in [19].

Now we return to the problem of the RMT level statistics.

2.3 Average DOS from supersymmetry

We have, instead of Eq. (2.6),

$$(E - \hat{H})^{-1}_{kl} = -i \int [\mathrm{d}\Phi^* \mathrm{d}\Phi] S_k S_l^* \exp\{i \sum_{ij} \Phi_i^\dagger [E\delta_{ij} - H_{ij}]\Phi_j\} \ , \tag{2.23}$$

where

$$\Phi_i = \begin{pmatrix} S_i \\ \chi_i \end{pmatrix}$$

is a two-component supervector, and we have included an infinitesimally small imaginary part $+i\eta$ in the definition of E, so that $\mathrm{Im} E > 0$. The averaging over $\hat{H}$ is now straightforward,

$$\left\langle \exp(i \sum_{ij} \Phi_i^\dagger H_{ij} \Phi_j) \right\rangle = \exp\left\{ -\frac{1}{2N} \sum_{ij} (\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) \right\} \ . \tag{2.24}$$

The next step of our strategy is to decouple the Φ^4 term in Eq. (2.24) by introducing an integration over a 2×2 supermatrix R,

$$\exp\left\{ -\frac{1}{2N} \sum_{ij} (\Phi_i^\dagger \Phi_j)(\Phi_j^\dagger \Phi_i) \right\} = \int \mathrm{d}R \exp\left\{ -\frac{N}{2} \mathrm{Str}\, R^2 - i \sum_i \Phi_i^\dagger R \Phi_i \right\} \ . \tag{2.25}$$

In order that the Gaussian integral (2.25) over the commuting components of the matrix R be convergent, the latter has to be chosen in the form[4]

$$R = \left(\begin{array}{cc} q_{\rm b} & \rho^* \\ \rho & iq_{\rm f} \end{array} \right) \quad ; \qquad q_{\rm b}, q_{\rm f} \in \mathbb{R} \ . \tag{2.26}$$

In other words, integration over the ff-component of the matrix R has to be performed along the imaginary axis. Indeed, due to the factor i in the fermion-fermion element, the quadratic form $\mathrm{Str} R^2 = q_{\rm b}^2 + q_{\rm f}^2$ is positively defined, ensuring the convergence.

Substituting (2.25), (2.24) in (2.23), we get

$$\langle \mathrm{Tr}(E-\hat{H})^{-1} \rangle = -i \int [\mathrm{d}\Phi^* \mathrm{d}\Phi] \sum_k S_k S_k^* \exp\left\{ iE \sum_i \Phi_i^\dagger \Phi_i - \frac{N}{2} \,\mathrm{Str}\, R^2 - i \sum_i \Phi_i^\dagger R \Phi_i \right\} \ . \tag{2.27}$$

Now we see what was the idea of the Hubbard-Stratonovich decoupling: the integral over the Φ-field, if taken first, is now Gaussian (and convergent due to $\mathrm{Im} E > 0$). Furthermore, since the matrix R does not has any structure in the N-dimensional space (where the matrices $\hat{H}$ act), the integral over each of Φ_i, $\Phi_i^\dagger$ $(i = 1, 2, \ldots, N)$ produces the same factor $\mathrm{Sdet}(E - R)$. As a result, we find

$$\langle \mathrm{Tr}(E - \hat{H})^{-1} \rangle = N \int \mathrm{d}R (E - R)^{-1}_{\rm bb} \exp\{-NS[R]\} \ , \tag{2.28}$$

$$S[R] = \frac{1}{2} \mathrm{Str} R^2 + \mathrm{Str} \ln(E - R) \ . \tag{2.29}$$

The presence of the factor $N \gg 1$ in the exponent of (2.28) allows us to apply the saddle-point method. The saddle point equation for the action (2.29) has the form

$$R = (E - R)^{-1} \ . \tag{2.30}$$

We look first for diagonal solutions of (2.30). According to (2.30), the eigenvalues r of the matrix R can take two values $r = \frac{E}{2} \pm i\sqrt{1 - E^2/4}$, yielding four diagonal solutions[5]

$$R = \frac{E}{2} - i\sqrt{1 - E^2/4} \left(\begin{array}{cc} s_{\rm b} & \\ & s_{\rm f} \end{array} \right) \quad ; \qquad s_{\rm b}, s_{\rm f} = \pm 1 \ . \tag{2.31}$$

Since these matrices do not belong to the original integration manifold (2.26), a shift of the integration contours over $R_{\rm bb} = q_{\rm b}$ and $R_{\rm ff} = iq_{\rm f}$ is necessary. However, at $q_{\rm b} = E$ the integrand of (2.28) is singular,

$$\exp\{-N \mathrm{Str} \ln(E - R)\} \equiv \mathrm{Sdet}^{-N}(E - R) \propto (E - q_{\rm b})^{-N} \to \infty \ . \tag{2.32}$$

[4]It does not play any role whether ρ and ρ^* are considered to be complex conjugate of each other or just two independent Grassmannian variables [see the footnote preceding Eq. (2.10)].

[5]it is implied in (2.31) that the first term is a constant $(E/2)$ times unit (super-)matrix. Likewise, we omit the unit matrix symbol in other formulas; e.g., in the l.h.s. (r.h.s.) of (2.28) E implicitly includes the $N \times N$ (resp. 2×2) unit matrix.

Recalling that $\mathrm{Im}E = +\eta > 0$, we conclude that integration contour over q_{b} can only be shifted to the half-plane $\mathrm{Im}q_{\mathrm{b}} < 0$, so that only the saddle points with $s_{\mathrm{b}} = +1$ are relevant. For q_{f} this argumentation is not valid, since according to the definition of the superdeterminant the integrand has at $iq_{\mathrm{f}} = E$ a zero $[\propto (E - iq_{\mathrm{f}})^N]$ rather than a singularity. Nevertheless, it turns out that $s_{\mathrm{f}} = +1$ is the proper choice for the fermionic sector as well. The reason is, however, different: the $s_{\mathrm{f}} = -1$ saddle-point, though being legitimate, gives a contribution suppressed by $1/N$, as will be explained below.

The leading (at $N \gg 1$) contribution is thus given by the vicinity of the saddle point

$$R_0 = \frac{E}{2} - i\sqrt{1 - E^2/4}\ . \tag{2.33}$$

To calculate the integral in the saddle-point approximation, we need the action at the saddle-point and the quadratic form around it,

$$S[R_0] = 0\ , \tag{2.34}$$

$$\delta^2 S[R_0] = C(\delta q_{\mathrm{b}}^2 + \delta q_{\mathrm{f}}^2 + 2\rho^*\rho)\ ; \qquad C = 2 - E\left(\frac{E}{2} - i\sqrt{1 - E^2/4}\right). \tag{2.35}$$

The Gaussian integration around R_0 yields

$$\int \mathrm{d}q_{\mathrm{b}}\mathrm{d}q_{\mathrm{f}}\mathrm{d}\rho^*\mathrm{d}\rho \exp\{-NC(\delta q_{\mathrm{b}}^2 + \delta q_{\mathrm{f}}^2 + 2\rho^*\rho)\} = \mathrm{Sdet}NC = 1\ . \tag{2.36}$$

[Superdeterminant of the unit matrix multiplied by a number (NC) is unity, since the contributions of bosons (π/NC) and fermions (NC/π) cancel each other.] Finally, the preexponential factor in (2.28) can be evaluated at the saddle point,

$$(E - R_0)^{-1}_{\mathrm{bb}} = (R_0)_{\mathrm{bb}} = \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}\ . \tag{2.37}$$

According to (2.5), we find thus the density of states

$$\nu(E) = \begin{cases} \frac{1}{\pi}\sqrt{1 - E^2/4}\ , & |E| \le 2 \\ 0\ , & |E| \ge 2\ , \end{cases} \tag{2.38}$$

which is Wigner's semicircle law.

Now we return (as was promised in the paragraph below Eq. (2.32)) to another possible choice of sign of s_{f} in (2.31), i.e. $s_{\mathrm{b}} = -s_{\mathrm{f}} = 1$. The corresponding diagonal saddle point,

$$R_1 = \frac{E}{2} - i\sqrt{1 - \frac{E^2}{4}}\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\ ,$$

generates in fact, by means of rotations with a Grassmannian generator, a whole manifold of saddle points,

$$R = UR_1U^{-1}\ ;\quad U = \exp\begin{pmatrix} & \alpha^* \\ \alpha & \end{pmatrix} = \begin{pmatrix} 1 + \frac{\alpha^*\alpha}{2} & \alpha^* \\ \alpha & 1 - \frac{\alpha^*\alpha}{2} \end{pmatrix}\ .$$

The quadratic form of the action around R_1 does not contain Grassmannians,

$$\delta^2 S[R_1] = C(\delta q_{\rm b}^2 + \delta q_{\rm f}^2) \ ,$$

since the action is the same on the whole manifold. Therefore, the above compensation of the bosonic gaussian integral (yielding a factor $\propto 1/N$) by the fermionic one (producing a factor $\propto N$) does not take place, and the result is suppressed by $1/N$ and can be neglected. We will see in Sec. 2.4 that in the problem of the level correlation function, a manifold of saddle points emerges as well; in contrast to the present case, however, the balance of fermionic and bosonic degrees of freedom will be preserved, so that the result will be of the leading order in $1/N$.

2.4 Level correlations.

The two-level correlation function characterizing the probability density to have two levels separated by an energy interval ω is defined as

$$R_2(\omega) = \frac{\langle \nu(E-\omega/2)\nu(E+\omega/2)\rangle}{\langle \nu(E)\rangle^2}. \tag{2.39}$$

As was found by Dyson and Mehta (see [7, 8]), for the GUE the two-level correlation function has the form

$$R_2(s) = \delta(s) + 1 - \frac{\sin^2(\pi s)}{(\pi s)^2} \ , \tag{2.40}$$

where $s = \omega/\Delta$, and Δ is the mean level spacing, $\Delta = 1/N\langle\nu(E)\rangle$. Note that we assume that ω is much smaller than the energy band width; in particular, we neglect the change of the mean level density on the scale of ω. This is justified by the fact that, according to (2.40), the scale for correlations is set by the mean level spacing, $\Delta \propto 1/N$. The delta function in the r.h.s of Eq. (2.40) corresponds to a "self-correlation" of an energy level, the second term (unity) is a disconnected part of R_2 (corresponding to the absence of correlations), and the last term is the non-trivial part of R_2 describing the correlations of different levels. We show below how Eq. (2.40) is obtained in the framework of the supersymmetric σ-model method.

We follow essentially the same strategy as was outlined in Sec. 2.1, 2.3 for the average DOS. The level corelation function is expressed in terms of the Green's functions as follows

$$R_2(\omega) = \frac{1}{2(\pi\nu N)^2}{\rm Re}\left\langle {\rm Tr}G_{\rm R}^{E+\omega/2}{\rm Tr}(G_{\rm R}^{E-\omega/2} - G_{\rm A}^{E-\omega/2})\right\rangle \ . \tag{2.41}$$

The product of two retarded Green's functions here is trivial; it can be calculated in the same way as the average DOS above, yielding

$$N^{-2}\langle {\rm Tr}G_{\rm R}^{E+\omega/2}{\rm Tr}G_{\rm R}^{E-\omega/2}\rangle \simeq N^{-2}\langle {\rm Tr}G_{\rm R}^{E}\rangle^2 = \left(E/2 - i\sqrt{1-E^2/4}\right)^2 \ . \tag{2.42}$$

In other words, the average of the product of retarded Green's functions decouples into the product of the averages (the same is valid for the product of the advanced Green's functions, of course). All the non-trivial information about correlations is contained therefore in the product of the type $\langle G_{\rm R}G_{\rm A}\rangle$,

$$T_2(E_1,E_2)=\frac{1}{N^2}\langle \mathrm{Tr}(E_1-\hat{H})^{-1}\mathrm{Tr}(E_2-\hat{H})^{-1}\rangle \ , \tag{2.43}$$

where

$$E_{1,2}=E\pm\left(\frac{\omega}{2}+i\eta\right)\equiv E\pm r/2N \ ; \quad \mathrm{Im}\, r>0 \ . \tag{2.44}$$

To represent $T_2(E_1,E_2)$ as a superintegral, we introduce supervectors of a double size,

$$\Phi_i=\begin{pmatrix} S_1(i)\\ \chi_1(i)\\ S_2(i)\\ \chi_2(i)\end{pmatrix} \ , \tag{2.45}$$

with the first two components corresponding to the advanced and the last two to the retarded sector. As before, the S-components are commuting, while the χ-components anticommuting; the index i running from 1 to N corresponds to the vector space in which the matrix $\hat{H}$ acts. We have then

$$T_2(E_1,E_2)=\frac{(-)^N}{N^2}\int[\mathrm{d}\Phi^*\mathrm{d}\Phi]\sum_{i,j}S_1^*(i)S_1(i)S_2^*(j)S_2(j)\exp\{-S[\Phi]\} \ , \tag{2.46}$$

where the action $S[\Phi]$ is given by (to make notations more compact, we combine all $S_1(i)$ into an N-component vector S_1, and the same for S_2, χ_1, χ_2)

$$\begin{aligned} S[\Phi] &= -iS_1^\dagger(E_1-\hat{H})S_1-i\chi_1^\dagger(E_1-\hat{H})\chi_1+iS_2^\dagger(E_2-\hat{H})S_2-i\chi_2^\dagger(E_2-\hat{H})\chi_2\\ &\equiv -i\Phi^\dagger L\left(E+\frac{r}{2N}\Lambda-\hat{H}\right)\Phi \ . \end{aligned} \tag{2.47}$$

Here the matrices Λ and L are defined as $\Lambda=\mathrm{diag}(1,1,-1,-1)$, $L=\mathrm{diag}(1,1,-1,1)$, with ordering of components according to (2.45), i.e. $(\mathrm{Rb},\mathrm{Rf},\mathrm{Ab},\mathrm{Af})$, where R,A correspond to the retarded-advanced and b,f to the boson-fermion decomposition. After averaging over the ensemble of matrices $\hat{H}$, the action acquires the form

$$S[\Phi]=-i\Phi^\dagger L\left(E+\frac{r}{2N}\Lambda\right)\Phi+\frac{1}{2N}\sum_{ij}(\Phi_i^\dagger L\Phi_j)(\Phi_j^\dagger L\Phi_i) \ . \tag{2.48}$$

Decoupling of the quartic term via the Hubbard-Stratonovich transformation requires the introduction of a 4×4 supermatrix variable R,

$$\begin{aligned} T_2(E,r) &= \frac{(-)^N}{N^2}\int[\mathrm{d}\Phi^*\mathrm{d}\Phi]\sum_{i,j}S_1^*(i)S_1(i)S_2^*(j)S_2(j)\int\mathrm{d}R\\ &\times\ \exp\left\{-i\Phi^\dagger L^{1/2}\left(E-R+\frac{r}{2N}\Lambda\right)L^{1/2}\Phi-\frac{N}{2}\mathrm{Str}R^2\right\} \ . \end{aligned} \tag{2.49}$$

The next step of our program is to interchange the integrations $\int\mathrm{d}R$ and $\int[\mathrm{d}\Phi^*\mathrm{d}\Phi]$. This leads, however, to a set of requirements of convergence of both the Φ-integral (when it is taken first) and the R-integral. It turns out that in order to satisfy these requirements, one has to choose a non-trivial manifold of integration over the matrix R. Indeed, a naive attempt to generalize Eq. (2.26) straightforwardly by choosing

$$R = \begin{pmatrix} R_{\rm b} & \bar{\chi} \\ \chi & iR_{\rm f} \end{pmatrix} \text{ in boson-fermion decomposition} \tag{2.50}$$

with Hermitian matrices $R_{\rm b}$ and $R_{\rm f}$ fails, since then the Φ-dependent part of the action, $i\Phi^\dagger L^{1/2}(E-R)L^{1/2}\Phi$, contains the term

$$S_1^\dagger R_{{\rm b};12}S_2 + S_2^\dagger R_{{\rm b};21}S_1 = 2{\rm Re}\, S_1^\dagger R_{{\rm b};12}S_2 \ ,$$

which is real and has an arbitrary sign, thus leading to a divergent integral over S_1, S_2. The natural idea to cure this problem by multiplying the components $R_{{\rm b};12}$ and $R_{{\rm b};21}$ by i is immediately seen to fail as well, since it leads to a divergent R-integral. The way out was found by Schäfer and Wegner [20] for the case of the bosonic replica model. Since the problem is pertinent to the bosonic sector of the supersymmetric model, the idea of Ref. [20] can be straightforwardly generalized to the supersymmetric case. For a thorough discussion of this issue, the reader is referred to the review paper by Verbaarschot, Weidenmüller, and Zirnbauer [14].

The solution of the problem, which we are going to formulate now, is suggested by the invariance of the quartic term in (2.48) with respect to the rotations $\Phi \to V\Phi$, where the matrices V satisfy $V^\dagger L V = L$, thus forming a pseudounitary supergroup[6] $U(1,1|2)$. The reason for the above problem with convergence of the integral over the commuting components of Φ lies in the fact that the set of hermitian matrices $R_{\rm b}$ is *not* invariant with respect to the pseudounitary group $U(1,1)$. The proper integration manifold is to be chosen as follows [1, 2, 14] (note that we return to the ordering of components according to Eq. (2.45), so that the external block structure shown explicitly in (2.51) corresponds to the retarded-advanced decomposition; each block being a 2×2 supermatrix):

$$R = T \begin{pmatrix} P_1 - i\delta_0 & \\ & P_2 + i\delta_0 \end{pmatrix} T^{-1} \ , \tag{2.51}$$

where $\delta_0 > 0$ is a constant (which is to be chosen below in such a way that the integration manifold passes through the saddle points) and $P_1 = P_1^\dagger$, $P_2 = P_2^\dagger$ are hermitian supermatrices. Further, the matrix T satisfies $T^\dagger L T = L$ and belongs to the coset space[7] $U(1,1|2)/U(1|1)\times U(1|1)$. This coset space is obtained from the group $U(1,1|2)$, if the elements of this group which commute with Λ (they form the subgroup

[6] Let us remind that $L = {\rm diag}(1,1,-1,1)$; in the notation $U(1,1|2)$ the content of the brackets to the left of the vertical bar refers to the $(+,-)$ metric in the bosonic sector, while that to the right corresponds to the $(+,+)$ metric in the fermionic sector. In other words, the group $U(1,1|2)$ represents a product $U(1,1)|_{\rm bb}\times U(2)|_{\rm ff}$, "dressed" by Grassmannian generators transforming the commuting components into anticommuting and vice versa.

[7] For a group G and its subgroup K the space G/K is formed by the set of (left) cosets gK with $g\in G$.

$U(1|1) \times U(1|1)$) are identified with unity. The matrices of this coset space can be parametrized in the following way

$$T = \begin{pmatrix} (1+t_{12}t_{21})^{1/2} & t_{12} \\ t_{21} & (1+t_{21}t_{12})^{1/2} \end{pmatrix} ; \tag{2.52}$$

$$t_{12} = \begin{pmatrix} a & -i\eta_1 \\ \eta_2^* & ib^* \end{pmatrix} ; \quad t_{21} = \begin{pmatrix} a^* & -\eta_2 \\ i\eta_1^* & ib \end{pmatrix} ; \quad |b|^2 \le 1 . \tag{2.53}$$

The matrix R has 16 real degrees of freedom (the same number as a 4×4 hermitian matrix); in the above parametrization 8 degrees of freedom are contained in matrices $P_{1,2}$ and the remaining 8 in t_{12}, t_{21}. The measure $\mathrm{d}R$ takes in this parametrization the following form (see [14] for details)

$$\mathrm{d}R = \mathcal{F}(P_1, P_2)\mathrm{d}P_1\mathrm{d}P_2\mathrm{d}\mu(T) , \tag{2.54}$$

where $\mathrm{d}P_i$ are conventional matrix measures (product of differentials of all elements) and $\mathcal{F}(P_1, P_2)$ is a function depending on eigenvalues of $P_1 - i\delta_0$ and $P_2 + i\delta_0$ and equal to unity at the saddle point manifold, for which $P_{1,2}$ are proportional to the unit matrix. Further, $\mathrm{d}\mu(T)$ is the invariant measure on the coset space; in the parametrization (2.52), (2.53) of T it is given explicitly by

$$\mathrm{d}\mu(T) = \mathrm{d}t_{12}\mathrm{d}t_{21} . \tag{2.55}$$

Having specified the manifold of matrices R over which the integration in (2.49) goes, we can interchange the order of integrals and evaluate the integral over Φ first. The result reads, similarly to (2.28),

$$T_2(E, r) = \int \mathrm{d}R \left(E - R + \frac{r}{2N}\Lambda\right)^{-1}_{\mathrm{bb},11} \left(E - R + \frac{r}{2N}\Lambda\right)^{-1}_{\mathrm{bb},22} e^{-NS[R]} ; \tag{2.56}$$

$$S[R] = \frac{1}{2}\mathrm{Str}R^2 + \mathrm{Str}\ln\left(E - R + \frac{r}{2N}\Lambda\right) . \tag{2.57}$$

As in the case of the average DOS calculation, the large factor $N \gg 1$ in the exponent allows us to use the saddle-point approximation. The saddle-point equation that is obtained by varying the action (2.57)[8] looks identical to Eq. (2.30) [the difference being that R is now a 4×4 supermatrix, while it was 2×2 in Sec. 2.3]. Similarly to Eq. (2.31), diagonal solutions of this equation have the form

$$R = \frac{E}{2} - i\sqrt{1 - E^2/4} \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & s_3 & \\ & & & s_4 \end{pmatrix} ; \qquad s_1, \ldots, s_4 = \pm 1 . \tag{2.58}$$

When fixing the signs of s_i, the same reasons as for the average DOS calculation are to be taken into account:

[8]We neglect the term proportional to r/N when deriving the saddle-point equation, since it gives a correction $\sim O(1)$ to $NS[R]$. We will take this term into account below when integrating over the saddle-point manifold.

- bosonic sector: shift of the integration contour should not cross singularities. This requires $s_1 = -s_3 = -1$;
- fermionic sector: the leading contribution is found to be given by the saddle points with $s_2 = -s_4 = -1$ and $s_2 = -s_4 = 1$; contributions of the two other saddle-points are suppressed by $1/N$.

Therefore, only 2 saddle-points out of 16 survive. In fact, it is sufficient to consider only one of them [say, with the signs $(-,-,+,+)$]; the second one [with $(-,+,+,-)$] will belong to the manifold obtained by rotating the first one by the matrices T. Indeed, if $R_0 = E/2 - i\sqrt{1-E^2/4}\Lambda$ is a solution of Eq. (2.30), then all matrices of the form

$$R = TR_0T^{-1} = \frac{E}{2} - i\sqrt{1-E^2/4}\,T\Lambda T^{-1} \equiv \frac{E}{2} - i\pi\nu Q \tag{2.59}$$

are solutions [and thus saddle points of the action (2.57)] as well. Here $\nu \equiv \langle \nu(E) \rangle$ [see (2.38)] and matrices Q defined as

$$Q = T\Lambda T^{-1} \tag{2.60}$$

(and obviously satisfying $Q^2 = 1$) form a manifold of the supermatrix σ-model.

Comparing (2.59) with (2.51), we see that δ_0 should be chosen as $\delta_0 = \pi\nu$ and that the saddle-point manifold corresponds to $P_1 = P_2 = E/2$ (and T running over the coset space). The Gaussian integral over P_1 and P_2 around this saddle-point value can be easily calculated, yielding unity (due to compensation of the bosonic and fermionic massive modes, *cf.* Eq. (2.36)). We are thus left with an integral over the manifold (2.60). Expanding the action (2.57) up to the first order in $r/N \ll 1$, we reduce it to the form[9]

$$T_2(E,r) = \int \mathrm{d}\mu(Q) \left(\frac{E}{2} - i\pi\nu Q_{\mathrm{bb},11}\right)\left(\frac{E}{2} - i\pi\nu Q_{\mathrm{bb},22}\right) \exp\left\{\frac{i\pi\nu r}{2}\mathrm{Str} Q\Lambda\right\} . \tag{2.61}$$

We have therefore reduced calculation of the level correlation function for an ensemble of matrices $\hat{H}$ (with $\sim N^2 \to \infty$ degrees of freedom) to evaluation of an integral over a supermatrix Q parametrized by a finite number (eight) variables and belonging to certain non-linear space. The obtained problem is known as zero-dimensional (0D) σ-model. The term "zero-dimensional" distinguishes integrals of the type (2.61) over a single matrix Q from a field-theoretical σ-model studied in Sec. 3 with the matrix Q depending on spatial coordinates.

For an explicit evaluation of integrals of the type (2.61) over the coset space the parametrization (2.52), (2.53) is inappropriate; a much more convenient parametrization was found by Efetov [1]:

$$Q = \begin{pmatrix} U_1 & \\ & U_2 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & i\mu_1 & 0 \\ 0 & \lambda_2 & 0 & \mu_2^* \\ i\mu_1^* & 0 & -\lambda_1 & 0 \\ 0 & \mu_2 & 0 & -\lambda_2 \end{pmatrix} \begin{pmatrix} U_1^{-1} & \\ & U_2^{-1} \end{pmatrix} ; \tag{2.62}$$

[9] since in Eq. (2.61) and below we use Q as a variable on the coset space, we denote the corresponding invariant measure as $\mathrm{d}\mu(Q)$; it is identical to $\mathrm{d}\mu(T)$ introduced earlier.

$$U_1 = \exp\left(\begin{array}{cc} 0 & -\alpha^* \\ \alpha & 0 \end{array}\right) \;; \qquad U_2 = \exp i\left(\begin{array}{cc} 0 & -\beta^* \\ \beta & 0 \end{array}\right) \;, \tag{2.63}$$

$$1 \le \lambda_1 < \infty, \quad -1 \le \lambda_2 \le 1, \quad |\mu_1|^2 = \lambda_1^2 - 1, \quad |\mu_2|^2 = 1 - \lambda_2^2 \;. \tag{2.64}$$

Alternatively, instead of the variables $\lambda_{1,2}$, $\mu_{1,2}$ defined in Eq. (2.64), one can introduce the set of variables θ_1, θ_2, ϕ_1, ϕ_2 via

$$\begin{aligned} &\lambda_1 = \cosh\theta_1 \;, \quad \mu_1 = \sinh\theta_1 e^{i\phi_1} \;, \quad \lambda_2 = \cos\theta_2 \;, \quad \mu_2 = \sin\theta_2 e^{i\phi_2} \;, \\ &0 \le \theta_1 < \infty \;, \quad 0 \le \theta_2 \le \pi \;, \quad 0 \le \phi_{1,2} < 2\pi \;. \end{aligned} \tag{2.65}$$

Note that $\lambda_{1,2}$ are eigenvalues of the boson-boson (and fermion-fermion) block of the matrix Q; we will call them simply "eigenvalues" for brevity. The integral (2.61) takes in this parametrization the form

$$\begin{aligned} T_2(E,r) \;\; &= \;\; -\int \mathrm{d}\mu(Q)\left[\frac{E}{2} - i\pi\nu(\lambda_1 - \alpha^*\alpha(\lambda_1-\lambda_2))\right] \\ &\times \;\; \left[\frac{E}{2} + i\pi\nu(\lambda_1 + \beta^*\beta(\lambda_1-\lambda_2))\right] e^{i\pi\nu r(\lambda_1-\lambda_2)} \;, \end{aligned} \tag{2.66}$$

with the measure $\mathrm{d}\mu(Q)$ given by

$$\mathrm{d}\mu(Q) = -\frac{\mathrm{d}\lambda_1 \mathrm{d}\lambda_2}{(\lambda_1-\lambda_2)^2}\mathrm{d}\phi_1\mathrm{d}\phi_2\mathrm{d}\alpha\mathrm{d}\alpha^*\mathrm{d}\beta\mathrm{d}\beta^* \;. \tag{2.67}$$

The crucial advantage of the above parametrization is that the exponent in (2.61) acquires a very simple form (see (2.66)) dependent on λ_1 and λ_2 only. This allows to integrate out straightforwardly the Grassmannian variables. The integration rules (2.8) imply that only the highest order term ($\sim \alpha^*\alpha\beta^*\beta$) in the expansion of the integrand in a polynomial over Grassmannians gives a non-zero contribution after integration over $\mathrm{d}\alpha^*\mathrm{d}\alpha\mathrm{d}\beta^*\mathrm{d}\beta$. The result is easily seen to be

$$(\pi\nu)^2\int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_2\, e^{i\pi\nu r(\lambda_1-\lambda_2)} = \frac{2i}{r^2}\sin(\pi\nu r)e^{i\pi\nu r} \;. \tag{2.68}$$

There exists, however, one more contribution to the integral (2.66), namely that of the term of the zeroth order in Grassmannians. Though naively it gives zero after the Grassmannian integration, the corresponding integral over $\lambda_{1,2}$ has a singularity $\int \mathrm{d}\lambda_1\mathrm{d}\lambda_2/(\lambda_1-\lambda_2)^2$ and thus diverges in the vicinity of $\lambda_1 = \lambda_2 = 1$. The reason for this ambiguity is in the singular character of Efetov's parametrization at $Q = \Lambda$, i.e at $\lambda_1 = \lambda_2 = 1$. The problem arises only for the term which does not contain Grassmannians; all higher order terms contain additional powers of $(\lambda_1 - \lambda_2)$ [see (2.66)], which make the integral over λ's convergent. The value of the integral of the zeroth order term is determined by the following formula

$$\int \mathrm{d}\mu(Q)F(\lambda_1,\lambda_2) = F(1,1) \equiv F(Q=\Lambda) \;, \tag{2.69}$$

where F is an arbitrary function (depending on eigenvalues λ_1, λ_2 only) which vanishes at $\lambda_1 \to \infty$. Equation (2.69) is a particular case of a theorem (often called Parisi-Sourlas-Efetov-Wegner theorem in the physical literature), which states that for a broad class of supersymmetric models an integral of an invariant function (in our case the fact that $F(Q)$ depends only on λ's means that it is invariant with respect to rotations $Q \to VQV^{-1}$ with $V \in U(1|1) \times U(1|1)$) is given by its value at the origin (in our case $Q = \Lambda$). For a discussion and a proof, see Refs. [21, 22]. A general mathematical treatment of such anomalous contributions to integrals over supermanifolds can be found in [23, 19].

Applying (2.69) to the zeroth order (in Grassmannians) term of (2.66) and adding the result to the contribution (2.68) of the highest order term, we finally get

$$T_2(E,r) = \left(\frac{E}{2} - i\pi\nu\right)\left(\frac{E}{2} + i\pi\nu\right) + \frac{2i}{r^2}\sin(\pi\nu r)e^{i\pi\nu r} \ . \tag{2.70}$$

Substituting this in (2.41), including the $\langle G_R G_R \rangle$ contribution (2.42), and taking into account that $\nu r = \nu\omega N = \omega/\Delta = s$, we find the two-level correlation function

$$\begin{aligned} R_2(\omega) &= \frac{1}{2\pi^2\nu^2}\mathrm{Re}\left[T_2(E,r) - \left(\frac{E}{2} - i\pi\nu\right)^2\right] \\ &= 1 + \delta(s) - \frac{\sin^2(\pi s)}{(\pi s)^2} \ , \end{aligned} \tag{2.71}$$

in agreement with (2.40).

2.5 Comments and generalizations

2.5.1 Structure of the saddle-point manifold.

We first note that the topology of the σ-model manifold was crucially important for the above calculation. Indeed, if we keep only the ordinary part (which does not contain Grassmannians) of each element Q of this supermanifold[10], we get a conventional (not super-) manifold[11], which is a product of the hyperboloid $U(1,1)/U(1) \times U(1)$ (parametrized by $\lambda_1 = \cosh\theta_1$ and ϕ_1) and the sphere $U(2)/U(1)\times U(1)$ (parametrized by $\lambda_2 = \cos\theta_2$ and ϕ_2). This combination of the compact (λ_2) and non-compact (λ_1) degrees of freedom is clearly reflected in the result, see Eq. (2.68).

It is appropriate here to add a comment concerning a somewhat subtle point of the calculation. The non-compact (hyperbolic) symmetry of the bb-sector originates from the opposite signs of the first and the third terms in Eq. (2.47). This sign choice was unambiguously dictated by the requirement of convergence of the integral over the bosonic components $S_{1,2}$ of the supervector Φ. On the other hand, the situation seems to be different for the two terms of (2.47) containing the Grassmannians $\chi_{1,2}$, since the integral over the anticommuting variables is always defined and no condition of

[10]the discarded part containing the Grassmannians is called nilpotent
[11]called the base of the supermanifold

convergence arises. Therefore, the choice of signs of the second and the fourth terms in (2.47), which has eventually led us to the compact symmetry of the ff-sector, seems to be arbitrary. A thorough analysis [14] shows, however, that this apparent freedom is spurious, and the symmetry of the ff-block is uniquely fixed to be compact on a later stage of the calculation by the requirement of convergence of the integral over the coset space (i.e. over the matrix T).

2.5.2 Gaussian ensembles of different symmetry.

Let us recall that for the sake of technical simplicity we considered the case of GUE (with the matrix $\hat{H}$ being hermitian without any further restrictions) while calculating the level statistics. An analogous (though technically more involved) calculation can be performed for the two other Gaussian ensembles of Wigner and Dyson, the Gaussian Orthogonal Ensemble (GOE) of real symmetric matrices and the Gaussian Symplectic Ensemble (GSE) of real quaternionic matrices [7, 8]. If the matrix $\hat{H}$ is considered as a Hamiltonian, the GUE describes systems without the (antiunitary) time reversal symmetry, while GOE and GSE correspond to systems with the time reversal invariance realized by an antiunitary operator $\mathcal{T}$ such that $\mathcal{T}^2 = 1$ (GOE) or $\mathcal{T}^2 = -1$ (GSE).

Calculation of the two-level correlation function for GOE within the supersymmetry approach is performed essentially in the same way as for GUE, see [1, 2]. However, due to the presence of an additional symmetry in the problem, one has to double the size of the supervectors Φ_i, combining Φ_i with its time reversal Φ_i^*. Correspondingly, one ends up with a 0D σ-model of 8×8 supermatrices Q parametrized by 16 independent variables. Now, instead of two for GUE, there are three eigenvalues, $1 \le \lambda_1, \lambda_2 < \infty$ and $-1 \le \lambda \le 1$. A similar structure is obtained for GSE with 8×8 Q-matrix as well, but with two "compact" and one "non-compact" eigenvalues, $-1 \le \lambda_1 \le 1$, $0 \le \lambda_2 \le 1$, $1 \le \lambda < \infty$. The evaluation of the corresponding integrals [1, 2] reproduces again the result for the two-level correlation function obtained by Mehta and Dyson within the orthogonal polynomial method,[12]

$$R_2(s) = 1 + \delta(s) - \frac{\sin^2(\pi s)}{(\pi s)^2} - \left[\frac{\pi}{2}\mathrm{sign}(s) - \mathrm{Si}(\pi s)\right]\left[\frac{\cos\pi s}{\pi s} - \frac{\sin\pi s}{(\pi s)^2}\right] \qquad \text{(GOE)}, \tag{2.72}$$

$$R_2(s) = 1 + \delta(s) - \frac{\sin^2(2\pi s)}{(2\pi s)^2} + \mathrm{Si}(2\pi s)\left[\frac{\cos 2\pi s}{2\pi s} - \frac{\sin 2\pi s}{(2\pi s)^2}\right] \qquad \text{(GSE)}, \tag{2.73}$$

$$\mathrm{Si}(x) = \int_0^x \frac{\sin y}{y} dy \ .$$

In the remaining part of the lecture course, we will continue considering explicitly the unitary class only; corresponding results for the other two classes will be sometimes quoted in the end of the calculation.

[12] Note that in the symplectic case all the levels are double degenerate (Kramers degeneracy). This degeneracy is disregarded in (2.73), which thus represents the correlation function of distinct levels only, normalized to the corresponding level spacing.

On top of the three "standard" Wigner-Dyson ensembles, seven other ensembles have been introduced more recently. They become relevant, in various physical situations, near a special point of the spectrum where the symmetry of the problem gets enlarged. Specifically, this happens (i) at the center of the band for a particle moving on a bipartite (AB) lattice with only off-diagonal (A→B or B→A) hopping allowed [24]; (ii) in the chiral random matrix ensembles [25], which model the massless Dirac operator in the quantum chromodynamics, near the zero point of the spectrum; (iii) near the Fermi energy in models of a mesoscopic metallic grain in contact with a superconductor [26]. We will not consider such ensembles in this lecture course. An interested reader is referred to Ref. [19] for a review of related algebraic structures.

A natural question that may be asked at this point is why is there a need in the supersymmetry formalism if the results discussed above have been obtained much earlier by the methods of "classical" RMT. In fact, Sec. 3-5 will give an answer to this question, since the supersymmetry approach will be used there to study statistical properties for a much broader class of problems (disordered and chaotic systems), essentially different in their formulation from the RMT ensembles. However, already here we want to give two examples of the random matrix ensembles which, while being direct generalizations of the Gaussian ensembles, are not accessible within the classical RMT methods.

2.5.3 Ensembles with non-Gaussian distributions of matrix elements.

Let us consider an ensemble of large ($N \to \infty$) $N \times N$ hermitian matrices with all matrix elements being independent and equally distributed, but (in contrast to GUE) with some non-Gaussian distribution function $f(z, z^*) \equiv f(|z|^2)$. In other words, the overall probability density for the matrix $\hat{H}$ is assumed to be

$$\mathcal{P}(\hat{H}) = \prod_{i \le j} f(|H_{ij}|^2) \ . \tag{2.74}$$

For the Gaussian distribution, $f(|z^2|) = \exp(-N|z|^2)$, this is reduced[13] to the Gaussian ensemble (2.1). However, for any other distribution function f the probability density $\mathcal{P}(\hat{H})$ cannot be written in the form $\exp \mathrm{Tr}\, F(\hat{H})$ any more. In other words, the ensemble (2.74) is no more invariant with respect to the unitary rotations, $\hat{H} \to \hat{U}\hat{H}\hat{U}^{-1}$ and the probability distributions of eigenvalues and eigenvectors of $\hat{H}$ do not decouple any more. This precludes the application of the orthogonal polynomial method, which uses an explicit form of the distribution of the eigenvalues.

On the other hand, the supersymmetry method can still be successfully used [27]. After averaging over $\hat{H}$ one gets, instead of the quartic term in the action (2.48),

$$\left\langle \exp\{-\sum_{ij} \Phi_i^\dagger H_{ij} L \Phi_j\} \right\rangle = \prod_{i<j} \int \mathrm{d}z \mathrm{d}z^* \exp\{i\Phi_i^\dagger L \Phi_j z + i\Phi_j^\dagger L \Phi_i z^*\} f(|z|^2)$$

[13] The distribution of the diagonal elements H_{ii} is not important in the leading order in $1/N$.

$$= \prod_{i<j} \left\{ 1 - (\Phi_i^\dagger L\Phi_j)(\Phi_j^\dagger L\Phi_i)\langle|z^2|\rangle_f + \frac{1}{4}(\Phi_i^\dagger L\Phi_j)^2(\Phi_j^\dagger L\Phi_i)^2\langle|z^4|\rangle_f + \ldots \right\} \ , \quad (2.75)$$

where $\langle\ldots\rangle_f$ denotes averaging over $f(|z|^2)$. As in the case of GUE, we will assume the normalization which keeps the band of eigenvalues of $\hat{H}$ finite in the limit $N \to \infty$. This implies $\langle|H_{ij}|^2\rangle \sim 1/N$, i.e. $\langle|z^2|\rangle_f \sim 1/N$. If f is a smooth, non-singular function, then $\langle|z^4|\rangle_f \sim 1/N^2$ etc. For this reason, one can drop all terms beyond the quartic one in the second line of (2.75). Indeed, the integral over Φ (taken after the Hubbard-Stratonovich decoupling) will be dominated by $\Phi \sim N^0$, see Eq. (2.49). Therefore, the second line of (2.75) is in fact an expansion in $1/N$. Dropping all higher order terms and reexponentiating the quartic term (which is $\sim 1/N$), we get

$$\left\langle \exp\{-\sum_{ij} \Phi_i^\dagger H_{ij} L\Phi_j\} \right\rangle \simeq \prod_{i,j} \exp\left\{ -\frac{1}{2}\langle|z^2|\rangle_f (\Phi_i^\dagger L\Phi_j)(\Phi_j^\dagger L\Phi_i) \right\} \ , \quad (2.76)$$

which is the same result as for the Gaussian distribution $f(z,z^*) = \exp\{-|z|^2/\langle|z^2|\rangle_f\}$. Therefore, the spectral statistics for the ensemble (2.74) has the same Wigner-Dyson form as for the GUE.

Sparse random matrix ensemble. A notable exception from this statement is formed by an ensemble of sparse random matrices [27, 28] for which the distribution function $f(|z|^2)$ is singular at $z = 0$. More specifically,

$$f(|z|^2) = \left(1 - \frac{p}{N}\right)\delta(|z|^2) + \frac{p}{N}h(|z|^2) \ , \quad (2.77)$$

where $p > 0$ is a constant of order unity (i.e. independent of N) and h is a smooth (in particular, non-singular at $z = 0$) distribution function with $\langle|z^2|\rangle_h \sim 1$. The distribution (2.77) implies that almost all matrix elements of $\hat{H}$ are zero; only p (in average) out of N elements in each row are non-zero (and distributed according to $h(|z|^2)$). Because of the singular character of $f(|z|^2)$ truncation of the series in (2.75) is no more legitimate. Indeed, we have now $\langle|z^2|\rangle_f \sim 1/N$, $\langle|z^4|\rangle_f \sim 1/N$, etc., so that all the terms have to be taken into account. To decouple all of them via a kind of the Hubbard-Stratonovich transformation, one has to introduce, in addition to the usual matrix $Q_{\alpha\beta}$, also higher order tensors $Q^{(4)}_{\alpha\beta\gamma\delta}$, $Q^{(6)}_{\alpha\beta\gamma\delta\mu\nu}, \ldots$. It turns out [27, 28] that they all can be combined in a function $Y(\Phi, \Phi^\dagger)$, so that the Hubbard-Stratonovich transformation acquires a functional form. The analysis show that the model exhibits an Anderson localization transition if one of the parameters (*e.g.* p) is changed. At $p < p_c$ all eigenvectors are localized and the level statistics is Poisonnian (uncorrelated levels) in the limit $N \to \infty$; at $p > p_c$ the states are delocalized and the Wigner-Dyson statistics applies[14]; finally $p = p_c$ is the critical point of the Anderson transition. We mention also that this model has effectively infinite-dimensional character and is closely related to the problem of Anderson localization on the tree-like Bethe lattice, see [29].

[14] To be precise, one should speak about the states belonging to the infinite cluster only, see [27, 28].

2.5.4 Random banded matrices.

The random banded matrix (RBM) ensemble is defined in the following way (see *e.g.* [30, 31]). As in the Gaussian ensemble, all the matrix elements are supposed to be independent and have a Gaussian distribution. The difference is that the variance of this distribution is now not the same for all the elements but rather depends on the distance from the main diagonal,

$$\langle |H_{ij}|^2 \rangle = F(|i-j|) \ . \tag{2.78}$$

The function $F(r)$ is supposed to be roughly constant for $r \lesssim b$ and negligibly small for $r \gg b$. Therefore, all the elements of the matrix which are essentially non-zero are located within a band of width $\sim b$ around the main diagonal (hence the term "banded matrices"). Again, the RBM ensemble is not rotationally-invariant and thus cannot be treated by the classical RMT methods. In contrast, the supersymmetry approach can be applied to this ensemble [32]. In particular, one finds that if b/N is kept constant while the limit $N \to \infty$ is considered, the level statistics acquire the universal Wigner-Dyson form. More generally, this ensemble describes a quasi-one-dimensional system, with statistical properties depending on a value of the ratio $X = N/b^2$. If $X \ll 1$ (as in the above limiting procedure), the level (and eigenfunction) statistics are essentially of the GUE form; in the opposite limit $X \gg 1$ the system is in the strong localization regime. This will be discussed in more detail in Sec. 3, 4.

2.5.5 Parametric level statistics.

We have just demonstrated that the supersymmetry approach is extremely useful when the matrix ensemble different from the Gaussian ones are considered. However, also for the Gaussian ensembles the supersymmetry method has produced a bulk of new results for quantities more complicated than the conventional two-level correlation function (2.39). As an important example, we mention here the parametric level statistics which is defined in the following way. Let us perturb the set of matrices $\hat{H}_0$ forming the GUE by adding some given traceless matrix $\hat{V}$ multiplied by a parameter α,

$$\hat{H}(\alpha) = H_0 + \alpha \hat{V} \ . \tag{2.79}$$

Consider now the following correlation function

$$R_2(\omega, \delta\alpha) = \langle \mathrm{Tr}\, \delta(E - \hat{H}(\alpha))\ \mathrm{Tr}\, \delta(E + \omega - \hat{H}(\alpha + \delta\alpha)) \rangle / \langle \nu \rangle^2 \ . \tag{2.80}$$

For $\delta\alpha = 0$ this is just the usual two-level correlation function (2.39). The parametric DOS-DOS correlation function was calculated via the supersymmetry method by Simons and Altshuler [33]. As for the simple level correlation function (2.39), the result has a universal form if the perturbation parameter $\delta\alpha$ is properly rescaled,

$$R_2(\omega, \delta\alpha) = 1 + \frac{1}{2}\mathrm{Re} \int_1^\infty \mathrm{d}\lambda_1 \int_{-1}^1 \mathrm{d}\lambda_2 \exp\left[i\pi s(\lambda_1 - \lambda_2) - \frac{\pi^2}{2} x^2 (\lambda_1^2 - \lambda_2^2) \right] \ , \tag{2.81}$$

where $x = \delta\alpha \langle (\partial_\alpha E_i)^2 \rangle^{1/2} / \Delta$ and, as before, $s = \omega/\Delta$. The corresponding formulas for the other two ensembles can be found in [33, 34, 2].

3 Level statistics in a disordered sample: Diffusive σ-model.

3.1 Derivation of the diffusive σ-model.

Having introduced the main ideas and ingredients of the supersymmetry formalism for the Gaussian Ensemble, we are prepared to proceed with consideration of a much richer problem of a particle moving in a random potential. The Hamiltonian has now the form[15]

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + U(\mathbf{r}) , \qquad \hat{\mathbf{p}} = -i\nabla \tag{3.1}$$

where $U(\mathbf{r})$ is a disorder potential. For simplicity we choose it to be of the white-noise type (this is not essential for the derivation of the σ-model)

$$\langle U(\mathbf{r})U(\mathbf{r}')\rangle = \frac{1}{2\pi\nu\tau}\delta(\mathbf{r}-\mathbf{r}') , \tag{3.2}$$

where τ is the mean free time. We will assume that the time reversal invariance is broken (say, by weak homogeneous or random magnetic field) and restrict our consideration to the (technically simpler) case of the unitary symmetry.[16] The DOS and the Green's functions are given by a direct generalization of Eqs. (2.3)–(2.5),

$$\nu(E) = \frac{1}{2\pi i V}\int \mathrm{d}^d\mathbf{r}\left[G_{\mathrm{A}}^E(\mathbf{r},\mathbf{r}) - G_{\mathrm{R}}^E(\mathbf{r},\mathbf{r})\right] , \tag{3.3}$$

$$G_{\mathrm{R,A}}^E(\mathbf{r}_1,\mathbf{r}_2) = \langle \mathbf{r}_1|(E-\hat{H}\pm i\eta)^{-1}|\mathbf{r}_2\rangle , \qquad \eta\to +0 . \tag{3.4}$$

In the two-level correlation function (2.39) the $\langle G_{\mathrm{R}}G_{\mathrm{R}}\rangle$ and $\langle G_{\mathrm{A}}G_{\mathrm{A}}\rangle$ terms are again trivial (decouple in the product of averages), and all the non-trivial information is contained in the $\langle G_{\mathrm{R}}G_{\mathrm{A}}\rangle$ terms. Introducing the supervector field

$$\Phi(\mathbf{r}) = \begin{pmatrix} S_1(\mathbf{r}) \\ \chi_1(\mathbf{r}) \\ S_2(\mathbf{r}) \\ \chi_2(\mathbf{r}) \end{pmatrix} , \tag{3.5}$$

we write, in analogy with (2.46), (2.47), the corresponding product of the Green's functions as a functional integral,

$$G_{\mathrm{R}}^{E+\omega/2}(\mathbf{r}_1,\mathbf{r}_1)G_{\mathrm{A}}^{E-\omega/2}(\mathbf{r}_2,\mathbf{r}_2) = \int \mathrm{D}\Phi\,\mathrm{D}\Phi^\dagger S_1(\mathbf{r}_1)S_1^*(\mathbf{r}_1)S_2(\mathbf{r}_2)S_2^*(\mathbf{r}_2)$$
$$\times\exp\left\{i\int \mathrm{d}^d\mathbf{r}\,\Phi^\dagger(\mathbf{r})L[E+(\omega/2+i\eta)\Lambda-\hat{H}]\Phi(\mathbf{r})\right\} . \tag{3.6}$$

[15] We set $\hbar = 1$.

[16] We will use a somewhat sloppy (but convenient) terminology and refer to systems having the symmetry of GOE, GUE, and GSE as to systems of orthogonal, unitary, and symplectic symmetry, respectively. For a disordered electronic problem, the orthogonal symmetry corresponds to purely potential scattering in the absence of magnetic fields (both time reversal and spin-rotation invariance preserved), the symplectic symmetry to the spin-orbit scattering (spin-rotation invariance broken, but time reversal preserved), while the unitary symmetry class is realized in the presence of homogeneous or random magnetic field breaking the time reversal invariance [35, 1, 2].

Averaging over the disorder with the correlator (3.2) produces a quartic term,

$$\left\langle \exp\{i\int \mathrm{d}^d\mathbf{r}\,\Phi^\dagger(\mathbf{r})LU(\mathbf{r})\Phi(\mathbf{r})\}\right\rangle = \exp\left\{-\frac{1}{4\pi\nu\tau}\int \mathrm{d}^d\mathbf{r}\,[\Phi^\dagger(\mathbf{r})L\Phi(\mathbf{r})]^2\right\}, \qquad (3.7)$$

the Hubbard-Stratonovich decoupling of which requires the introduction of an $\mathbf{r}$-dependent supermatrix field $R(\mathbf{r})$,

$$\begin{aligned}\exp\left\{-\frac{1}{4\pi\nu\tau}\int \mathrm{d}^d\mathbf{r}\,[\Phi^\dagger(\mathbf{r})L\Phi(\mathbf{r})]^2\right\} &= \int \mathrm{D}R\,\exp\{-i\int \mathrm{d}^d\mathbf{r}\,\Phi^\dagger(\mathbf{r})L^{1/2}R(\mathbf{r})L^{1/2}\Phi(\mathbf{r}) \\ &- \pi\nu\tau\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}R^2(\mathbf{r})\}\ . \qquad (3.8)\end{aligned}$$

For any given $\mathbf{r}$ the matrix $R(\mathbf{r})$ has the structure specified by Eqs. (2.51)–(2.53). Substituting (3.8), (3.7) in (3.6) and performing the Φ-integral, we obtain, similarly to (2.57), an integral over the R-field with the action

$$S[R] = \pi\nu\tau\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}R^2 + \mathrm{Str}\ln\left(E + \frac{\omega}{2}\Lambda - \frac{\hat{\mathbf{p}}^2}{2m} - R\right)\ . \qquad (3.9)$$

The corresponding saddle-point equation reads (for $\omega \to 0$)

$$R(\mathbf{r}) = \frac{1}{2\pi\nu\tau}g(\mathbf{r},\mathbf{r})\ ; \qquad g = \left(E - \frac{\hat{\mathbf{p}}^2}{2m} - R\right)^{-1}\ . \qquad (3.10)$$

We look first for a diagonal, $\mathbf{r}$-independent solution of (3.10), $R = \mathrm{diag}(q_1, q_2, q_3, q_4)$, which has, in the weak-disorder regime $E\tau \gg 1$, the form

$$q_j = \frac{1}{2\pi\nu\tau}\mathrm{Re}\langle G_\mathrm{R}(\mathbf{r},\mathbf{r})\rangle \pm \frac{i}{2\tau}\ , \qquad (3.11)$$

with

$$\langle G_\mathrm{R}(\mathbf{r},\mathbf{r})\rangle = \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}(E - \mathbf{p}^2/2m + i/2\tau)^{-1}. \qquad (3.12)$$

The first term in (3.11) gives a non-interesting constant real contribution to R, which can be absorbed[17] into the energy E.

The choice of signs in the second term of (3.11) is dictated by the same considerations as in the GUE case (see the text below Eq. (2.58)), leaving us with

$$R_0 = \frac{1}{2\pi\nu\tau}\mathrm{Re}\langle G_\mathrm{R}(\mathbf{r},\mathbf{r})\rangle - \frac{i}{2\tau}\Lambda \equiv \sigma - \frac{i}{2\tau}\Lambda. \qquad (3.13)$$

The manifold of saddle-points is generated from R_0 by rotations with matrices T defined in (2.52), (2.53),

$$R = \sigma - \frac{i}{2\tau}T\Lambda T^{-1} \equiv \sigma - \frac{i}{2\tau}Q\ , \qquad (3.14)$$

[17] As it stands, Eq. (3.12) has an ultraviolet divergence at $p \to \infty$ in $d \geq 2$. This is related to the white-noise (zero correlation length) character of the disorder potential. Assuming a finite correlation length of the disorder would introduce an UV-cutoff and make the integral (3.12) finite. In a realistic model of a disordered metal the correlation length is set by the screening length, which is of order of the Fermi wave length. Introducing a cutoff $\sim p_F = (2mE)^{1/2}$ in (3.12), we find that the first term in (3.12) is much smaller than E in the considered weak-disorder regime.

with Q from Eq. (2.60). The set of matrices (3.14) constitutes a manifold of degenerate (at $\omega \to 0$) saddle points of the action (3.9).

We allow now the matrix T (and consequently Q) to vary slowly in space,

$$R(\mathbf{r}) = \sigma - \frac{i}{2\tau} Q(\mathbf{r}) , \qquad Q(\mathbf{r}) = T(\mathbf{r}) \Lambda T^{-1}(\mathbf{r}) . \tag{3.15}$$

While (3.15) with an $\mathbf{r}$-dependent Q is not an exact saddle-point any more, the fields of this form constitute the low-lying excitations for the functional integral $\int \mathrm{D}R \ldots e^{-S[R]}$ with the action (3.9). Performing the gradient expansion of the second term in Eq. (3.9) for the configurations defined by Eq. (3.15)(and also expanding in ω up to the linear term), we find the action for these low-lying modes

$$S[Q] = \frac{\pi\nu}{4} \int \mathrm{d}^d\mathbf{r}\, \mathrm{Str}[-D(\nabla Q)^2 - 2i\omega\Lambda Q] , \tag{3.16}$$

where $D = v_F^2\tau/d$ is the diffusion constant (here $v_F = (2E/m)^{1/2}$ is the particle velocity and d the spatial dimensionality). The two-level correlation function is thus reduced to a functional integral over the Q-matrix field slowly varying on the coset space,

$$R_2(\omega) = \left(\frac{1}{4V}\right)^2 \mathrm{Re} \int \mathrm{D}Q(\mathbf{r}) \left[\int \mathrm{d}^d\mathbf{r}\, \mathrm{Str} Q\Lambda k\right]^2 e^{-S[Q]} , \tag{3.17}$$

where $k = \mathrm{diag}(1, -1, 1, -1)$, i.e k is equal to 1 (-1) in the boson-boson (resp. fermion-fermion) block.

The field theory characterized by the action (3.16) is called a (d-dimensional) non-linear σ-model. It characterizes the low-frequency long-wavelength physics of the original electronic problem. The fact that such a matrix σ-model in the replica limit $n \to 0$ is the effective field theory for the problem of a particle in a random potential was first realized by Wegner [36]; a derivation (in the replica formulation) was given by Schäfer and Wegner [37] and by Efetov, Larkin, and Khmelnitskii [38]. The supersymmetric version of the model was presented by Efetov [1]. To elucidate the physical content of the σ-model (3.16), it is instructive to draw a parallel with a model of classical magnetic moments with a ferromagnetic interaction in an external magnetic field,

$$H[\mathbf{S}] = -\sum_{\mathbf{rr}'} J_{\mathbf{rr}'} \mathbf{S}(\mathbf{r})\mathbf{S}(\mathbf{r}') - \sum_{\mathbf{r}} \mathbf{BS}(\mathbf{r}) , \tag{3.18}$$

where $\mathbf{S}(\mathbf{r})$ are n-component vectors with $\mathbf{S}^2(\mathbf{r}) = 1$. At low temperatures, only the low-energy sector is important, which corresponds to a slowly varying vector $\mathbf{S}(\mathbf{r})$. The Hamiltonian (3.18) is then reduced to a vector σ-model,

$$H[\mathbf{S}] = \int \mathrm{d}^d\mathbf{r} \left[\frac{\kappa}{2}(\nabla\mathbf{S}(\mathbf{r}))^2 - \mathbf{BS}(\mathbf{r})\right] , \tag{3.19}$$

where κ is the spin stiffness. Let us now demonstrate the analogies between (3.16) and (3.19). The n-component unit vector $\mathbf{S}$ sweeps the $(n-1)$-dimensional sphere S^{n-1} isomorphic to the coset space $O(n)/O(n-1)$. If the external magnetic field $\mathbf{B}$

is directed, say, along $\mathbf{e}_1 = (1, 0, \ldots, 0)$, then $\mathbf{e}_1$ plays in (3.19) the same role as Λ in (3.16). Indeed, $O(n-1)$ is the subgroup of $O(n)$ which does not rotate $\mathbf{e}_1$; for this reason the space of matrices from $O(n)$ which generate different vectors when acting on $\mathbf{e}_1$ is the coset space $O(n)/O(n-1)$. Fully analogously, $U(1|1) \times U(1|1)$ is the subgroup of those matrices from $U(1,1|2)$ which do not rotate (i.e commute with) Λ, so that the manifold of matrices Q is generated by rotations T belonging to the coset space $U(1,1|2)/U(1|1)\times U(1|1)$. Furthermore, while the first term of (3.19) is invariant with respect to a global rotation of $\mathbf{S}(\mathbf{r})$, the second term breaks this $O(n)/O(n-1)$ symmetry. In the ferromagnetic phase the symmetry is broken spontaneously at $B \to 0$; the corresponding Goldstone modes are the spin waves. To write their Hamiltonian explicitly, one should represent H in terms of independent degrees of freedom, *i.e.* in terms of the transverse (with respect to $\mathbf{e}_1$) part $\mathbf{S}_\perp$ (the longitudinal component being $S_1 = (1 - \mathbf{S}_\perp^2)^{1/2}$),

$$H[\mathbf{S}_\perp] = \frac{1}{2} \int \mathrm{d}^d\mathbf{r} \, \left[\kappa[\nabla \mathbf{S}_\perp(\mathbf{r})]^2 + B\mathbf{S}_\perp^2(\mathbf{r}) + O(\mathbf{S}_\perp^4(\mathbf{r}))\right] \ , \tag{3.20}$$

with the last term describing interaction of the spin waves. Equation (3.20) implies that the correlation function of the transverse magnetization has the Goldstone-type low-momentum behavior,

$$\int \mathrm{d}^d\mathbf{r}\, e^{-i\mathbf{q}\mathbf{r}} \langle \mathbf{S}_\perp(0)\mathbf{S}_\perp(\mathbf{r})\rangle \propto \frac{1}{\kappa \mathbf{q}^2 + B} \ . \tag{3.21}$$

Comparing (3.19) with (3.16), we see that the frequency ω plays, in the case of a disordered electronic system, the same role of a symmetry breaking field as the magnetic field $\mathbf{B}$ for a ferromagnet. The soft modes for (3.16) are the diffusion modes. To write their action explicitly, one should choose some parametrization of Q in terms of independent degrees of freedom. Two parametrizations, most often used for perturbative calculations, are

$$\begin{aligned} Q &= (1 - W/2)\Lambda(1 - W/2)^{-1} = \Lambda(1 + W/2)(1 - W/2)^{-1} \\ &= \Lambda \left(1 + W + \frac{W^2}{2} + \frac{W^3}{4} + \frac{W^4}{8} + \ldots\right) \end{aligned} \tag{3.22}$$

and

$$Q = \Lambda \left(W + \sqrt{1 + W^2/2}\right) = \Lambda \left(1 + W + \frac{W^2}{2} - \frac{W^4}{8} + \ldots\right) \ , \tag{3.23}$$

with W having the off-diagonal block structure in the RA space,

$$W = \begin{pmatrix} 0 & W_{12} \\ W_{21} & 0 \end{pmatrix} \ ; \qquad W_{12}^\dagger = kW_{21} \ . \tag{3.24}$$

In either case the action (3.16) takes the form

$$S[W] = \frac{\pi\nu}{4} \int \mathrm{d}^d\mathbf{r}\, \mathrm{Str}[D(\nabla W)^2 - i\omega W^2 + O(W^3)] \ , \tag{3.25}$$

with the last term on the r.h.s. describing the interaction of the diffusion modes. The analog of (3.21) is the diffusion propagator (see below),

$$\int \mathrm{d}^d\mathbf{r}\, e^{-i\mathbf{qr}} \langle \mathrm{Str} kW_{12}(0)kW_{21}(\mathbf{r})\rangle \sim \frac{1}{\pi\nu(D\mathbf{q}^2 - i\omega)} \,. \tag{3.26}$$

After this digression we return to the analysis of the two-level correlation function (3.17).

3.2 Reduction to the 0D σ-model: Universal limit.

According to (3.25), the "energies" (eigenvalues of the quadratic form of the action) of the diffusion modes are given by

$$D\mathbf{q}_\mu^2 - i\omega \equiv \epsilon_\mu - i\omega \ , \tag{3.27}$$

with $\mathbf{q}_\mu$ being the allowed values of wave vectors. One can show that in an isolated sample the boundary condition for the Q-field is[18]

$$\nabla_\mathbf{n} Q|_{\mathrm{boundary}} = 0 \ , \tag{3.28}$$

where $\nabla_\mathbf{n}$ is the normal derivative. If we consider a rectangular sample of a size $L_1 \times \ldots \times L_d$, the wave vectors $\mathbf{q}_\mu$ are quantized according to

$$\mathbf{q}_\mu = \pi\left(\frac{n_1}{L_1}, \ldots, \frac{n_d}{L_d}\right) \ , \qquad n_i = 0, 1, 2, \ldots . \tag{3.29}$$

For periodic boundary conditions, which are often used by theoreticians, we have instead

$$\mathbf{q}_\mu = 2\pi\left(\frac{n_1}{L_1}, \ldots, \frac{n_d}{L_d}\right) \ , \qquad n_i = 0, \pm1, \pm2, \ldots . \tag{3.30}$$

In either case, one can distinguish the zero mode with $\mathbf{q}_0 = 0$ (and thus $\epsilon_0 = 0$) from all other modes with $\epsilon_\mu \geq D(\pi/L)^2$, where $L = \max\{L_1, \ldots, L_d\}$. The energy $E_c = D/L^2$ is called the Thouless energy; it is of the order of the inverse time of diffusion through the sample. Equation (3.27) suggests that for $\omega \ll E_c$ one can neglect all non-zero modes when calculating the level correlation function. This approximation used by Efetov [1] is known as the zero-mode approximation. As a result, the functional integral (3.17) reduces to an integral over a single supermatrix Q, acquiring the form of the correlation function of the 0D σ-model, Eq. (2.61). Therefore, in the zero-mode approximation the level statistics is described by the 0D σ-model and thus has (according to Sec. 2.4) the RMT form (2.71).

For systems with preserved time-reversal invariance the same consideration leads [1, 2] to the σ-models of the orthogonal or symplectic symmetry (depending on the

[18] To understand the physical meaning of (3.28), one should remember that Q was introduced as a field conjugate to the product $\Phi\Phi^\dagger$, i.e. it is a kind of a density field. Equation (3.28) is thus the condition of the absence of current through the sample boundary, which is precisely what should be imposed on a boundary with vacuum or with an insulator.

presence of spin-dependent scattering). In the zero-mode approximation they reduce to corresponding 0D σ-models yielding the level statistics (2.72), (2.73) of Gaussian ensembles of the corresponding symmetries.

In order for the zero-mode approximation to make sense, the Thouless energy E_c should be much larger than the mean level spacing Δ. It is easy to see that the ratio $g = 2\pi E_c/\Delta$ is the dimensionless (measured in units of e^2/h) conductance of the sample (for a current flowing in the direction of L). Therefore, the above condition is equivalent to $g \gg 1$; we will term the corresponding situation "metallic regime" (or "weak localization regime"). The opposite case, $g \lesssim 1$, corresponds to the strong localization of a particle and is realized in 2D or 3D if the disorder is not weak[19], $E\tau \sim 1$, or in sufficiently long systems of quasi-1D geometry (see below). The notion of the dimensionless conductance plays a central role in the scaling theory of localization, see [39] for review.

3.3 Deviations from universality

3.3.1 Perturbation theory.

Deviations from the universal (RMT) behavior are due to the non-zero diffusion modes. The first calculation of the level statistics in a diffusive grain taking into account the non-zero modes was performed by Altshuler and Shklovskii [40]. They used the perturbation theory, which amounts to keeping only the terms of the leading order in W in the expansion of the preexponential factor and of the action (3.25) in Eq. (3.17). This yields (for the unitary symmetry)

$$
\begin{aligned}
R_{2,\mathrm{AS}}(\omega) - 1 &\simeq \frac{1}{(4V)^2}\mathrm{Re}\int \mathrm{D}W(\mathbf{r}) \left[\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}Wk\right]^2 \\
&\qquad \times \exp\left\{\frac{\pi\nu}{4}\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}W[D\nabla^2 + i\omega]W\right\} \\
&= \frac{\Delta^2}{2\pi^2}\mathrm{Re}\sum_\mu \frac{1}{(\epsilon_\mu - i\omega)^2}. \qquad (3.31)
\end{aligned}
$$

The same result is obtained for the other symmetry classes, with an additional factor $2/\beta$, where β is the commonly used parameter equal to 1, 2, and 4 for the orthogonal, unitary, and symplectic symmetry, respectively.

We analyze first the perturbative expression (3.31) in the limit $\omega \ll E_c$ (or, equivalently, $s \equiv \omega\Delta \ll g$), when the zero-mode approximation applies. Keeping only the term with $\epsilon_0 = 0$ in (3.31) yields

$$
R_{2,\mathrm{AS}}(\omega) - 1 = -\frac{1}{2\pi^2 s^2}\,, \qquad s \ll g\,. \qquad (3.32)
$$

To compare this perturbative formula with the exact result (2.71), we note that the connected part of the RMT correlator (2.71) can be decomposed into the smooth and

[19] Strictly speaking, in 2D even for a weak disorder all states are localized, but the corresponding localization length is exponentially large.

the oscillatory contributions,

$$-\frac{\sin^2(\pi s)}{(\pi s)^2} = -\frac{1}{2(\pi s)^2} + \frac{\cos(2\pi s)}{2(\pi s)^2} \ . \tag{3.33}$$

We see that the Altshuler-Shklovskii calculation reproduces the smooth term, but fails to give the oscillatory contribution. This is because their method is perturbative in $1/s$, while the oscillations $\propto \cos(2\pi s)$ are of non-perturbative (in $1/s$) character. Furthermore, the Altshuler-Shklovskii result gives no information about the actual small-s behavior of $R_2(\omega)$.

We consider now the opposite regime, $\omega \gg E_c$ ($s \gg g$), in which the summation in (3.31) can be replaced by integration, yielding[20]

$$R_{2,\mathrm{AS}}(\omega) - 1 \simeq c_d g^{-d/2} s^{d/2-2} \ , \qquad s \gg g \ , \tag{3.34}$$

with a numerical coefficient c_d. Therefore, in the domain $s \gg g$ the level statistics differs completely from its universal (RMT) form. The slower (compared to RMT) decay of DOS correlations at $s \gg g$ corresponds to the diffusive motion of the particle at times $t \ll t_D \sim E_c^{-1}$. In contrast, at long times, $t \gg t_D$, the trajectory covers ergodically the whole sample volume, and the correlations acquire their universal form.

It should be mentioned here that Altshuler and Shklovskii used in fact not the σ-model formalism but the impurity diagram technique, with the ladder diagrams representing the diffusion modes. This diffuson-cooperon diagrammatics (completely equivalent to the perturbative expansion of the σ-model) is by now a standard tool, which has allowed to discover and to study a number of remarkable effects in mesoscopic physics, in particular the weak localization, the Altshuler-Aronov effect of interplay of the Coulomb interaction and disorder, and the universal conductance fluctuations. Several excellent reviews of these subjects are available, see Refs. [39, 42, 43, 33]. However, as the above discussion shows, the perturbative methods are not sufficient for studying the statistics of energy levels (and also of eigenfunctions, see Sec. 4), and the non-perturbative σ-model approach has to be used.

3.3.2 Deviations from universality at $\omega \ll E_c$.

We return now to the region $s \ll g$ and consider deviations from the universal (RMT) behavior. The method that allows us to calculate such deviations from the universality was developed in [44] and can be outlined as follows. We first decompose Q (in a proper way taking into account the non-linear constraint $Q^2 = 1$) into the constant part Q_0 and the contribution $\tilde{Q}$ of higher modes with non–zero momenta. Then we integrate out all non-zero modes. This can be done perturbatively provided the dimensionless conductance $g \gg 1$. As a result, we get an integral over the matrix Q_0 only, which has to be calculated non-perturbatively.

[20] We assume the spatial dimensionality $d < 4$ (otherwise, the sum in (3.31) has an ultraviolet divergence). We also note that in 2D the Altshuler-Shklovskii calculation gives the coefficient $c_2 = 0$; a more careful consideration [41] taking into account higher order terms of the perturbative expansion yields $c_2 \sim 1/g^2$ for the unitary and $c_2 \sim 1/g$ for the orthogonal and symplectic symmetry.

We begin by presenting the correlator $R_2(\omega)$ in the form

$$R_2(\omega) = \frac{1}{(2\pi i)^2}\frac{\partial^2}{\partial u^2}\int \mathrm{D}Q \exp\{-S[Q]\}|_{u=0} \ ,$$
$$S[Q] = -\frac{\pi\nu D}{4}\int \mathrm{Str}(\nabla Q)^2 + \tilde{s}\int \mathrm{Str}\Lambda Q + \tilde{u}\int \mathrm{Str}Q\Lambda k \ , \qquad (3.35)$$

where $\tilde{s} = \pi s/2iV$, $\tilde{u} = \pi u/2iV$. Now we decompose Q in the following way:

$$Q(\mathbf{r}) = T_0\tilde{Q}(\mathbf{r})T_0^{-1} \ , \qquad (3.36)$$

where T_0 is a spatially uniform matrix from the coset space and $\tilde{Q}$ describes all modes with non-zero momenta. When $\omega \ll E_c$, the matrix $\tilde{Q}$ fluctuates only weakly near the origin Λ of the coset space. In the leading order, $\tilde{Q} = \Lambda$, thus reducing (3.35) to the 0D σ–model, which leads to the Wigner–Dyson result (2.71). To find the corrections, we should expand the matrix $\tilde{Q}$ around the origin Λ. We use the parametrization (3.22),

$$\tilde{Q} = \Lambda(1+W/2)(1-W/2)^{-1} = \Lambda\left(1 + W + \frac{W^2}{2} + \frac{W^3}{4} + \dots\right) \ . \qquad (3.37)$$

To keep the number of degrees of freedom unchanged, we should exclude the zero mode from $\tilde{Q}$, which is achieved by the constraint $\int \mathrm{d}^d\mathbf{r}\, W = 0$. Substituting the expansion (3.37) into Eq. (3.35), we get

$$S = S_0[Q_0] + S_W[W] + S_1[Q_0, W] \ ;$$
$$S_0 = \int \mathrm{Str}\left[\tilde{s}Q_0\Lambda + \tilde{u}Q_0\Lambda k\right] \ ,$$
$$S_W = \frac{\pi\nu D}{4}\int \mathrm{Str}(\nabla W)^2 \ ,$$
$$S_1 = \frac{1}{2}\int \mathrm{Str}[\tilde{s}U_0\Lambda W^2 + \tilde{u}U_{0k}\Lambda W^2] + O(W^3) \ , \qquad (3.38)$$

where $Q_0 = T_0\Lambda T_0^{-1}$, $U_0 = T_0^{-1}\Lambda T_0$, $U_{0k} = T_0^{-1}\Lambda k T_0$. Let us define $S_{\mathrm{eff}}[Q_0]$ as a result of elimination of the fast (non-zero) modes:

$$e^{-S_{\mathrm{eff}}[Q_0]} = e^{-S_0[Q_0]}\langle e^{-S_1[Q_0,W]+\ln J[W]}\rangle_W \ , \qquad (3.39)$$

where $\langle\dots\rangle_W$ denote the integration over W with the Gaussian weight $\exp\{-S_W[W]\}$, and $J[W]$ is the Jacobian of the transformation (3.36), (3.37) from the variable Q to $\{Q_0, W\}$ (the Jacobian does not contribute to the leading order correction calculated here, but is important for higher-order calculations [45, 46]). Expanding up to the order W^4, we get

$$S_{\mathrm{eff}} = S_0 + \langle S_1\rangle - \frac{1}{2}\langle S_1^2\rangle + \frac{1}{2}\langle S_1\rangle^2 + \dots \ . \qquad (3.40)$$

The integral over the fast modes W can be calculated now using the Wick theorem with the contraction rules induced by the action S_W,

$$\langle \mathrm{Str}W(\mathbf{r})PW(\mathbf{r}')R\rangle = \Pi(\mathbf{r},\mathbf{r}')(\mathrm{Str}P\mathrm{Str}R - \mathrm{Str}P\Lambda\mathrm{Str}R\Lambda) \ ;$$
$$\langle \mathrm{Str}[W(\mathbf{r})P]\mathrm{Str}[W(\mathbf{r}')R]\rangle = \Pi(\mathbf{r},\mathbf{r}')\mathrm{Str}(PR - P\Lambda R\Lambda), \qquad (3.41)$$

where P and R are arbitrary supermatrices. The diffusion propagator Π is the solution of the diffusion equation

$$-D\nabla^2\Pi(\mathbf{r}_1,\mathbf{r}_2) = (\pi\nu)^{-1}[\delta(\mathbf{r}_1-\mathbf{r}_2) - V^{-1}] \tag{3.42}$$

with the Neumann boundary condition (normal derivative equal to zero at the sample boundary) and can be presented in the form

$$\Pi(\mathbf{r},\mathbf{r}') = \frac{1}{\pi\nu}\sum_{\mu;\ \epsilon_\mu\neq 0}\frac{1}{\epsilon_\mu}\phi_\mu(\mathbf{r})\phi_\mu(\mathbf{r}')\ , \tag{3.43}$$

where $\phi_\mu(\mathbf{r})$ are the eigenfunctions of the diffusion operator $-D\nabla^2$ corresponding to the eigenvalues ϵ_μ (equal to $D\mathbf{q}^2$ for a rectangular geometry). As a result, we find

$$\begin{aligned}
&\langle S_1\rangle = 0\ ,\\
&\langle S_1^2\rangle = \frac{1}{2}\int \mathrm{d}^d\mathbf{r}\mathrm{d}^d\mathbf{r}'\Pi^2(\mathbf{r},\mathbf{r}')(\tilde{s}\mathrm{Str}Q_0\Lambda + \tilde{u}\mathrm{Str}Q_0\Lambda k)^2\ .
\end{aligned} \tag{3.44}$$

Substitution of Eq. (3.44) into Eq. (3.40) yields

$$\begin{aligned}
S_{\mathrm{eff}}[Q_0] &= \frac{\pi}{2i}s\mathrm{Str}Q_0\Lambda + \frac{\pi}{2i}u\mathrm{Str}Q_0\Lambda k + A\frac{\pi^2}{16}(s\mathrm{Str}Q_0\Lambda + u\mathrm{Str}Q_0\Lambda k)^2\ ;\\
A &= \frac{1}{V^2}\int \mathrm{d}^d\mathbf{r}\mathrm{d}^d\mathbf{r}'\Pi^2(\mathbf{r},\mathbf{r}') = \frac{1}{\pi^2}\sum_{\mu\neq 0}\left(\frac{\Delta}{\epsilon_\mu}\right)^2 = \frac{4a_d}{g^2}\ ,
\end{aligned} \tag{3.45}$$

where a_d is a numerical coefficient depending on the spatial dimensionality d and on the sample geometry. Assuming a cubic sample with hard-wall boundary conditions, we find

$$a_d = \frac{1}{\pi^4}\sum_{\substack{n_1,\ldots,n_d=0\\ n_1^2+\ldots+n_d^2>0}}^{\infty}\frac{1}{(n_1^2+\ldots+n_d^2)^2}\ , \tag{3.46}$$

yielding for $d=1,2,3$ the values $a_1 = 1/90 \simeq 0.0111$, $a_2 \simeq 0.0266$, and $a_3 \simeq 0.0527$ respectively.[21] In the case of a cubic sample with periodic boundary conditions we get instead

$$a_d = \frac{1}{(2\pi)^4}\sum_{\substack{n_1,\ldots,n_d=-\infty\\ n_1^2+\ldots+n_d^2>0}}^{\infty}\frac{1}{(n_1^2+\ldots+n_d^2)^2}\ , \tag{3.47}$$

so that $a_1 = 1/720 \simeq 0.00139$, $a_2 \simeq 0.00387$, and $a_3 \simeq 0.0106$. Note that for $d<4$ the sum in Eqs. (3.46), (3.47) converges, so that no ultraviolet cutoff is needed.

[21] While speaking about $d=1$, we mean a sample of a quasi-1D geometry, i.e. either a 2D strip $b\times L$ with $b\ll L$ or a 3D wire $b_1\times b_2\times L$ with $b_1,b_2\ll L$. For a strictly 1D sample (a chain) the diffusive σ-model is not applicable.

Using now Eq.(3.35) and calculating the remaining integral over the supermatrix Q_0, we finally get the following expression for the correlator to the $1/g^2$ order:

$$R_2(s) = 1 + \delta(s) - \frac{\sin^2(\pi s)}{(\pi s)^2} + A\sin^2(\pi s)\ ; \qquad A = \frac{4a_d}{g^2}\ . \tag{3.48}$$

The last term in Eq.(3.48) represents the sought correction of order $1/g^2$ to the Wigner–Dyson distribution.

The important feature of Eq.(3.48) is that it relates corrections to the smooth and oscillatory parts of the level correlation function,

$$\delta R_2(s) = A\sin^2\pi s = \frac{A}{2}(1-\cos 2\pi s)\ . \tag{3.49}$$

While appearing naturally in the framework of the supersymmetric σ-model, this fact is highly non-trivial from the point of view of semiclassical theory [47], which represents the level structure factor $K(\tau)$ (Fourier transform of $R_2(s)$) in terms of a sum over periodic orbits. The smooth part of $R_2(s)$ corresponds then to the small-τ behavior of $K(\tau)$, which is related to the properties of short periodic orbits. On the other hand, the oscillatory part of $R_2(s)$ is related to the behavior of $K(\tau)$ in the vicinity of the Heisenberg time $\tau = 2\pi$ ($t = 2\pi/\Delta$ in dimensionful units), and thus to the properties of long periodic orbits. We will return to the non-universal corrections to $K(\tau)$ below.

The calculation presented above can be straightforwardly generalized to the other symmetry classes. The result can be presented in a form valid for all the three cases:

$$R_2^{(\beta)}(s) = \left(1 + \frac{A}{2\beta}\frac{d^2}{ds^2}s^2\right) R^{(\beta)}_{2,RMT}(s)\ , \qquad A = \frac{4a_d}{g^2}\ , \tag{3.50}$$

where $\beta = 1(2,4)$ for the orthogonal (unitary, symplectic) symmetry; $R^{(\beta)}_{2,\mathrm{RMT}}$ denotes the corresponding RMT correlation function (2.71)–(2.73).

For $s \to 0$ the RMT correlation functions have the following behavior:

$$\begin{aligned} &R^{(\beta)}_{2,\mathrm{RMT}} \simeq C_\beta s^\beta\ , \qquad s\to 0\ ; \\ &C_1 = \frac{\pi^2}{6}\ , \quad C_2 = \frac{\pi^2}{3}\ , \quad C_4 = \frac{(2\pi)^4}{135}\ . \end{aligned} \tag{3.51}$$

As is clear from Eq.(3.50), the found correction does not change the exponent β, but renormalizes the prefactor C_β:

$$R_2^{(\beta)}(s) = \left(1 + \frac{(\beta+2)(\beta+1)}{2\beta}A\right) C_\beta s^\beta\ ; \quad s\to 0 \tag{3.52}$$

The correction to C_β is positive, which means that the level repulsion becomes weaker. This is related to a tendency of eigenfunctions to localization with decreasing g.

3.3.3 Stationary-point method.

Let us return now to the behavior of the level correlation function in its high-frequency tail $s \gg g$. We have already discussed the non-oscillatory part of $R_2(s)$ in this region, see Eq. (3.34). What is the fate of the oscillations in $R_2(s)$ in this regime? The answer to this question was given by Andreev and Altshuler [48] who calculated $R_2(s)$ using the stationary-point method for the σ-model integral (3.17) and treating the zero mode and the higher modes on equal footing. Their crucial observation was that on top of the trivial stationary point $Q = \Lambda$ (expansion around which is just the usual perturbation theory), there exists another one, $Q = k\Lambda$, whose vicinity generates the oscillatory part of $R_2(s)$. (In the case of symplectic symmetry there exists an additional family of stationary points, see [48]). The saddle-point approximation[22] of Andreev and Altshuler is valid for $s \gg 1$; at $1 \ll s \ll g$ it reproduces the above results of Ref. [44] (we remind that the method of [44] works for all $s \ll g$). The result of [48] has the following form:

$$R^{\mathrm{U}}_{2,\mathrm{osc}}(s) = \frac{\cos 2\pi s}{2\pi^2} D(s) \ , \tag{3.53}$$

$$R^{\mathrm{O}}_{2,\mathrm{osc}}(s) = \frac{\cos 2\pi s}{2\pi^4} D^2(s) \ , \tag{3.54}$$

$$R^{\mathrm{Sp}}_{2,\mathrm{osc}}(s) = \frac{\cos 2\pi s}{4} D^{1/2}(s) + \frac{\cos 4\pi s}{32\pi^4} D^2(s) \ , \tag{3.55}$$

where $D(s)$ is the spectral determinant

$$D(s) = \frac{1}{s^2} \prod_{\mu \neq 0} \left(1 + \frac{s^2 \Delta^2}{\epsilon_\mu^2}\right)^{-1} . \tag{3.56}$$

The product in Eq.(3.56) goes over the non-zero eigenvalues ϵ_μ of the diffusion operator. This demonstrates again the relation between $R_{2,\mathrm{osc}}(s)$ and the perturbative part (3.31), which can be also expressed through $D(s)$,

$$R^{(\beta)}_{2,\mathrm{AS}}(s) - 1 = -\frac{1}{2\beta\pi^2} \frac{\partial^2 \ln D(s)}{\partial s^2} \ . \tag{3.57}$$

In the high-frequency region $s \gg g$ the spectral determinant is found to have the following behavior:

$$D(s) \sim \exp\left\{ -\frac{\pi}{\Gamma(d/2) d \sin(\pi d/4)} \left(\frac{2s}{g}\right)^{d/2} \right\} \ , \tag{3.58}$$

so that the amplitude of the oscillations vanishes exponentially with s in this region.

[22] To avoid possible confusion, we remind that the matrices of the manifold (3.14) are exact saddle-points of the action (3.9) (*i.e.* they all have exactly the same action) for $\omega = 0$ only. At finite ω this becomes a manifold of quasi-saddle-points, with the action difference determined by the term $(-i\pi s/2)\mathrm{Str} Q\Lambda$, see Eqs. (2.61) and (3.16). The corresponding soft (σ-model) modes should be contrasted to massive modes (describing fluctuations in $P_{1,2}$, see Eq. (2.51)), which have been integrated out in the course of derivation of the σ-model. Now, on this manifold of almost-saddle-points Q there are two true (even at non-zero ω) stationary points, namely, $Q = \Lambda$ and $Q = k\Lambda$. In fact, we have already mentioned the existence of the second diagonal saddle point below Eq. (2.58). It is easy to see that the choice of signs $s_2 = -s_4 = 1$ produces there precisely the matrix $Q = k\Lambda$.

Taken together, the results of [44] and [48] provide a complete description of the deviations of the level correlation function from universality in the metallic regime $g \gg 1$. They show that in the whole region of frequencies these deviations are controlled by the classical (diffusion) operator governing the dynamics in the corresponding classical system.

3.4 Spectral characteristics related to $R_2(s)$.

3.4.1 Spectral formfactor.

The spectral formfactor is defined as the Fourier transform of the connected part $R_2^{(c)}(s) = R_2(s) - 1$ of the two-level correlation function,

$$K(\tau) = \int_{-\infty}^{\infty} R_2^{(c)}(s) e^{is\tau} \mathrm{d}s \ . \tag{3.59}$$

By definition, $K(\tau)$ (as well as $R_2(s)$) is an even function, so that it is sufficient to discuss it at $\tau > 0$. In GUE it has the form

$$K(\tau) = \begin{cases} \tau/2\pi \ , & 0 \le \tau/2\pi \le 1 \\ 1 \ , & \tau/2\pi \ge 1 \ . \end{cases} \tag{3.60}$$

Let us analyze, what kind of corrections to $K_2(\tau)$ is implied by the deviations of $R_2(s)$ from universality studied in Sec. 3.3. For this purpose, let us use Eq. (3.49), forgetting for a moment about the condition of its validity ($s \ll g$). The Fourier transformation of (3.49) then yields

$$\delta K(\tau) = \frac{A\pi}{2}[2\delta(\tau) - \delta(\tau - 2\pi) - \delta(\tau + 2\pi)] \ . \tag{3.61}$$

Taking now into account the existence of the cutoff $s \sim g$ for (3.49) leads to smearing of the δ-functions in (3.61) over an interval $\sim 1/g$ around $\tau = 0$ and $\tau = \pm 2\pi$, respectively. Thus, we conclude that

$$\delta K(\tau) = \delta K_0(\tau) + \delta K_{2\pi}(\tau) \ , \qquad \tau > 0 \ , \tag{3.62}$$

where $\delta K_0(\tau)$ is located in the interval between $\tau = 0$ and $\tau \sim 1/g$, while $\delta K_{2\pi}(\tau)$ is concentrated in the interval of a width $\sim 1/g$ around $\tau = 2\pi$. Since $\delta R_2(0) = 0$, the integrals of $\delta K_0(\tau)$ and $\delta K_{2\pi}(\tau)$ are equal up to a sign,

$$\int \mathrm{d}\tau \, \delta K_0(\tau) = -\int \mathrm{d}\tau \, \delta K_{2\pi}(\tau) \ . \tag{3.63}$$

Furthermore, using the identity

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \delta R_2(s)|_{s=0} = -\int \mathrm{d}\tau \, \tau^2 \delta K(\tau) \ ,$$

we find

$$\int \mathrm{d}\tau \, \delta K_{2\pi}(\tau) \simeq -\frac{A}{4} \ , \tag{3.64}$$

so that the correction around the Heisenberg time $\tau = 2\pi$ is negative. Since the decay of $R_{2,\mathrm{osc}}(s)$ at $s > g$ is exponential, $\delta K_{2\pi}(\tau)$ is a smooth function. Using the fact that it is essentially located in an interval of width $\sim 1/g$ and that $A \sim 1/g^2$, we conclude that the magnitude of $\delta K_{2\pi}(\tau)$ in the above interval is $\sim 1/g$. The correction $K_{2\pi}(\tau)$ has thus both the magnitude and the width of the order of $1/g$ and leads to a rounding of the singularity in the spectral form-factor at the Heisenberg time $\tau = 2\pi$, as was first realized by Andreev and Altshuler [48]. In the quasi-1D case, the spectral determinant $D(s)$, Eq. (3.56), and thus the correction $\delta K_{2\pi}(\tau)$ can be calculated analytically, see [48].

As to $\delta K_0(\tau)$, its small-τ behavior depends on spatial dimensionality, in view of the the Altshuler-Shklovskii "tail" (3.34). Taking the Fourier transform, we find[23]

$$\delta K_0(\tau) \sim \frac{1}{g}(g\tau)^{1-d/2}\ , \qquad \tau \lesssim 1/g\ . \tag{3.65}$$

For semiclassical treatment of the spectral form-factor applicable for $\tau/2\pi \ll 1$ [where it reproduces the formula (3.65)] see Ref. [49].

3.4.2 Level number variance.

Consider the variance $\Sigma_2(s)$ of the number of energy levels in a spectral window of a width $\delta E = s\Delta$ (the average number of levels in this interval being equal to s). It is easy to see that $\Sigma_2(s)$ is related to the two-level correlation function as follows:

$$\Sigma_2(s) = \int_{-s}^{s} \mathrm{d}\tilde{s}\,(s - |\tilde{s}|) R_2^{(\mathrm{c})}(\tilde{s})\ . \tag{3.66}$$

The relation (3.66) can be also presented in the following way

$$\frac{d}{ds}\Sigma_2(s) = \int_{-s}^{s} d\tilde{s}\, R_2^{(\mathrm{c})}(\tilde{s})\ . \tag{3.67}$$

The level number variance is commonly used to characterize the long-range behavior of spectral correlations. The $1/s^2$ decay of the smooth part[24] of the two-level correlation function at $s \gg 1$ in RMT implies the $\ln s$ behavior of $\Sigma_2(s)$. In particular, for GUE the large-s asymptotics reads [7, 8]

$$\Sigma^{\mathrm{U}}_{2,\mathrm{RMT}}(s) = \frac{1}{\pi^2}[1 + \gamma + \ln(2\pi s)]\ , \qquad s \gg 1\ . \tag{3.68}$$

According to (3.49), the correction to the RMT form of $\Sigma_2(s)$ in a diffusive sample is small (and positive) at $s \ll g$ and has the form

$$\delta\Sigma_2(s) = \frac{A}{2}s^2 = 2a_d\left(\frac{s}{g}\right)^2\ , \qquad s \ll g\ . \tag{3.69}$$

[23]In 3D the spectral formfactor (3.65) seems to diverge as $\tau \to 0$. It should be taken into account, however, that the above considerations are valid only for frequencies ω corresponding to diffusive motion, *i.e.* $\omega \ll \tau_{\mathrm{e}}^{-1}$, where τ_{e} is the elastic mean free path (denoted by τ in Sec. 3.1). Correspondingly, the applicability of (3.65) is restricted by the condition $\tau \gtrsim \tau_e\Delta$.

[24]The oscillatory part of $R_2(s)$ is not important for the behavior of $\Sigma_2(s)$ at $s \gg 1$, because it gives a negligible contribution after integration (3.66).

On the other hand, at $s \gg g$ the level number variance is determined by the Altshuler-Shklovskii behavior of $R_2(s)$ and is totally different from its RMT form:

$$\Sigma_2(s) \sim \left(\frac{s}{g}\right)^{d/2} \gg \Sigma_{2,\mathrm{RMT}}(s) \ , \qquad s \gg g \ . \tag{3.70}$$

4 Eigenfunction statistics.

Not only the energy levels statistics but also the statistical properties of wave functions are of considerable interest. In the case of nuclear spectra (the statistical description of which was the original motivation for the development of the RMT), they determine fluctuations of widths and heights of the resonances [50, 51]. A more recent growth of interest to statistical properties of eigenfunctions in disordered and chaotic systems has been motivated, on the experimental side, by the possibility of fabrication of small systems (quantum dots) with well resolved electron energy levels [52, 53, 54, 55]. Fluctuations in the tunneling conductance of such a dot measured in recent experiments [56, 57] are related to statistical properties of wavefunction amplitudes [58, 59, 60, 61]. Furthermore, the eigenfunction fluctuations determine the statistics of matrix elements of the Coulomb interaction, which is important for understanding the properties of excitation and addition spectra of the dot [62, 63, 64, 65]. It is also worth mentioning that the microwave cavity technique allows one to observe experimentally spatial fluctuations of the wave amplitude in chaotic and disordered cavities [66, 67, 68] (though in this case one considers the intensity of a classical wave rather than of a quantum particle, all the results are equally applicable).

4.1 Random matrix theory

Within the RMT, the eigenfunction statistics has a very simple form in the limit $N \gg 1$. The components of an eigenvector $\psi^{(i)}$ of a matrix $\hat{H}$ from the Gaussian ensemble become then uncorrelated Gaussian random numbers (real, complex, or quaternionic for $\beta = 1$, 2, and 4, respectively) with the distributions

$$\mathcal{P}\left(\psi_k^{(i)}\right) \propto \exp\left\{-\frac{N\beta}{2}|\psi_k^{(i)}|^2\right\} \ . \tag{4.1}$$

If we introduce the "intensity"

$$y_k^{(i)} = N|\psi_k^{(i)}|^2 \tag{4.2}$$

(we have chosen the normalization $\langle y\rangle = 1$), its distribution will then have the form of the χ^2-distribution with β degrees of freedom. In particular, for GUE and GOE we have[25] [50, 51]

$$\mathcal{P}^{\mathrm{U}}(y) = e^{-y} \ , \tag{4.3}$$

$$\mathcal{P}^{\mathrm{O}}(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \ . \tag{4.4}$$

Equation (4.4) is known as the Porter-Thomas distribution; it was originally introduced to describe fluctuations of widths and heights of resonances in nuclear spectra [50].

4.2 Eigenfunction statistics in terms of the supersymmetric σ-model

Theoretical study of the eigenfunction statistics in a d-dimensional disordered system is again possible with making use of the supersymmetry method [69, 70, 45, 71]. We consider the local intensity of an eigenfunction at some point $\mathbf{r}_0$ of the sample, $y_i(\mathbf{r}_0) = V|\psi_i^2(\mathbf{r}_0)|$ (we have again normalized it to $\langle y\rangle = 1$). In order to calculate the distribution function $\mathcal{P}(y)$, we introduce its moments,

$$I_q(\mathbf{r}_0) = \Delta\langle\sum_i |\psi_i(\mathbf{r}_0)|^{2q}\delta(E-E_i)\rangle \equiv \langle|\psi(\mathbf{r}_0)|^{2q}\rangle_E \ . \tag{4.5}$$

The next step is to observe that $I_q(\mathbf{r}_0)$ is related for $q = 2, 3, \ldots$ to the following product of the Green's functions[26]

$$\begin{aligned} K_{l,m}(\mathbf{r}_0,\eta) &= \langle\mathbf{r}_0|(E-\hat{H}+i\eta)^{-1}|\mathbf{r}_0\rangle^l\langle\mathbf{r}_0|(E-\hat{H}-i\eta)^{-1}|\mathbf{r}_0\rangle^m \\ &\equiv [G_{\mathrm{R}}^E(\mathbf{r}_0,\mathbf{r}_0)]^l[G_{\mathrm{A}}^E(\mathbf{r}_0,\mathbf{r}_0)]^m \ , \end{aligned} \tag{4.6}$$

where $l, m \geq 1$ and $l + m = q$. Indeed, in the limit $\eta \to 0$ $\langle K_{l,m}\rangle$ possesses a singularity $\propto \eta^{1-l-m}$ determined by the probability $\propto \eta$ to find an eigenvalue of $\hat{H}$ within a distance $\sim \eta$ from E, in which case $K_{l,m} \propto \eta^{-l-m}$. Extracting this singularity, we get

$$I_q(\mathbf{r}_0) = \frac{\Delta}{2\pi} i^{l-m}\frac{(l-1)!(m-1)!}{(l+m-2)!}\lim_{\eta\to+0}(2\eta)^{l+m-1}\langle K_{l,m}(\mathbf{r}_0,\eta)\rangle \ . \tag{4.7}$$

Note that at this stage we have for $q > 2$ the freedom in choosing l and m (constrained only by $l + m = q$). The final result will not, of course, depend on this choice.

We express now the product of Green's functions as an integral over the supervector field

$$\begin{aligned} K_{l,m}(\mathbf{r}_0,\eta) = \frac{i^{m-l}}{l!m!}\int & D\Phi\, D\Phi^\dagger[S_1(\mathbf{r}_0)S_1^*(\mathbf{r}_0)]^l[S_2(\mathbf{r}_0)S_2^*(\mathbf{r}_0)]^m \\ & \times\exp\left\{i\int \mathrm{d}^d\mathbf{r}\,\Phi^\dagger(\mathbf{r})L^{1/2}(E+i\eta\Lambda-\hat{H})L^{1/2}\Phi(\mathbf{r})\right\} \ . \end{aligned} \tag{4.8}$$

[25] For GSE this consideration gives $\mathcal{P}(y) = 4ye^{-2y}$. Note, however, that in terms of the electronic wave function this corresponds to defining y as the total (summed over the spin projections) intensity, $y = V(|\psi_\uparrow|^2 + |\psi_\downarrow|^2)$. For the distribution of the spin-projected intensity $y = 2V|\psi_\uparrow|^2$ the RMT result would have the same form (4.4) as for GUE. We do not consider the symplectic symmetry below; deviations from universality have in the symplectic symmetry class a form similar to the results for the systems of the unitary and the orthogonal symmetry.

[26] For $q = 0$ and $q = 1$ the moments are trivial, $I_0(\mathbf{r}_0) = 1$, $I_1(\mathbf{r}_0) = 1/V$.

Proceeding in the same way as in the case of the level correlation function (Sec. 2.1), we represent the r.h.s. of Eq. (4.8) in terms of a σ-model correlation function. In the case of the unitary symmetry the results reads

$$\begin{aligned} K_{l,m}(\mathbf{r}_0,\eta) &= (-i\pi\nu)^{l+m}\int \mathrm{D}Q \sum_j \left(\begin{array}{c} l \\ j \end{array}\right)\left(\begin{array}{c} m \\ j \end{array}\right) \\ &\times\; Q^{l-j}_{11,\mathrm{bb}}(\mathbf{r}_0)Q^{m-j}_{22,\mathrm{bb}}(\mathbf{r}_0)Q^{j}_{12,\mathrm{bb}}(\mathbf{r}_0)Q^{j}_{21,\mathrm{bb}}(\mathbf{r}_0)e^{-S[Q]}\;, \end{aligned} \tag{4.9}$$

where $S[Q]$ is the σ-model action,

$$S[Q] = -\int \mathrm{d}^d\mathbf{r}\,\mathrm{Str}\left[\frac{\pi\nu D}{4}(\nabla Q)^2 - \pi\nu\eta\Lambda Q\right]\;. \tag{4.10}$$

We remind that the two pairs of indices of the Q-matrices refer to the retarded-advanced (1, 2) and the boson-fermion (b, f) decomposition, respectively.

Since the preexponential factor in (4.9) depends on the Q-field at the point $\mathbf{r}_0$ only, it is convenient to introduce the function $Y(Q_0)$ as the result of integrating out all other degrees of freedom,

$$Y(Q_0) = \int_{Q(\mathbf{r}_0)=Q_0} \mathrm{D}Q(\mathbf{r})\exp\{-S[Q]\}\;. \tag{4.11}$$

With this definition, Eq.(3.4) takes the form of an integral over the single matrix Q_0,

$$\begin{aligned} K_{l,m}(\mathbf{r}_0,\eta) &= (-i\pi\nu)^{l+m}\int \mathrm{d}\mu(Q_0) \sum_j \left(\begin{array}{c} l \\ j \end{array}\right)\left(\begin{array}{c} m \\ j \end{array}\right) \\ &\times\; Q^{l-j}_{0;11,\mathrm{bb}}Q^{m-j}_{0;22,\mathrm{bb}}Q^{j}_{0;12,\mathrm{bb}}Q^{j}_{0;21,\mathrm{bb}}Y(Q_0)\;. \end{aligned} \tag{4.12}$$

For invariance reasons, the function $Y(Q_0)$ turns out to be dependent in the unitary symmetry case on the two eigenvalues $1 \le \lambda_1 < \infty$ and $-1 \le \lambda_2 \le 1$ only, when the parametrization (2.62), (2.63) for the matrix Q_0 is used. Moreover, in the limit $\eta \to 0$ (at a fixed value of the system volume, and thus of the level spacing Δ) only the dependence on λ_1 persists,

$$Y(Q_0) \equiv Y(\lambda_1,\lambda_2) \to Y_{\mathrm{a}}(2\pi\frac{\eta}{\Delta}\lambda_1)\;, \tag{4.13}$$

with relevant values of λ_1 being $\lambda_1 \sim \Delta/\eta \gg 1$. One can further show that in this asymptotic domain

$$Q_{11,\mathrm{bb}}Q_{22,\mathrm{bb}} \simeq Q_{12,\mathrm{bb}}Q_{21,\mathrm{bb}}\;,$$

so that all terms in $\sum_j$ in Eq. (4.12) are equal. Evaluating the integral over all coordinates but λ_1 in the parametrization (2.62), (2.63) of Q_0, we get

$$I_q(\mathbf{r}_0) = \frac{1}{V^q}q(q-1)\int \mathrm{d}z\, z^{q-2}Y_{\mathrm{a}}(z). \tag{4.14}$$

Consequently, the distribution function of the eigenfunction intensity is given by [69]

$$\mathcal{P}^{\mathrm{U}}(y) = \frac{\mathrm{d}^2}{\mathrm{d}y^2}Y_{\mathrm{a}}(y)\;. \tag{4.15}$$

In the case of the orthogonal symmetry, $Y(Q_0) \equiv Y(\lambda_1, \lambda_2, \lambda)$, where $1 \le \lambda_1, \lambda_2 < \infty$ and $-1 \le \lambda \le 1$. In the limit $\eta \to 0$, the relevant region of values is $\lambda_1 \gg \lambda_2, \lambda$, where

$$Y(Q_0) \to Y_{\rm a}(\pi \frac{\eta}{\Delta} \lambda_1) \ . \tag{4.16}$$

The distribution of eigenfunction intensities is expressed in this case through the function $Y_{\rm a}$ as follows [69]:

$$\begin{aligned} \mathcal{P}^{\rm O}(y) &= \frac{1}{\pi y^{1/2}} \int_{y/2}^{\infty} {\rm d}z (2z - y)^{-1/2} \frac{{\rm d}^2}{{\rm d}z^2} Y_{\rm a}(z) \\ &= \frac{2\sqrt{2}}{\pi y^{1/2}} \frac{{\rm d}^2}{{\rm d}y^2} \int_0^{\infty} \frac{{\rm d}z}{z^{1/2}} Y_{\rm a}(z + y/2) \ . \end{aligned} \tag{4.17}$$

In a metallic sample, typical configurations of the Q–field are nearly constant in space, so that the zero-mode approximation is expected to be a good starting point (see Sec. 3.2). It amounts to approximating the functional integral (4.9), (4.11) by an integral over a single supermatrix Q, yielding

$$Y_{\rm a}(z) \simeq e^{-z} \qquad ({\rm O}, {\rm U}) \ , \tag{4.18}$$

which reproduces [after substitution in (4.15), (4.17)] the RMT results (4.3) and (4.4). Therefore, like for the level correlations, the zero mode approximation yields the RMT results for the distribution of the eigenfunction amplitudes. To calculate deviations from RMT, one has to go beyond the zero-mode approximation and to evaluate the function $Y_{\rm a}(z)$ determined by Eqs. (4.11), (4.13) for a d-dimensional diffusive system. In the case of a quasi-1D geometry this can be done exactly via the transfer-matrix method [69, 70]. The result depends crucially on the ratio L/ξ, where L is the system size and $\xi = 2\pi\nu AD$ the localization length (A being the transverse cross-section of the sample). The two limiting cases correspond to the metallic regime $L \ll \xi$ (with the dimensionless conductance $g = \xi/L \gg 1$) and to the strong localization regime $L \gg \xi$. For higher d, the exact solution is not available any more, and one should rely on approximate methods. Corrections to the "main body" of the distribution can be found by treating the non-zero modes perturbatively [45], while the asymptotic "tail" can be found [71] via an instanton method [72]. For the quasi-1D geometry these approximate methods reproduced the results obtained earlier [69] from the exact solution.

4.3 Quasi-one-dimensional geometry

4.3.1 Exact solution of the σ-model

In the case of quasi-1D geometry an exact solution of the σ-model is possible due to the transfer-matrix method. The idea of the method, quite general for one-dimensional problems, is in reducing the functional integral of the type (4.11) to solution of a differential equation. This is completely analogous to constructing the Schrödinger equation from the quantum-mechanical Feynman path integral. In the present case, the role of the time is played by the coordinate along the wire, while the role of the particle coordinate is played by the supermatrix Q. In general, at a finite value of the frequency η in Eq. (4.10) (more precisely, η plays the role of imaginary frequency), the corresponding differential equation is too complicated and cannot be solved analytically. However, a simplification appearing in the limit $\eta \to 0$, when only the non-compact variable λ_1 survives, allows to find an analytical solution [32, 69, 70] of the 1D σ-model[27].

There are several different microscopic models which can be mapped onto the 1D supermatrix σ-model. First of all, this is a model of a particle in a random potential (discussed above) in the case of a quasi-1D sample geometry. Then one can neglect the transverse variation of the Q-field in the σ-model action, thus reducing it to the 1D form [73, 1]. Secondly, the random banded matrix (RBM) model (already mentioned in Sec. 2.5.4) has been mapped onto the 1D σ-model [32, 69, 70]. The RBM model is relevant to various problems in the field of quantum chaos. In particular, the evolution operator of a kicked rotor (paradigmatic model of a periodically driven quantum system) has a structure of a quasi-random banded matrix, which makes this system belong to the "quasi-1D universality class" (see Sec. 5.4 for a more detailed discussion). Finally, the Iida-Weidenmüller-Zuk random matrix model [74] of the transport in a disordered wire can be also reduced to the 1D σ-model, which allows to study analytically the wire conductance and its fluctuations [74, 75].

The result for the function $Y_{\rm a}(y)$ determining the distribution of the eigenfunction intensity reads (for the unitary symmetry)

$$Y_{\rm a}(y) = W^{(1)}(y\xi/L, \tau_+) W^{(1)}(y\xi/L, \tau_-) \ . \tag{4.19}$$

Here $\xi = 2\pi\nu DA$ is the localization length (where A is the wire cross-section), and $\tau_- = x/\xi$, $\tau_+ = (L-x)/\xi$, with $0 < x < L$ being the coordinate of the observation point $\mathbf{r}_0$ along the sample. For the orthogonal symmetry ξ is replaced by $\xi/2$. The function $W^{(1)}(z,\tau)$ satisfies the equation

$$\frac{\partial W^{(1)}(z,\tau)}{\partial \tau} = \left(z^2 \frac{\partial^2}{\partial z^2} - z \right) W^{(1)}(z,\tau) \tag{4.20}$$

and the boundary condition

$$W^{(1)}(z, 0) = 1 \ . \tag{4.21}$$

The solution to Eqs.(4.20), (4.21) can be found in terms of the expansion in eigenfunctions of the operator $z^2 \frac{\partial^2}{\partial z^2} - z$. The functions $2z^{1/2} K_{i\mu}(2z^{1/2})$, with $K_\nu(x)$ being the modified Bessel function (Macdonald function), form the proper basis for such an expansion [76], which is known as the Lebedev–Kontorovich expansion; the corresponding eigenvalues are $-(1+\mu^2)/4$. The result is

$$W^{(1)}(z,\tau) = 2z^{1/2} \left\{ K_1(2z^{1/2}) + \frac{2}{\pi} \int_0^\infty d\mu \frac{\mu}{1+\mu^2} \sinh \frac{\pi\mu}{2} K_{i\mu}(2z^{1/2}) e^{-\frac{1+\mu^2}{4}\tau} \right\} \ . \tag{4.22}$$

[27] Let us stress that we consider a sample with the hard-wall (not periodic) boundary conditions in the longitudinal direction, i.e a wire with two ends (not a ring).

The formulas (4.15), (4.17), (4.19), (4.22) give therefore the exact solution for the eigenfunction statistics for arbitrary value of the parameter $X = L/\xi$. The form of the distribution function $\mathcal{P}(y)$ is essentially different in the metallic regime $X \ll 1$ (in this case $X = 1/g$) and in the insulating one, $X \gg 1$. We discuss these two limiting cases below.

4.3.2 Short wire

In the case of a short wire, $X = 1/g \ll 1$, Eq. (4.15), (4.17), (4.19), (4.22) yield [69, 70, 77]

$$\mathcal{P}^{(\mathrm{U})}(y) = e^{-y}\left[1 + \tfrac{\alpha X}{6}(2 - 4y + y^2) + \ldots\right] \; ; \qquad y \lesssim X^{-1/2} \tag{4.23}$$

$$\mathcal{P}^{(\mathrm{O})}(y) = \tfrac{e^{-y/2}}{\sqrt{2\pi y}}\left[1 + \tfrac{\alpha X}{6}\left(\tfrac{3}{2} - 3y + \tfrac{y^2}{2}\right) + \ldots\right] \; ; \qquad y \lesssim X^{-1/2} \tag{4.24}$$

$$\mathcal{P}^{(\mathrm{U})}(y) = \exp\left\{-y + \tfrac{\alpha}{6}y^2 X + \ldots\right\} \; ; \qquad X^{-1/2} \lesssim y \lesssim X^{-1} \tag{4.25}$$

$$\mathcal{P}^{(\mathrm{O})}(y) = \tfrac{1}{\sqrt{2\pi y}}\exp\left\{\tfrac{1}{2}\left[-y + \tfrac{\alpha}{6}y^2 X + \ldots\right]\right\}; \qquad X^{-1/2} \lesssim y \lesssim X^{-1} \tag{4.26}$$

$$\mathcal{P}(y) \sim \exp\left[-2\beta\sqrt{y/X}\right] \; ; \qquad y \gtrsim X^{-1} \, . \tag{4.27}$$

Here the coefficient α is equal to $\alpha = 2[1 - 3x(L-x)/L^2]$. We see that there exist three different regimes of the behavior of the distribution function. For not too large amplitudes y, Eqs.(4.23), (4.24) are just the RMT results with relatively small corrections. In the intermediate range (4.25), (4.26) the correction *in the exponent* is small compared to the leading term but much larger than unity, so that $\mathcal{P}(y) \gg \mathcal{P}_{RMT}(y)$ though $\ln \mathcal{P}(y) \simeq \ln \mathcal{P}_{RMT}(y)$. Finally, in the large-amplitude region, (4.27), the distribution function $\mathcal{P}(y)$ differs completely from the RMT prediction.[28]

4.3.3 Long wire

In the limit of a long sample, $X = L/\xi \gg 1$, Eqs. (4.15), (4.17), (4.19), (4.22) reduce to

$$\mathcal{P}^{(\mathrm{U})}(u) \simeq \frac{8\xi^2 A}{L}\left[K_1^2(2\sqrt{uA\xi}) + K_0^2(2\sqrt{uA\xi})\right] \, , \tag{4.28}$$

$$\mathcal{P}^{(\mathrm{O})}(u) \simeq \frac{2\xi^2 A}{L}\frac{K_1(2\sqrt{uA\xi})}{\sqrt{uA\xi}} \, , \tag{4.29}$$

with $\xi = 2\pi\nu A D$ as before, and $u = |\psi^2(\mathbf{r}_0)|$. Note that in this case it is not appropriate to use $y = uV$ as a variable, since typical intensity of a localized wave function is $u \sim 1/A\xi$ in contrast to $u \sim 1/V$ for a delocalized one. The asymptotic behavior of Eqs. (4.28), (4.29) at $u \gg 1/A\xi$ has precisely the same form,

$$\mathcal{P}(u) \sim \exp(-2\beta\sqrt{uA\xi}) \, , \tag{4.30}$$

[28] Note that Eq. (4.27) is not valid when the observation point is located close to the sample boundary, in which case the exponent of (4.27) becomes smaller by a factor of 2, see [77].

as in the region of very large amplitude in the metallic sample, Eq. (4.27). On this basis, it was conjectured in [70] that the asymptotic behavior (4.27) is controlled by the probability to have a quasi-localized eigenstate with an effective spatial extent much less than ξ ("anomalously localized state"). This conjecture was proven rigorously in [77] where the shape of the anomalously localized state (ALS) responsible for the large-u asymptotics was calculated via the transfer-matrix method.

The transfer-matrix method allows to study, in the quasi-1D geometry, not only statistics of the eigenfunction amplitude in a given point, but also correlation functions of amplitudes in different points. Relegating a more extensive discussion of this issue to Sec. 4.5, we mention here a distribution function which characterizes fluctuations in the rate of exponential decay of eigenfunctions (Lyapunov exponent). Specifically, let us consider the product of the eigenfunction intensity in the two points close to the opposite edges of the sample $x_1 \to 0$, $x_2 \to L$,

$$v = (2\pi\nu DA^2)^2 |\psi_\alpha^2(\mathbf{r}_1)\psi_\alpha^2(\mathbf{r}_2)| \ . \tag{4.31}$$

The corresponding distribution function is found to be [78, 70]

$$\begin{aligned} \mathcal{P}(-\ln v) &= F^{(\beta)}[-(\beta \ln v)/2X] \left(\frac{\beta}{8\pi X}\right)^{1/2} \exp\left\{-\frac{\beta}{8X}\left(\frac{2X}{\beta} + \ln v\right)^2\right\} \\ F^{\mathrm{U}}(u) &= u\frac{\Gamma^2[(3-u)/2]}{\Gamma(u)} \ , \qquad F^{\mathrm{O}}(u) = \frac{u\Gamma^2[(1-u)/2]}{\pi\Gamma(u)} \end{aligned} \tag{4.32}$$

Therefore, $\ln v$ is asymptotically distributed according to the Gaussian law with mean value $\langle -\ln v\rangle = (2/\beta)X = L/\beta\pi\nu AD$ and variance $\mathrm{var}(-\ln v) = 2\langle -\ln v\rangle$. The same log-normal distribution is found for the conductance and for transmission coefficients of a quasi-1D sample from the Dorokhov-Mello-Pereyra-Kumar formalism [79, 80].

Note that the formula (4.32) is valid in the region of $v \ll 1$ (i.e. negative $\ln v$) only, which contains almost all normalization of the distribution function. In the region of still higher values of v the log-normal form of $\mathcal{P}(v)$ changes into the much faster stretched-exponential fall-off $\propto \exp\{-2\sqrt{2}\beta v^{1/4}\}$, as can be easily found from the exact solution given in [78, 70]. The decay rate of all the moments $\langle v^k\rangle$, $k \geq 1/2$, is four times less than $\langle -\ln v\rangle$ and does not depend on k: $\langle v^k\rangle \propto e^{-X/2\beta}$. This is because the moments $\langle v^k\rangle$, $k \geq 1/2$ are determined by the probability to find an "anomalously delocalized state" with $v \sim 1$.

4.4 Metallic regime (arbitrary d).

For arbitrary dimensionality d, deviations from the RMT distribution $\mathcal{P}(y)$ for not too large y can be calculated [45] via the method of Ref. [44] described in Section 3.3.2. Applying this method to the moments (4.7), (4.9), one gets

$$I_q(\mathbf{r}_0) = \frac{q!}{V^q}\left[1 + \frac{\kappa}{2}q(q-1) + \ldots\right] \qquad (\mathrm{U}) \ , \tag{4.33}$$

where

$$\kappa = \Pi(\mathbf{r}_0, \mathbf{r}_0) = \sum_{\mu \neq 0} \frac{\phi_\mu^2(\mathbf{r}_0)}{\pi \nu \epsilon_\mu} \ . \tag{4.34}$$

Correspondingly, the correction to the distribution function reads

$$\mathcal{P}^{(\mathrm{U})}(y) = e^{-y} \left[1 + \frac{\kappa}{2}(2 - 4y + y^2) + \ldots \right] \ . \tag{4.35}$$

Similar results are obtained for the orthogonal symmetry class,

$$I_q(\mathbf{r}_0) = \frac{(2q-1)!!}{V^q} [1 + \kappa q(q-1) + \ldots] \qquad (\mathrm{O}) \ , \tag{4.36}$$

$$\mathcal{P}^{(\mathrm{O})}(y) = \frac{e^{-y/2}}{\sqrt{2\pi y}} \left[1 + \frac{\kappa}{2} \left(\frac{3}{2} - 3y + \frac{y^2}{2} \right) + \ldots \right] \ . \tag{4.37}$$

Numerical studies of the statistics of eigenfunction amplitudes in the weak-localization regime have been performed in Ref. [81] for the 2D and in Ref. [82] for the 3D case. The found deviations from RMT are well described by the above theoretical results. Experimentally, statistical properties of the eigenfunction intensity have been studied for microwaves in a disordered cavity [67]. For a weak disorder the found deviations are in good agreement with (4.37) as well.

As we see from the above formulas, the magnitude of the corrections is governed by the parameter $\kappa = \Pi(\mathbf{r}_0, \mathbf{r}_0)$ (the one-diffuson loop in the diagrammatic language). In the quasi-one-dimensional case (with hard wall boundary conditions in the longitudinal direction), it is equal to

$$\kappa \equiv \Pi(\mathbf{r}_0, \mathbf{r}_0) = \frac{2}{g} \left[\frac{1}{3} - \frac{x}{L} \left(1 - \frac{x}{L} \right) \right] \ , \qquad 0 \leq x \leq L \ , \tag{4.38}$$

where x is the longitudinal coordinate of the observation point $\mathbf{r}_0$, so that Eqs. (3.12), (3.13) agree with the results (4.23), (4.24) obtained from the exact solution. For the periodic boundary conditions in the longitudinal direction (a ring) we have $\kappa = 1/6g$. In the case of 2D geometry,

$$\Pi(\mathbf{r}, \mathbf{r}) = \frac{1}{\pi g} \ln \frac{L}{l} \ , \tag{4.39}$$

with $g = 2\pi\nu D$. Finally, in the 3D case the sum over the momenta $\Pi(\mathbf{r}, \mathbf{r}) = (\pi\nu V)^{-1} \sum_{\mathbf{q}} (D\mathbf{q}^2)^{-1}$ diverges linearly at large q. The diffusion approximation is valid up to $q \sim l^{-1}$; the corresponding cutoff gives $\Pi(\mathbf{r}, \mathbf{r}) \sim 1/2\pi\nu D l = g^{-1}(L/l)$. This divergency indicates that more accurate evaluation of $\Pi(\mathbf{r}, \mathbf{r})$ requires taking into account also the contribution of the ballistic region ($q > l^{-1}$) which depends on microscopic details of the random potential; see [4] for details.

The formulas (4.35), (4.37) are valid in the region of not too large amplitudes, where the perturbative correction is smaller than the RMT term, i.e. at $y \ll \kappa^{-1/2}$. In the region of large amplitudes, $y > \kappa^{-1/2}$ the distribution function was found by Fal'ko and Efetov [71] who applied to Eqs. (4.15), (4.17) the saddle-point method suggested by Muzykantskii and Khmelnitskii [72]. We relegate the discussion of the method to Sec. 4.6 and only present the results here:

$$\mathcal{P}(y) \simeq \exp\left\{\frac{\beta}{2}(-y + \frac{\kappa}{2}y^2 + \ldots)\right\} \times \left\{ \begin{array}{ll} 1 & (U) \\ \frac{1}{\sqrt{2\pi y}} & (O) \end{array} \right. , \ \kappa^{-1/2} \lesssim y \lesssim \kappa^{-1}, \quad (4.40)$$

$$\mathcal{P}(y) \sim \exp\left\{-\frac{\beta}{4\kappa}\ln^d(\kappa y)\right\} , \qquad y \gtrsim \kappa^{-1}. \quad (4.41)$$

Again, as in the quasi-one-dimensional case, there is an intermediate range where a correction in the exponent is large compared to unity, but small compared to the leading RMT term [Eq. (4.40)] and a far asymptotic region (4.41), where the decay of $\mathcal{P}(y)$ is much slower than in RMT. Similarly to the quasi-1D result (4.27), the asymptotic behavior (4.41) is determined by anomalously localized states (see [77, 4] for a review).

4.4.1 2D: Weak multifractality of eigenfunctions

Since $d = 2$ is the lower critical dimension for the Anderson localization problem, metallic 2D samples (with $g \gg 1$) share many common properties with systems at the critical point of the metal-insulator transition. Although the localization length ξ in 2D is not infinite (as for truly critical systems), it is exponentially large, and the criticality takes place in the very broad range of the system size $L \ll \xi$.

The criticality of eigenfunctions shows up via their multifractality. The multifractal structures first introduced by Mandelbrot [83] are characterized by an infinite set of critical exponents describing the scaling of the moments of a distribution of some quantity. Since then, this feature has been observed in various objects, such as the energy-dissipating set in turbulence [84, 85, 86], strange attractors in chaotic dynamical systems [87, 88, 89, 90], and the growth probability distribution in diffusion-limited aggregation [91, 92, 93]; see Ref. [94] for a review.

The fact that an eigenfunction at the mobility edge has the multifractal structure was emphasized in [95] on the basis of renormalization group calculations done by Wegner several years earlier [96]. For this problem, the probability distribution is just the eigenfunction intensity $|\psi^2(\mathbf{r})|$ and the corresponding moments are the inverse participation ratios (IPR's),

$$P_q = \int d^d\mathbf{r}|\psi^{2q}(\mathbf{r})| \, . \quad (4.42)$$

The multifractality is characterized by the anomalous scaling of P_q with the system size L,

$$P_q \propto L^{-D_q(q-1)} \equiv L^{-\tau(q)}, \quad (4.43)$$

with D_q different from the spatial dimensionality d and dependent on q. Equivalently, the eigenfunctions are characterized by the singularity spectrum $f(\alpha)$ describing the measure $L^{f(\alpha)}$ of the set of those points $\mathbf{r}$ where the eigenfunction takes the value $|\psi^2(\mathbf{r})| \propto L^{-\alpha}$. The two sets of exponents $\tau(q)$ and $f(\alpha)$ are related via the Legendre transformation,

$$\tau(q) = q\alpha - f(\alpha) \; ; \quad f'(\alpha) = q \; ; \quad \tau'(q) = \alpha \, . \quad (4.44)$$

By now, the multifractality of critical wave functions is confirmed by numerical simulations [97, 98, 99]; for more recent reviews see Refs. [100, 101, 102].

We turn now to the wave function statistics in 2D. We note first that the formulas (4.33), (4.36) for the IPR's with $q \lesssim \kappa^{-1/2}$ can be rewritten in the 2D case (with (4.39) taken into account) as

$$\frac{\langle P_q \rangle}{P_q^{\mathrm{RMT}}} \simeq \left(\frac{L}{l}\right)^{\frac{1}{\beta\pi g}q(q-1)} , \tag{4.45}$$

where P_q^{RMT} is the RMT value of P_q equal to $q!L^{-2(q-1)}$ for GUE and $(2q-1)!!L^{-2(q-1)}$ for GOE. We see that (4.45) has precisely the form (4.43) with

$$D_q = 2 - \frac{q}{\beta\pi g} \tag{4.46}$$

As was found in [71], the eigenfunction amplitude distribution (4.40), (4.41) leads to the same result (4.46) for all $q \ll 2\beta\pi g$. Since the deviation of D_q from the normal dimension 2 is proportional to the small parameter $1/\pi g$, it can be termed "weak multifractality" (in analogy with weak localization). The result (4.46) was in fact obtained for the first time by Wegner [96] via the renormalization group calculations.

The limits of validity of Eq. (4.46) are not unambiguous and should be commented here. The singularity spectrum $f(\alpha)$ corresponding to (4.46) has the form

$$f(\alpha) = 2 - \frac{\beta\pi g}{4}\left(2 + \frac{1}{\beta\pi g} - \alpha\right)^2 , \tag{4.47}$$

so that $f(\alpha_\pm = 0)$ for

$$\alpha_\pm = 2\left[1 \pm \frac{1}{(2\beta\pi g)^{1/2}}\right]^2 . \tag{4.48}$$

If α lies outside the interval (α_-, α_+), the corresponding $f(\alpha) < 0$, which means that the most likely the singularity α will not be found for a given eigenfunction. However, if one considers the *average* $\langle P_q \rangle$ over a sufficiently large ensemble of eigenfunctions, a negative value of $f(\alpha)$ makes sense (see a related discussion in [103]). This is the definition which was assumed in [71] where Eq. (4.46) was obtained for all positive $q \ll 2\beta\pi g$.

In contrast, if one studies a *typical* value of P_q, the regions $\alpha > \alpha_+$ and $\alpha < \alpha_-$ will not contribute. In this case, Eq. (4.46) is valid only within the interval $q_- \leq q \leq q_+$ with $q_\pm = \pm(2\beta\pi g)^{1/2}$; outside this region one finds [104, 105]

$$\tau(q) \equiv D_q(q-1) = \begin{cases} q\alpha_- , & q > q_+ \\ q\alpha_+ , & q < q_- . \end{cases} \tag{4.49}$$

Therefore, within this definition the multifractal dimensions D_q saturate at the values α_+ and α_- for $q \to +\infty$ and $q \to -\infty$ respectively. This is in agreement with results of numerical simulations [97, 98, 99, 100, 101].

4.5 Spatial correlations of eigenfunction amplitudes.

Correlations of amplitudes of an eigenfunction in different spatial points are characterized by a set of correlation functions (we consider, as usual, the unitary symmetry for definiteness)

$$\Delta\langle\sum_i \psi_i^*(\mathbf{r}_1)\psi_i(\mathbf{r}_1')\ldots\psi_i^*(\mathbf{r}_q)\psi_i(\mathbf{r}_q')\delta(E-E_i)\rangle \equiv \langle\psi^*(\mathbf{r}_1)\psi(\mathbf{r}_1')\ldots\psi^*(\mathbf{r}_q)\psi(\mathbf{r}_q')\rangle\ . \quad (4.50)$$

Using the supersymmetry formalism and performing the same transformations that have led us to Eq. (4.9), we get

$$\begin{aligned}&\langle\psi^*(\mathbf{r}_1)\psi(\mathbf{r}_1')\ldots\psi^*(\mathbf{r}_q)\psi(\mathbf{r}_q')\rangle\\ &= -\frac{1}{2V(q-1)!}\lim_{\eta\to 0}(2\pi\nu\eta)^{q-1}\int \mathrm{D}Q\sum_\sigma\prod_{i=1}^{q}\frac{1}{\pi\nu}g_{p_ip_{\sigma(i)},\mathrm{bb}}(\mathbf{r}_i,\mathbf{r}_{\sigma(i)}')e^{-S[Q]}\ ,\end{aligned} \quad (4.51)$$

where the summation goes over all transpositions σ of the set $\{1,2,\ldots,q\}$, p_i is equal to 1 for $i=1,\ldots,q-1$ and to 2 for $i=q$, and g is the Green's function in the field $Q(\mathbf{r})$,

$$g=\left(E-\frac{\hat{\mathbf{p}}^2}{2m}+i\frac{Q}{2\tau}\right)^{-1}\ . \quad (4.52)$$

Taking into account that the field $Q(\mathbf{r})$ varies only weakly on the scale of the mean free path l yields[29]

$$g(\mathbf{r}_1,\mathbf{r}_2)\simeq \mathrm{Re}\,G_\mathrm{A}(\mathbf{r}_1-\mathbf{r}_2)-i\,\mathrm{Im}\,G_\mathrm{A}(\mathbf{r}_1-\mathbf{r}_2)Q(\mathbf{r}_1)\ , \quad (4.53)$$

$$G_\mathrm{A}(\mathbf{r})=\int\frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d}\frac{e^{i\mathbf{pr}}}{E-\mathbf{p}^2/2m-i/2\tau}\ . \quad (4.54)$$

For $|\mathbf{r}_1-\mathbf{r}_2|\gg l$ the Green's function $g(\mathbf{r}_1,\mathbf{r}_2)$ vanishes exponentially, in view of $G_\mathrm{A}(\mathbf{r})\propto e^{-r/2l}$. Since in the limit $\eta\to 0$ the characteristic magnitude of Q is $\propto \Delta/\eta\gg 1$ [see the text around Eq. (4.13)], the real part $\mathrm{Re}\,G_\mathrm{A}(\mathbf{r})$ can be neglected in all the Green's function factors in (4.51), and only the products of the imaginary parts survive. The imaginary part $\mathrm{Im}G_\mathrm{A}(\mathbf{r})$ is given explicitly by

$$\frac{\mathrm{Im}G_\mathrm{A}(\mathbf{r})}{\pi\nu}\equiv f_\mathrm{F}(r)\simeq e^{-r/2l}\times\left\{\begin{array}{ll} J_0(p_\mathrm{F}r)\ , & \text{2D}\\ \frac{\sin(p_\mathrm{F}r)}{p_\mathrm{F}r}\ , & \text{3D}\end{array}\right. \quad (4.55)$$

Let us note that $f_\mathrm{F}(r)$ is determined by microscopic (short-scale) dimensionality of the sample rather than by its global geometry. In particular, a quasi-1D sample may be microscopically of 2D (strip) or 3D (wire) nature.

[29] Since Q is a slowly varying field, the argument of Q in the second term of (4.53) can be chosen to be either $\mathbf{r}_1$ or $\mathbf{r}_2$, or $(\mathbf{r}_1+\mathbf{r}_2)/2$.

4.5.1 Zero-mode approximation.

In the zero-mode approximation, Eq. (4.51) reduces to

$$V^q\langle\psi^*(\mathbf{r}_1)\psi(\mathbf{r}'_1)\ldots\psi^*(\mathbf{r}_q)\psi(\mathbf{r}'_q)\rangle = \sum_\sigma\prod_{i=1}^q f_{\rm F}(\mathbf{r}_i,\mathbf{r}'_{\sigma(i)})\ . \tag{4.56}$$

Equation (4.56) implies that the wave function $\psi(\mathbf{r})$ has a global Gaussian distribution,

$$\mathcal{P}\{\psi(\mathbf{r})\} \propto \exp\left\{-\frac{\beta}{2}\int {\rm d}^d\mathbf{r}{\rm d}^d\mathbf{r}'\psi^*(\mathbf{r})K(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')\right\} \tag{4.57}$$

determined by the correlation function

$$V\langle\psi^*(\mathbf{r})\psi(\mathbf{r}')\rangle = f_{\rm F}(|\mathbf{r}-\mathbf{r}'|)\ , \tag{4.58}$$

the kernel $K(\mathbf{r},\mathbf{r}')$ being the operator inverse of $V^{-1}f_{\rm F}(|\mathbf{r}-\mathbf{r}'|)$. An analogous consideration for the orthogonal symmetry class (when $\psi(\mathbf{r})$ is real) leads, in the zero-mode approximation, to the same conclusion. The result (4.57), (4.58) was first obtained by Berry [106] from the conjecture that a wave function in a classically chaotic system is given by a random superposition of plane waves.[30]

As has been explained above, quite generally there are corrections to the zero-mode approximation induced by the diffusion modes. They change the eigenfunction correlations qualitatively by inducing correlations on long scales $r \gg l$ (which are exponentially small in the zero-mode approximation). Similarly to our discussion of the wave function fluctuations (Sec. 4.3, 4.4), we proceed by first presenting results of the exact solution in the quasi-1D case and then consider the metallic regime for an arbitrary dimensionality.

4.5.2 Quasi-1D geometry.

In the case of the quasi-1D geometry of the sample the method described in Sec. 4.3 allows to calculate analytically all the multipoint correlation functions (4.50); the results are presented in terms of multiple integrals of the type (4.22), see reviews [70, 4]. Without going into technical details, we quote here some important conclusions concerning the global statistics of eigenfunctions. A wave function $\psi(\mathbf{r})$ can be represented as a product

$$\psi(\mathbf{r}) = \Phi(\mathbf{r})\Psi(x)\ , \tag{4.59}$$

where x is the coordinate of the point $\mathbf{r}$ along the sample. Here $\Phi(\mathbf{r})$ is a quickly fluctuating in space function, which has the Gaussian statistics (4.57), (4.58) obtained

[30]Historically, the equivalence of the Berry's conjecture and the zero-mode supersymmetry calculation has been established in a somewhat convoluted and less general way. Prigodin *et al* [107] calculated, in the zero-mode approximation for the σ-model, the joint distribution function $\mathcal{P}(u,v)$ of the wave function amplitudes in two different spatial points, $u=|\psi^2(\mathbf{r})|$, $v=|\psi^2(\mathbf{r}')|$. Srednicki then demonstrated [108] that the same results for $\mathcal{P}(u,v)$ are obtained from the Berry's conjecture of Gaussian statistics of $\psi(\mathbf{r})$. We have demonstrated this equivalence above in a more transparent and general form.

above from the 0D σ-model. The (real) function $\Psi(x)$ determines, in contrast, a smooth envelope of the wave function. Its fluctuations are long-range correlated and are described by the probability density

$$\mathcal{P}\{\theta(x)\} \propto e^{-L/2\beta\xi} e^{-[\theta(0)+\theta(L)]/2} \exp\left\{-\frac{\beta}{8}\xi \int_0^L \mathrm{d}x \left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2\right\} \delta\left(L^{-1}\int \mathrm{d}x\, e^{\theta} - 1\right) , \tag{4.60}$$

where $\theta(x) = \ln \Psi^2(x)$.

The physics of these results is as follows. The short-range fluctuations of the wave function (described by the function $\Phi(\mathbf{r})$) originate, as explained above, from the superposition of plane waves with random amplitudes and phases leading to the Gaussian fluctuations of eigenfunctions with the correlation function (4.55). The second factor $\Psi(x)$ in the decomposition (4.59) describes the smooth envelope of the eigenfunction (changing on a scale $\gg l$), whose statistics given by (4.60) is determined by diffusion and localization effects. This factor is responsible for the long-range correlations (*i.e.* those on scales $\gg l$) of the wave amplitude. In the metallic regime, $\xi/L = g \gg 1$, $\Psi(x)$ fluctuates relatively weakly around unity and the above long-range correlations of $\psi(\mathbf{r})$ are parametrically small (see Sec. 4.5.3). In the opposite regime of a long wire, $L/\xi \gg 1$, the strong localization manifests itself in extremely strong spatial correlations of eigenfunctions.

Finally, we compare the eigenfunction statistics in the quasi-1D case with that in a strictly 1D disordered system. In the latter case, the eigenfunction can be written as

$$\psi_{1\mathrm{D}}(x) = \sqrt{\frac{2}{L}}\cos(kx+\delta)\Psi(x) , \tag{4.61}$$

where $\Psi(x)$ is again a smooth envelope function. The local statistics of $\psi_{1\mathrm{D}}(x)$ (i.e. the moments (4.5)) was studied in [109], while the global statistics (the correlation functions of the type (4.50)) in [110]. Comparison of the results for the quasi-1D and 1D systems shows that the statistics of the smooth envelopes Ψ is exactly the same in the two cases, for a given value of the ratio of the system length L to the localization length (equal to $\beta\pi\nu AD$ in quasi-1D and to the mean free path l in 1D). The equivalence of the statistics of the eigenfunction envelopes implies, in particular, that the distribution of the inverse participation ratios P_q [see Eq. (4.42)] is identical in the 1D and quasi-1D cases. This is confirmed by explicit calculations of the distribution function of P_2, see Refs. [111, 112] (1D) and Refs. [113, 70] (quasi-1D).

4.5.3 Metallic regime (arbitrary d).

Correlations of eigenfunction amplitudes in the regime of a good conductor can be again studied via the method of Ref. [44] described in Sec. 3.3.2; see Refs. [45, 114]. The result has the form of the expansion in powers of the diffusion propagator Π. In particular, for the simplest correlation function showing long-range correlations we find

(up to the linear-in-Π terms)

$$V^2\langle|\psi(\mathbf{r}_1)|^2|\psi(\mathbf{r}_2)|^2\rangle = 1+\frac{2}{\beta}f_{\rm F}^2(|\mathbf{r}_1-\mathbf{r}_2|)\left[1+\frac{2}{\beta}\Pi(\mathbf{r}_1,\mathbf{r}_1)\right]+\frac{2}{\beta}\Pi(\mathbf{r}_1,\mathbf{r}_2)\ . \qquad (4.62)$$

The last term on the r.h.s. of (4.62) describes the long-range correlations between $|\psi(\mathbf{r}_1)|^2$ and $|\psi(\mathbf{r}_2)|^2$ induced by diffusion (or, in other words, by classical dynamics).

In a similar way one can calculate also higher order correlation functions of the eigenfunction amplitudes. In particular, the correlation function $\langle|\psi^4(\mathbf{r}_1)||\psi^4(\mathbf{r}_2)|\rangle$ determines fluctuations of the inverse participation ratio P_2, the result for the relative variance of $\delta(P_2)=\mathrm{var}(P_2)/\langle P_2\rangle^2$ being [45]

$$\delta(P_2)=\frac{8}{\beta^2}\int\frac{\mathrm{d}^d\mathbf{r}\mathrm{d}^d\mathbf{r}'}{V^2}\Pi^2(\mathbf{r},\mathbf{r}')=\frac{32a_d}{\beta^2g^2}, \qquad (4.63)$$

with the numerical coefficient a_d defined in Sec. 3.3.2 (see Eqs. (3.46), (3.47)). The fluctuations (4.63) have the same relative magnitude ($\sim 1/g$) as the famous universal conductance fluctuations. Note also that extrapolating Eq.(4.63) to the Anderson transition point, where $g\sim 1$, we find $\delta(P_2)\sim 1$, so that the magnitude of IPR fluctuations is of the order of its mean value [which is, in turn, much larger than in the metallic regime; see Eq. (4.43)].

Equation (4.63) can be generalized onto higher IPR's P_q with $q>2$,

$$\frac{\mathrm{var}(P_q)}{\langle P_q\rangle^2}\simeq\frac{2}{\beta^2}q^2(q-1)^2\int\frac{\mathrm{d}^d\mathbf{r}\mathrm{d}^d\mathbf{r}'}{V^2}\Pi^2(\mathbf{r},\mathbf{r}')=\frac{8q^2(q-1)^2a_d}{\beta^2g^2}, \qquad (4.64)$$

so that the relative magnitude of the fluctuations of P_q is $\sim q(q-1)/g$. Furthermore, the higher irreducible moments (cumulants) $\langle\langle P_q^n\rangle\rangle$, $n=2,3,\ldots$, have the form

$$\begin{aligned}\frac{\langle\langle P_q^n\rangle\rangle}{\langle P_q\rangle^n} &= \frac{(n-1)!}{2}\left[\frac{2}{\beta}q(q-1)\right]^n\int\frac{\mathrm{d}^d\mathbf{r}_1\ldots\mathrm{d}^d\mathbf{r}_n}{V^n}\Pi(\mathbf{r}_1,\mathbf{r}_2)\ldots\Pi(\mathbf{r}_n,\mathbf{r}_1)\\ &= \frac{(n-1)!}{2}\mathrm{Tr}\left[\frac{2}{\beta}q(q-1)\Pi\right]^n\\ &= \frac{(n-1)!}{2}\left[\frac{2}{\beta}q(q-1)\right]^n\sum_{\mu\neq0}\left(\frac{\Delta}{\pi\epsilon_\mu}\right)^n\ , \end{aligned}\qquad (4.65)$$

where Π is the integral operator with the kernel $\Pi(\mathbf{r},\mathbf{r}')/V$. This is valid provided $q^2n\ll 2\beta\pi g$. Prigodin and Altshuler [115] obtained Eq. (4.65) starting from the assumption that the eigenfunction statistics is described by the Liouville theory. According to (4.65), the "central body" of the distribution function $\mathcal{P}(P_q)$ of the IPR P_q (with $q^2/\beta\pi g\ll 1$) is determined by the spectrum of eigenvalues ϵ_μ of the diffusion operator $-D\nabla^2$. For a more detailed study of this distribution function see [115, 4].

The perturbative calculations show that the cumulants of the IPR's are correctly reproduced (in the leading order in $1/g$) if one assumes [115] that the statistics of the eigenfunction envelopes $|\psi^2(\mathbf{r})|_{\rm smooth}=e^{\theta(\mathbf{r})}$ is governed by the Liouville theory (see *e.g.* [116, 117] and references therein) defined by the functional integral

$$\int\mathrm{D}\theta\,\delta\left(\int\frac{\mathrm{d}^d\mathbf{r}}{V}e^\theta-1\right)\exp\left\{-\frac{\beta\pi\nu D}{4}\int\mathrm{d}^d\mathbf{r}(\nabla\theta)^2\right\}\ldots \qquad (4.66)$$

The "tails" of the IPR distribution function governed by rare realizations of disorder are described by saddle-point solutions which can also be obtained from the Liouville theory description (4.66), see [4]. The multifractal dimensions (4.46) in 2D can be reproduced by starting from the Liouville theory as well [116, 117]. It should be stressed, however, that this agreement between the supermatrix σ-model governing the eigenfunctions statistics and the Liouville theory is not exact, but only holds in the leading order in $1/g$. In particular, the Liouville theory does not describe the wave function localization by weak disorder in 2D.

Up to now, we have considered correlations between amplitudes of one and the same eigenfunction at different spatial points. One can also study correlations of different eigenfunctions, see [114]. Understanding of both types of correlations is important for evaluation of fluctuations of matrix elements (*e.g.* those of Coulomb interaction) computed on eigenfunctions ψ_k of the one-particle Hamiltonian in a random potential. Such a problem naturally arises when one investigates the effect of interaction onto statistical properties of excitations in a mesoscopic sample (see Refs. [62, 64, 63]).

4.6 Anomalously localized states and long-time relaxation.

In this subsection we discuss one more method that can be used within the supermatrix σ-model formalism to investigate statistical properties of eigenfunctions. This is the instanton method introduced by Muzykantskii and Khmelnitskii [72] in order to calculate the long-time dispersion of the average conductance $G(t)$. Soon after the paper [72] appeared, it was realized that the method allows one to study the asymptotic behavior of distribution functions of different quantities, including relaxation times, eigenfunction intensities, local density of states, inverse participation ratio, and level curvatures. These asymptotics are determined by rare realization of disorder producing the states which show much stronger localization features than typical states in the system – anomalously localized states already mentioned in Sec. 4.3.3 and 4.4. For a review of the obtained results and relevant references the reader is referred to [77, 4]. In fact, we have already quoted the results obtained in this way by Fal'ko and Efetov [71] for the "tail" of the eigenfunction statistics, see Eqs. (4.40), (4.41). Here we wish to present the ideas of the method by discussing the original problem considered by Muzykantskii and Khmelnitskii.

Let us consider (following Refs. [72, 118]) the asymptotic (long-time) behavior of the relaxation processes in an open disordered conductor. One possible formulation of the problem is to consider the time-dependence of the average conductance $G(t)$ defined by the non-local (in time) current-voltage relation

$$I(t) = \int_{-\infty}^{t} \mathrm{d}t' G(t-t')V(t') \tag{4.67}$$

Alternatively, one can study the decay law, i.e. the survival probability $P_{\mathrm{s}}(t)$ for a particle injected into the sample at $t=0$ to be found there after a time t. Classically, $P_{\mathrm{s}}(t)$ decays according to the exponential law, $P_{\mathrm{s}}(t) \sim e^{-t/t_{\mathrm{D}}}$, where t_{D}^{-1} is the lowest eigenvalue of the diffuson operator $-D\nabla^2$ with the proper boundary conditions. The time $t_{\rm D}$ has the meaning of the time of diffusion through the sample, and $t_{\rm D}^{-1}$ is of the order of the Thouless energy. The same exponential decay holds for the conductance $G(t)$, where it is induced by the weak-localization correction. The quantities of interest can be expressed in the form of the σ-model correlation function

$$G(t),\ P_{\rm s}(t) \sim \int \frac{{\rm d}\omega}{2\pi} e^{-i\omega t} \int {\rm D}Q({\bf r}) A\{Q\} e^{-S[Q]}\ , \tag{4.68}$$

where $S[Q]$ is given by Eq. (3.16). The preexponential factor $A\{Q\}$ depends on the specific formulation of the problem but is not important for the leading exponential behavior studied here.

The main idea of the method is that the asymptotic behavior is determined by a non-homogeneous (*i.e.* ${\bf r}$-dependent) stationary point of the action $S[Q]$ (instanton). Such a stationary point is found by varying the exponent in Eq. (4.68) with respect to Q and ω, which yields the equations [72]

$$2D\nabla(Q\nabla Q) + i\omega[\Lambda, Q] = 0 \tag{4.69}$$

$$\frac{\pi\nu}{2}\int {\rm d}^d{\bf r}\, {\rm Str}(\Lambda Q) = t \tag{4.70}$$

[We assume unitary symmetry; in the orthogonal symmetry case the calculation is applicable with minor modifications.] It remains

i) to find a solution Q_ω of Eq.(4.69) (which will depend on ω);

ii) to substitute it into the self-consistency equation (4.70) and thus to fix ω as a function of t;

iii) to substitute the found solution Q_t into Eq.(4.68),

$$P_{\rm s}(t) \sim \exp\left\{ \frac{\pi\nu D}{4} {\rm Str} \int (\nabla Q_t)^2 \right\}\ . \tag{4.71}$$

Note that Eq. (4.69) is to be supplemented by the boundary conditions $Q = \Lambda$ at the open part of the boundary (*i.e.* boundary with an ideal metal) and (3.28) at the insulating part of the boundary (if it exists).

It is not difficult to show [72] that the solution of Eq. (4.69) has in the standard parametrization (2.62), (2.63) the only non-trivial variable – bosonic eigenvalue $\lambda_1 = \cosh\theta_1$, all other coordinates being equal to zero. As a result, Eq. (4.69) reduces to an equation for $\theta_1({\bf r})$ (we drop the subscript "1" below):

$$\nabla^2\theta + \frac{i\omega}{D}\sinh\theta = 0\ , \tag{4.72}$$

the self-consistency condition (4.70) takes the form

$$\pi\nu\int {\rm d}^d{\bf r}(\cosh\theta - 1) = t\ , \tag{4.73}$$

and Eq. (4.71) can be rewritten as

$$\ln P_{\rm s}(t) = -\frac{\pi\nu D}{2}\int {\rm d}^d{\bf r}(\nabla\theta)^2\ . \tag{4.74}$$

For sufficiently small times, θ is small according to (4.73), so that Eqs. (4.72), (4.73) can be linearized:

$$\nabla^2\theta + 2\gamma\theta = 0\ ; \qquad 2\gamma = i\omega/D\ ; \tag{4.75}$$

$$\frac{\pi\nu}{2}\int {\rm d}^d{\bf r}\theta^2 = t\ . \tag{4.76}$$

This yields

$$\theta({\bf r}) = \left(\frac{2t}{\pi\nu}\right)^{1/2}\phi_1({\bf r})\ , \tag{4.77}$$

where ϕ_1 is the eigenfunction of the diffusion operator corresponding to the lowest eigenvalue $\epsilon_1 = t_{\rm D}^{-1}$. The survival probability (4.74) reduces thus to

$$\ln P_{\rm s}(t) = \frac{\pi\nu D}{2}\int {\rm d}^d{\bf r}\,\theta\nabla^2\theta = -\frac{\pi\nu\epsilon_1}{2}\int {\rm d}^d{\bf r}\,\theta^2 = -t/t_{\rm D}\ , \tag{4.78}$$

as expected. Eq. (4.78) is valid (up to relatively small corrections) as long as $\theta \ll 1$, i.e. for $t\Delta \ll 1$. To find the behavior at $t \gtrsim \Delta^{-1}$, as well as the corrections at $t < \Delta^{-1}$, one should consider the exact (non-linear) equation (4.72), the solution of which depends on the sample geometry.

4.6.1 Quasi-1D geometry

We consider a wire of length L and a cross-section A with open boundary conditions at both edges, $\theta(-L/2) = \theta(L/2) = 0$. Equations (4.72), (4.73) take the form

$$\theta'' + 2\gamma\sinh\theta = 0\ , \qquad \int_{-L/2}^{L/2} {\rm d}x(\cosh\theta - 1) = t/\pi\nu A\ . \tag{4.79}$$

The solution of (4.79) yields the log-normal asymptotic behavior of $P_{\rm s}(t)$ at large times [72]:

$$\ln P_{\rm s}(t) \simeq -\frac{\beta g}{2}\ln^2(t\Delta)\ ; \qquad t\Delta \gg 1\ , \tag{4.80}$$

with $g = 2\pi\nu AD/L \gg 1$ being the dimensionless conductance.

Equation (4.80) has essentially the same form as the asymptotic formula for $G(t)$ found by Altshuler and Prigodin [119] for a *strictly* 1D sample with a length much *exceeding* the localization length:

$$G(t) \sim \exp\left\{-\frac{l}{L}\ln^2(t/\tau)\right\} \tag{4.81}$$

If we replace in Eq. (4.81) the 1D localization length l by the quasi-1D localization length $\beta\pi\nu AD$, we reproduce the asymptotics (4.80). This is one more manifestation of the equivalence of statistical properties of smooth envelopes of the wave functions in 1D and quasi-1D samples (see Sec. 4.5.2). Furthermore, the agreement of the results for the metallic and the insulating samples demonstrates clearly that the asymptotic "tail" (4.80) in a metallic sample is indeed due to anomalously localized eigenstates.

As another manifestation of this fact, Eq. (4.80) can be represented as a superposition of the simple relaxation processes with mesoscopically distributed relaxation times [120]:

$$P_{\rm s}(t) \sim \int {\rm d}t_\phi \, e^{-t/t_\phi} \mathcal{P}(t_\phi) \tag{4.82}$$

The distribution function $\mathcal{P}(t_\phi)$ then behaves as follows:

$$\mathcal{P}(t_\phi) \sim \exp\left\{ -\frac{\beta g}{2} \ln^2(g\Delta t_\phi) \right\} \; ; \qquad t_\phi \gg \frac{1}{g\Delta} \equiv t_{\rm D} \; . \tag{4.83}$$

Since the Thouless energy $t_{\rm D}^{-1}$ determines the typical width of a level of an open system, the formula (4.83) concerns indeed the states with anomalously small energy widths t_ϕ^{-1}.

The saddle-point method allows us also to find the corrections to Eq. (4.78) in the intermediate region $t_{\rm D} \ll t \ll \Delta^{-1}$ [121, 4],

$$-\ln P_{\rm s}(t) = \frac{t}{t_{\rm D}} \left(1 - \frac{1}{\beta\pi^2 g} \frac{t}{t_{\rm D}} + \ldots \right) \; , \tag{4.84}$$

with $t_{\rm D} = L^2/\pi^2 D$. Equation (4.84) is completely analogous to the formula (4.25), (4.26) for the statistics of eigenfunction amplitudes. It shows that the correction to the leading term $-t/t_{\rm D}$ in $\ln P_{\rm s}$ becomes large compared to unity at $t \gtrsim t_{\rm D}\sqrt{g}$, though it remains small compared to the leading term up to $t \sim g t_{\rm D} \sim \Delta^{-1}$. The result (4.84) was also obtained by Frahm [122] from rather involved calculations based on the equivalence between the 1D σ-model and the Fokker-Planck approach and employing the approximate solution of the Dorokhov-Mello-Pereyra-Kumar equations in the metallic limit. The fact that the logarithm of the quantum decay probability, $\ln P_{\rm s}(t)$, starts to deviate strongly (compared to unity) from the classical law, $\ln P_{\rm s}^{\rm cl}(t) = -t/t_{\rm D}$ at $t \sim t_{\rm D}\sqrt{g}$ was observed in numerical simulations by Casati, Maspero, and Shepelyansky [123]. For related results in the framework of a random matrix model see Sec. 4.6.3.

4.6.2 2D geometry

Considering a 2D disk-shaped sample of a radius R, one gets the following long-time asymptotics of $P_{\rm s}(t)$ (or $G(t)$) [72, 118]:

$$P_{\rm s}(t) \sim (t\Delta)^{-2\pi\beta g} \, , \qquad 1 \ll t\Delta \ll (R/l)^2 \tag{4.85}$$

$$P_{\rm s}(t) \sim \exp\left\{ -\frac{\pi\beta g}{4} \frac{\ln^2(t/g\tau)}{\ln(R/l)} \right\} \, , \qquad t\Delta \gg (R/l)^2 \tag{4.86}$$

where $g = 2\pi\nu D$ is the dimensionless conductance per square in 2D and τ is the mean free time. Equivalently, these results can be represented in terms of the distribution function $\mathcal{P}(t_\phi)$ of relaxation times,

$$\mathcal{P}(t_\phi) \sim \begin{cases} (t_\phi/t_\mathrm{D})^{-2\pi\beta g} \ , & t_\mathrm{D} \ll t_\phi \ll t_\mathrm{D}\left(\frac{R}{l}\right)^2 \\ \exp\left\{-\frac{\pi\beta g}{4}\frac{\ln^2(t_\phi/\tau)}{\ln(R/l)}\right\} \ , & t_\phi \gg t_\mathrm{D}\left(\frac{R}{l}\right)^2 \ , \end{cases} \tag{4.87}$$

where $t_\mathrm{D} \simeq R^2/D$ is the time of diffusion through the sample.

The far log-normal asymptotics (4.86) agrees with the one obtained earlier by Altshuler, Kravtsov, and Lerner from the renormalization-group (RG) treatment. Analogous agreement between the instanton and the RG calculations was found for the asymptotics of the local DOS distribution function [124]. Note that the instanton method is superior to the RG treatment in several respects: (i) it is not restricted to 2D or $2+\epsilon$ dimensions; (ii) it is much more transparent physically, since the stationary-point solution $\theta(\mathbf{r})$ describes directly the spatial shape of the anomalously localized state, $|\psi^2(\mathbf{r})|_\mathrm{smooth} \propto e^{\theta(\mathbf{r})}$; (iii) in some cases it allows to find intermediate asymptotics [see *e.g.* Eq. (4.85)] missed by the RG calculation.

4.6.3 Random matrix model

Here we mention briefly the results on the quantum decay law obtained by Savin and Sokolov [125] within the RMT model. This will allow us to see the similarities and the differences between the diffusive systems and the random matrix model. The model describes a Hamiltonian of an open chaotic system by a Gaussian random matrix coupled to M external (decay) channels. The found decay law has the form

$$P_\mathrm{s}(t) \sim (1+\Gamma t/M)^{-M} \ , \tag{4.88}$$

where $\Gamma = MT\Delta/2\pi$ is a typical width of the eigenstate, with T characterizing the channel coupling ($T=1$ for ideal coupling). In this case, the product MT plays the role of the dimensionless conductance g (in contrast to the diffusive case where g is governed by the bulk of the system, here it is determined by the number of decay channels and the strength of their coupling). For not too large t ($t\Delta T \ll 1$), Eq. (4.88) yields the classical decay law, $P_\mathrm{s}(t) \sim e^{-t\Gamma}$, with corrections of the form

$$\ln P_\mathrm{s}(t) = -t\Gamma(1-\Gamma t/2M+\ldots) \ , \tag{4.89}$$

which is similar to the results found for the diffusive systems (see Eq. (4.84)). At large $t \gg (\Delta T)^{-1}$, the decay takes the power-law asymptotic form [126]

$$\ln P_\mathrm{s}(t) \simeq -M\ln(\Gamma t/M) \ , \tag{4.90}$$

which is to be compared with Eqs. (4.80) and (4.85), (4.86).

5 Supersymmetry approach to the Quantum Chaos.

The aim of this section is to review recently emerged ideas concerning application of the supersymmetry formalism to chaotic ballistic systems ("billiards"). It should be stressed that this field is quite young, and most of the calculations performed so far are less rigorous than in the diffusive case; the limits of applicability of the obtained results are not well understood yet. In this sense the status of this section is somewhat different from the preceding part of this lecture course.

5.1 Introduction: What have we learned from the diffusive problem.

It is instructive to begin by summarizing what we have learned concerning the diffusive problem. As has been explained in Sec. 3, 4, the statistical properties of energy levels and eigenfunctions in a diffusive disordered sample are described by the diffusive supermatrix σ-model (3.16). In the metallic regime, when the dimensionless conductance is large, $g \gg 1$, and the eigenfunctions are not localized (*i.e.* cover roughly uniformly the whole sample volume), the zero-mode approximation is a good starting point for treating the σ-model. It reduces the problem to the 0D σ-model, yielding the RMT results for the level and eigenfunction statistics. Deviations from RMT (which are typically small in the "body" of the distribution functions but become large in the "tail") are controlled by the diffusion modes. In contrast to the universal RMT results, these deviations are *system-specific*, since they depend on the sample dimensionality, shape and size, as well as on the disorder strength. More specifically, the deviations are controlled by the diffusion operator $-D\nabla^2$, which determines the quadratic part of the action of the diffusion modes,

$$S[W] = \frac{\pi\nu}{2} \int \mathrm{d}^d\mathbf{r}\, \mathrm{Str} W_{21}[-D\nabla^2 - i\omega] W_{12}\ . \tag{5.1}$$

The magnitude of the deviations from RMT is controlled by the small parameter $g^{-1} \sim \Delta/\epsilon_1$, where ϵ_1 is the lowest non-zero eigenvalue of the diffusion operator (the Thouless energy).

5.2 Ballistic σ-model.

Let us now turn to the case of ballistic chaotic systems, *i.e.* to the quantum chaos. We will first guess what the field-theoretical description of such a problem should look like and then will discuss how it can be derived.

5.2.1 Heuristic arguments.

A bulk of numerical simulations data have unambiguously demonstrated that *generically* the statistics of energy levels and eigenfunctions amplitudes in a ballistic system (billiard) whose classical counterpart is chaotic is well described by RMT (this is known as Bohigas-Giannoni-Schmit conjecture [127]) . Since we already know that the 0D σ-model is essentially equivalent to RMT, we expect that the sought field theory of quantum chaos should reduce in the leading (zero-mode) approximation to the 0D

σ-model described in Sec. 2.4. Therefore, the field variable of this theory should be the supermatrix field belonging to the same coset space as the field $Q(\mathbf{r})$ of the σ-model.

Furthermore, in analogy with the diffusive case, we expect deviations from RMT to be controlled by the modes of density relaxation in the system. Indeed, deviations from ergodicity in a diffusive sample are physically due to the fact that the process of a particle spreading over the whole sample volume is not instantaneous but requires a time $\sim t_{\mathrm{D}}$, the relaxation being governed by the diffusion operator. Since the momentum relaxation in a diffusive sample takes place on a much shorter time scale (of the order of the mean free time τ), the diffusion operator (describing the dynamics on time scales $\gg \tau$) is an operator in $\mathbf{r}$-space only. This is why the Q-field depends on the coordinate but not on the velocity in the diffusive case. Such a separation of the fast and slow dynamics is not applicable to a generic clean sample. Therefore, in this case the supermatrix field Q should depend both on the coordinate $\mathbf{r}$ and the velocity direction $\mathbf{n} = \mathbf{v}/v_{\mathrm{F}}$, $Q = Q(\mathbf{r}, \mathbf{n})$ (the absolute value of the velocity $|\mathbf{v}| = v_{\mathrm{F}}$ being fixed by the energy conservation). The classical dynamics in the phase space is governed by the Liouville operator

$$\mathcal{L} = \{\cdot, H\} = \frac{\partial H}{\partial \mathbf{p}}\frac{\partial}{\partial \mathbf{r}} - \frac{\partial H}{\partial \mathbf{r}}\frac{\partial}{\partial \mathbf{p}} \,, \tag{5.2}$$

which reduces for the case of a billiard (no potential energy inside) to

$$\mathcal{L} = v_{\mathrm{F}}\mathbf{n}\frac{\partial}{\partial \mathbf{r}} \tag{5.3}$$

supplemented by the boundary conditions corresponding to the particle reflection by the sample boundary. It is clear from what has been said above that the Liouville operator $\mathcal{L}$ is expected to replace the diffusion operator in the σ-model description. Therefore, the analog of (5.1) should read[31]

$$S[W(\mathbf{r},\mathbf{n})] = \frac{\pi\nu}{2}\int \mathrm{d}^d\mathbf{r}\mathrm{d}\mathbf{n}\, \mathrm{Str} W_{21}(\mathbf{r},\mathbf{n})[\mathcal{L} - i\omega]W_{12}(\mathbf{r},\mathbf{n}) \,. \tag{5.4}$$

It remains to restore the action in terms of the Q-field from the quadratic form (5.4). It turns out, however, that in the ballistic case the corresponding action cannot be written in a simple form in terms of the Q-field,[32] and it is more convenient to write it in terms of the T-matrix field parametrizing the σ-model manifold according to [*cf.* (2.60), (3.15)]

$$Q(\mathbf{r},\mathbf{n}) = T(\mathbf{r},\mathbf{n})\Lambda T^{-1}(\mathbf{r},\mathbf{n}) \,. \tag{5.5}$$

Using that $T = 1 - W/2 + \ldots$, that $\mathcal{L}$ is the first-order differential operator and that the action should be invariant at $\omega \to 0$ with respect to the global rotations $Q(\mathbf{r},\mathbf{n}) \to UQ(\mathbf{r},\mathbf{n})U^{-1}$ (*i.e.* with respect to $T(\mathbf{r},\mathbf{n}) \to UT(\mathbf{r},\mathbf{n})$), one concludes that the only allowed form is

$$S[Q] = \frac{\pi\nu}{2}\int \mathrm{d}^d\mathbf{r}\mathrm{d}\mathbf{n}\, \mathrm{Str}[-2T^{-1}(\mathbf{r},\mathbf{n})\mathcal{L}T(\mathbf{r},\mathbf{n})\Lambda - i\omega Q(\mathbf{r},\mathbf{n})\Lambda] \,. \tag{5.6}$$

[31]The angular integral $\int \mathrm{d}\mathbf{n}\ldots$ is assumed to be normalized to unity, $\int \mathrm{d}\mathbf{n} = 1$.

[32]The action can be expressed in terms of the Q-field only at the expense of introduction of an additional coordinate, with the action taking the Wess-Zumino-Witten form, see [128].

5.2.2 Ballistic σ-model from disorder averaging.

The ballistic σ-model action was derived for the first time by Muzykantskii and Khmelnitskii (MK) [128]. Their starting point was a disordered system, and they followed the route outlined in Sec. 3.1 up to Eq. (3.9). Their further aim was to derive a theory which describes the physics at all momenta $q \ll k_F$ and not only at $q \ll l^{-1}$, *i.e.* which is quasiclassical but is not restricted to the diffusion approximation. Clearly, in this case one cannot use the gradient expansion leading to the diffusive σ-model (3.16). Instead of this, MK employed an analogy with the Eilenberger quasiclassical approach in the kinetic theory of disordered superconductors. Along these lines, they defined the field $Q(\mathbf{r},\mathbf{n})$ as a result of application of the following two operations to the Green's function $g(\mathbf{r},\mathbf{r}')$ [defined in Eq. (3.10)]:
(i) the Wigner transformation

$$g(\mathbf{r},\mathbf{r}') = \int \frac{\mathrm{d}^d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \tilde{g}(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{p}) \ ; \tag{5.7}$$

(ii) integration over the kinetic energy $\xi = v_{\mathrm{F}}(|\mathbf{p}| - p_{\mathrm{F}})$,

$$Q(\mathbf{r},\mathbf{n}) = \frac{1}{\pi}\int d\xi\, \tilde{g}(\mathbf{r},\mathbf{n}\frac{\xi}{v_{\mathrm{F}}}) \ . \tag{5.8}$$

They were then able to derive the ballistic σ-model[33] (5.6) with an additional term describing the scattering by impurities,

$$S_{\mathrm{imp}}[Q] = \frac{\pi\nu}{4}\int \mathrm{d}^d\mathbf{r}\mathrm{d}\mathbf{n}d\mathbf{n}' w(\mathbf{r};\mathbf{n},\mathbf{n}')\mathrm{Str}Q(\mathbf{r},\mathbf{n})Q(\mathbf{r},\mathbf{n}') \ , \tag{5.9}$$

where $w(\mathbf{r};\mathbf{n},\mathbf{n}')$ is the differential cross-section of the scattering $\mathbf{n}\to\mathbf{n}'$ at the point $\mathbf{r}$ (for the isotropic scattering $w(\mathbf{r};\mathbf{n},\mathbf{n}')$ is equal to $1/\tau(\mathbf{r})$).

MK conjectured further that the derived σ-model makes sense also in the limit $\tau\to\infty$, when it describes the clean system. Let us note, however, that one cannot simply set $\tau^{-1} = 0$ (*i.e.* remove the disorder completely). Indeed, then we would get a particular clean system with uniquely defined energy levels, so that the DOS will be a sum of δ-functions. This is certainly not what we want (or what we are able) to calculate in the σ-model approach. The correlation or distribution functions that we are discussing imply necessarily some averaging. The MK derivation seems to remain meaningful if $\Delta\tau \ll 1$. In this case the disorder is strong enough from the quantum point of view, *i.e.* we study not a particular quantum system but rather a large ensemble of systems. On the other hand, the condition that the disorder does not influence the classical dynamics is $\tau \gg L/v_{\mathrm{F}}$, where L is the system size. Since $L/v_F\Delta \sim (k_{\mathrm{F}}L)^{d-1} \gg 1$ in the semiclassical limit, the double inequality $L/v_{\mathrm{F}} \ll \tau \ll \Delta^{-1}$ can be satisfied. It means that the disorder is classically negligible, while it mixes strongly energy levels of the quantum system.

[33] As has already benn mentioned, MK obtained the ballistic action in a different (Wess-Zumino-Witten) form, which is however equivalent to (5.6).

In a diffusive sample the density relaxation is governed by the diffusion operator with eigenvalues $\epsilon_\mu > 0$ (and $\epsilon_0 = 0$ corresponding to the particle number conservation in a closed sample). The positiveness of ϵ_μ corresponds to the exponential decay of a density perturbation with time. What are their counterparts γ_μ in a chaotic sample? One might naively think that, since the evolution operator $e^{-\mathcal{L}t}$ is unitary, the corresponding eigenvalues $e^{-\gamma_\mu t}$ are of absolute value unity, so that γ_μ are purely imaginary. This is, however, incorrect. It is known that an ultraviolet regularization (projection onto the subspace of smooth functions) shifts all γ_μ (except $\gamma_0 = 0$) from the imaginary axis, giving them a positive real part, $\mathrm{Re}\gamma_\mu > 0$. These eigenvalues are known as Ruelle resonances [129]. The corresponding regularized evolution operator (called Perron-Frobenius operator) describes irreversible classical dynamics in a chaotic system. The above ultraviolet regularization may be physically understood as an infinitesimally weak noise introduced in the system to make the dynamics irreversible. The identification of the eigenvalues γ_μ determining the non-universal corrections to the spectral statistics in a chaotic system with the Ruelle resonances was done in [130].

5.2.3 σ-model from energy averaging.

In a subsequent paper, Andreev, Agam, Simons, and Altshuler (AASA) [131] proposed another derivation of the ballistic σ-model. Instead of averaging over disorder, they started from a completely clean system and performed the *energy* averaging in a certain spectral window. After the Hubbard-Stratonovich transformation, they arrived at the action of the form

$$S[\hat{Q}] \propto \mathrm{Str}_{\mathbf{r}}(-2\hat{T}^{-1}i[\hat{H},\hat{T}]\Lambda - i\omega\hat{Q}\Lambda) \ , \qquad (5.10)$$

where $\hat{H}$ is the quantum Hamiltonian, $\hat{T}$ and $\hat{Q} = \hat{T}\Lambda\hat{T}^{-1}$ are operators in the Hilbert space (and have on top of this the usual supermatrix structure), and $\mathrm{Str}_{\mathbf{r}}$ includes the supertrace over the supermatrix indices and the trace over the Hilbert space. The next (and crucial) step is the semiclassical expansion. Going to the Wigner representation and using the fact that in the semiclassical limit the Wigner transform of a commutator is a Poisson bracket of the corresponding Wigner transforms, AASA reduced (5.10) to the form (5.6). More recently Zirnbauer [18] demonstrated, however, that an additional averaging is necessary to guarantee the condition of a slow variation of the Wigner transform $T(\mathbf{r},\mathbf{p})$ required by the semiclassical expansion.

More specifically, Zirnbauer considered the level correlations for a unitary map (rather than for a Hermitian Hamiltonian). Similarly to the AASA approach, he averaged over the quasienergy $e^{i\phi}$. This can be done via the "color-flavor transformation" [19],

$$\int \mathrm{d}\phi \exp i(\Psi_1^\dagger e^{i\phi}\Psi_1 + \Psi_2^\dagger e^{-i\phi}\Psi_2) = \int \mathrm{d}\mu(Z,\bar{Z}) \exp(\Psi_1^\dagger Z\Psi_2 + \Psi_2^\dagger \bar{Z}\Psi_1) \ , \qquad (5.11)$$

where Ψ_1, Ψ_2 are supervectors, $T = \begin{pmatrix} 1 & Z \\ \bar{Z} & 1 \end{pmatrix}$ is the matrix from the coset space and $\mathrm{d}\mu(Z,\bar{Z})$ is the corresponding invariant measure.[34] This transformation replaces (in the case of unitary maps) the energy averaging and the Hubbard-Stratonovich transformation. Moreover, the saddle-point approximation is not needed in this case, since the r.h.s. of (5.11) is already an integral over the coset space. Zirnbauer showed further that, in order to justify the semiclassical expansion, one has to average over an ensemble of maps,

$$U(\xi) = \exp\left(i\sum_{k=1}^{s}\xi_k X_k/\hbar\right) U \ , \tag{5.12}$$

where ξ_k are random variables with the disorder strength scaling as $\langle\xi_k^2\rangle \propto \hbar^\alpha$, $0<\alpha<1$, in the limit $\hbar\to 0$. Since $\alpha>0$, all the members of the introduced ensemble have the same classical limits. On the other hand, the condition $\alpha<1$ ensures that the disorder is strong from the quantum point of view. This procedure is thus physically similar to the derivation of MK with $L/v_{\mathrm{F}} \ll \tau \ll \Delta^{-1}$ (see above).

5.2.4 Non-universal corrections and statistical noise.

Let us discuss now the implications of the ballistic σ-model (5.6) for the level statistics. Since the system is assumed to be chaotic, the only zero-mode [*i.e.* a field $Q(\mathbf{r})$ yielding zero when substituted in the first (kinetic) term of (5.6)] is $Q = \mathrm{const}$, so that in the zero-mode approximation the 0D σ-model and thus the RMT are reproduced, as expected.[35] Corrections to the RMT have the same form as discussed in Sec. 3.3, 3.4, with the diffusion eigenvalues ϵ_μ replaced by the Perrron-Frobenius eigenvalues γ_μ. In a strongly chaotic system of a characteristic size L, a typical relaxation time (the counterpart of the diffusion time t_{D}) is of the order of the flight time, $t_{\mathrm{B}} \sim L/v_{\mathrm{F}}$. Therefore, the ballistic counterpart of the dimensionless conductance $g \sim 1/\Delta t_{\mathrm{D}}$ can be estimated as $g \sim 1/\Delta t_{\mathrm{B}} \sim (k_F L)^{d-1} \sim N^{(d-1)/d}$, where $N \sim E/\Delta$ is the level number around which the statistics is studied. In particular, for the most often considered case of a 2D billiard $g \sim N^{1/2}$. We will discuss the deviations in more detail in Sec. 5.3, where important differences compared to the diffusive case will be demonstrated.

Prange [132] has discussed conditions of observability of the non-universal corrections to the spectral form-factor $K(\tau)$ around $\tau = 2\pi$ predicted by the σ-model. He pointed out that $K(\tau)$ is a strongly fluctuating function with the r.m.s. deviation equal to the mean value. After averaging over a window of width $\sim 1/g$ around $\tau = 2\pi$ the noise amplitude is reduced to $(g/N)^{1/2}$. Therefore, to detect the deviation $\delta K_{2\pi}(\tau)$, whose amplitude is $\sim 1/g$, the following inequality should be satisfied:

$$\left(\frac{g}{N}\right)^{1/2} \ll \frac{1}{g} \qquad \Longrightarrow \qquad N \gg g^3 \ . \tag{5.13}$$

However, as we have just discussed, $g \sim N^{1/2}$ for a generic 2D chaotic billiard, so that the condition (5.13) is not fulfilled. Therefore, the non-universal correction to

[34] Z and $\bar{Z}$ are identical to $-W_{12}/2$ and $-W_{21}/2$ respectively if the parametrization (3.22) is used.

[35] For an integrable system any function Q depending on integrals of motion only would be a zero mode, thus invalidating this consideration.

the spectral form-factor is not observable for an individual quantum system, when the only averaging available is the energy averaging. This points again to the necessity of the additional averaging over an ensemble of quantum systems having the same classical limit [132]. Let us note that for a diffusive system, Eq. (5.13) can be satisfied without problems. In this case g and N are two independent parameters, and one can consider the limit of arbitrarily large N at fixed g. Therefore, it is in principle possible to extract the non-universal correction from a single disordered sample by using the energy averaging.

5.2.5 Problem of repetitions.

The following subtle point concerning the non-universal correction to $R_2(\omega)$ predicted by the ballistic σ-model is worth mentioning here. The smooth (Altshuler-Shklovskii) part of the correction, Eqs. (3.31), (3.57), can be written as follows

$$\begin{aligned} R^{(c)}_{2,\mathrm{AS}}(\omega) &= \frac{1}{2\pi^2}\mathrm{Re}\sum_{\mu}\frac{\Delta^2}{(-i\omega+\gamma_\mu)^2} \\ &= \frac{\Delta^2}{2\pi^2}\mathrm{Re}\int_0^\infty dt\, t e^{i\omega t}\,\mathrm{Tr}\, e^{-\mathcal{L}t}\ , \end{aligned} \tag{5.14}$$

which is easily checked by using the fact that the eigenvalues of the evolution operator $e^{-\mathcal{L}t}$ are equal to $e^{-\gamma_\mu t}$. Expressing the trace in terms of the Gutzwiller sum over periodic orbits, one gets [133, 134]

$$\mathrm{Tr}\, e^{-\mathcal{L}t} = \sum_p T_p \sum_{r=1}^{\infty}\frac{\delta(t-rT_p)}{|\det(M_p^r-1)|}\ , \tag{5.15}$$

where the index p labels primitive orbits with periods T_p, the summation over r takes into account repetitions of the primitive orbits, and M_p is the monodromy matrix characterizing the dynamics in the vicinity of the orbit p. Substitution of (5.15) into (5.14) readily yields

$$R^{(c)}_{2,\mathrm{AS}}(\omega) = \frac{\Delta^2}{2\pi^2}\mathrm{Re}\sum_p T_p^2\sum_{r=1}^{\infty}\frac{r\exp(i\omega T_p r)}{|\det(M_p^r-1)|}\ . \tag{5.16}$$

On the other hand, one can calculate the two-level correlation function semiclassically, by starting from the Gutzwiller trace formula for the DOS [135, 134]

$$\frac{\nu(E)}{\langle\nu\rangle} = 1+\mathrm{Re}\frac{\Delta}{\pi}\sum_p T_p\sum_{r=1}^{\infty}\frac{\exp\{ir[S_p(E)-\mu_p\pi/2]\}}{|\det(M_p^r-1)|^{1/2}}\ , \tag{5.17}$$

where S_p is the action and μ_p the Maslov index of the orbit p. The two-level correlation function is thus represented as a double sum $\sum_{pp'}$ over the periodic orbits. Assuming that the terms with $p\neq p'$ vanish upon energy averaging due to randomly varying phase factors ("diagonal approximation" introduced by Berry), one finds [47]

$$R^{(c)}_{2,\mathrm{diag}}(\omega) = \frac{\Delta^2}{2\pi^2}\mathrm{Re}\sum_p T_p^2\sum_{r=1}^{\infty}\frac{\exp(i\omega T_p r)}{|\det(M_p^r-1)|}\ . \tag{5.18}$$

Comparing equations (5.16) and (5.18), we see that they almost agree, the only difference being in the extra factor r in (5.18). This discrepancy between the results of the ballistic σ-model and the diagonal approximation was emphasized by Bogomolny and Keating [136]. The same semiclassical analysis was performed earlier by Argaman, Imry, and Smilansky [49] in the context of diffusive systems. In that case, however, the relevant periodic orbits determining the non-universal behavior have a length $\sim v_{\rm F}t_{\rm D} \gg v_{\rm F}\tau$, *i.e.* they are much longer than the shortest periodic orbits. For this reason and in view of the exponential proliferation of the primitive periodic orbits with length, one can neglect the repetitions, keeping only the $r = 1$ term in the trace formulas. Then Eq. (5.16) and (5.18) are in full agreement with each other. On the other hand, in the ballistic case the shortest orbits of the length $\sim L$ are relevant, and there is no parameter which would justify neglecting the repetitions. There exists thus a real discrepancy between the two formulas, which still awaits its explanation.

5.3 Billiard with diffuse surface scattering

According to Sec. 5.2, the level and eigenfunction statistics of a clean chaotic system (with an ensemble averaging discussed above) are described by the formulas of Sec. 3, 4 with the Perron-Frobenius operator substituted for the diffusion operator. However, straightforward application of these results to a given chaotic billiard is complicated by the fact that the eigenvalues of the Perron-Frobenius operator are unknown, while its eigenfunctions are extremely singular. For this reason the σ-model approach has so far failed to provide explicit results for any particular ballistic system. In this subsection, we consider a ballistic system with surface disorder (a rough boundary) leading to diffusive scattering of a particle in each collision with the boundary [137, 138]. Since the particle loses memory of its direction of motion after a single collision, this model describes a limit of an "extremely chaotic" ballistic system, with typical relaxation time being of order of the flight time. (This should be contrasted with the case of a relatively slight distortion of an integrable billiard [139, 140].) This is a natural problem to be studied by the ballistic σ-model approach. While the assumption of the diffuse surface scattering makes possible an explicit analytic treatment of the problem [137, 138][36], the obtained results seem to reflect generic features of ballistic systems in the regime of hard chaos. To simplify the calculations, a circular geometry of the billiard is assumed. A similar problem was studied numerically in Ref. [142] for a square geometry. As usual, we consider the case of unitary symmetry; generalization to the orthogonal case is straightforward.

Inside the billiard, the motion is free and the Liouville operator $\mathcal{L}$ is given by Eq. (5.3). Clearly, it should be supplemented by a boundary condition, which depends on the form of the surface roughness. As a model approximation we consider purely diffuse scattering [143, 144] for which the distribution function $\varphi(\mathbf{r}, \mathbf{n})$ of the outgoing

[36] Very recently, the same approach was used [141] to calculate the persistent current in a ring with diffusive scattering.

particles is constant and is fixed by flux conservation[37]:

$$\varphi(\mathbf{r},\mathbf{n}) = \pi \int_{(\mathbf{N}\mathbf{n}')>0} (\mathbf{N}\mathbf{n}')\, \varphi(\mathbf{r},\mathbf{n}')\mathrm{d}\mathbf{n}', \quad (\mathbf{N}\mathbf{n}) < 0. \tag{5.19}$$

Here the point $\mathbf{r}$ lies at the surface, and $\mathbf{N}$ is an outward normal to the surface. This boundary condition should be satisfied by the eigenfunctions of $\mathcal{L}$. The boundary condition breaks the naive anti-hermiticity of $\mathcal{L}$, and all its eigenvalues (except the zero one) acquire a positive real part, as expected for the Perron-Frobenius operator of a chaotic system.

Specifically, the eigenvalues γ of the operator $\mathcal{L}$ corresponding to the angular momentum l obey the equation

$$\tilde{J}_l(\xi) \equiv -1 + \frac{1}{2}\int_0^\pi \mathrm{d}\theta \sin\theta \exp\left[2il\theta + 2\xi\sin\theta\right] = 0, \tag{5.20}$$

where $\xi \equiv R\gamma/v_F$, and R is the radius of the circle. For each value of $l = 0, \pm 1, \pm 2, \dots$ Eq.(5.20) has a set of solutions ξ_{lk} with $\xi_{lk} = \xi_{-l,k} = \xi^*_{l,-k}$, which can be labeled with $k = 0, \pm 1, \pm 2, \dots$ (even l) or $k = \pm 1/2, \pm 3/2, \dots$ (odd l). For $l = k = 0$ we have $\xi_{00} = 0$, corresponding to the zero mode $\varphi(\mathbf{r},\mathbf{n}) = \mathrm{const}$. All other eigenvalues have a positive real part $\mathrm{Re}\,\xi_{lk} > 0$ and govern the relaxation of the corresponding classical system to the homogeneous distribution in the phase space. The asymptotic form of the solutions of Eq. (5.20) for large $|k|$ and/or $|l|$ can be obtained by using the saddle-point method,

$$\xi_{kl} \approx \begin{cases} 0.66l + 0.14\ln l + 0.55\pi i k, & 0 \le k \ll l \\ (\ln k)/4 + \pi i(k + 1/8), & 0 \le l \ll k \end{cases} . \tag{5.21}$$

Note that for $k = 0$ all eigenvalues are real, while for high values of k they lie close to the imaginary axis and do not depend on l (see Fig. 1).

5.3.1 Level statistics.

The characteristic frequency separating the regions of the close-to-RMT and the fully non-universal behavior (analog of the Thouless energy) for the considered problem is $\omega \sim t_{\mathrm{B}}^{-1} \sim v_{\mathrm{F}}/R$. In the low-frequency range, $\omega \ll v_{\mathrm{F}}/R$, the level correlation function is given by Eq. (3.48), where we expect $A \sim 1/g^2 \sim (R\Delta/v_{\mathrm{F}})^2$. Indeed, the calculation yields [137, 138]

$$A = a\left(\frac{R\Delta}{\pi v_F}\right)^2, \qquad a = -19/27 - 175\pi^2/1152 + 64/(9\pi^2) \approx -1.48. \tag{5.22}$$

[37]The exact form of the boundary condition depends on the underlying microscopic model. In particular, the diffuse scattering can be modelled by surrounding the cavity by a disordered layer with a bulk mean free path l and a thickness $d \gg l$. The corresponding boundary condition [145, 146] differs from Eq. (5.19) by a parameterless function of order unity. For a review of the boundary conditions corresponding to various microscopic realizations of the rough surface see [147].

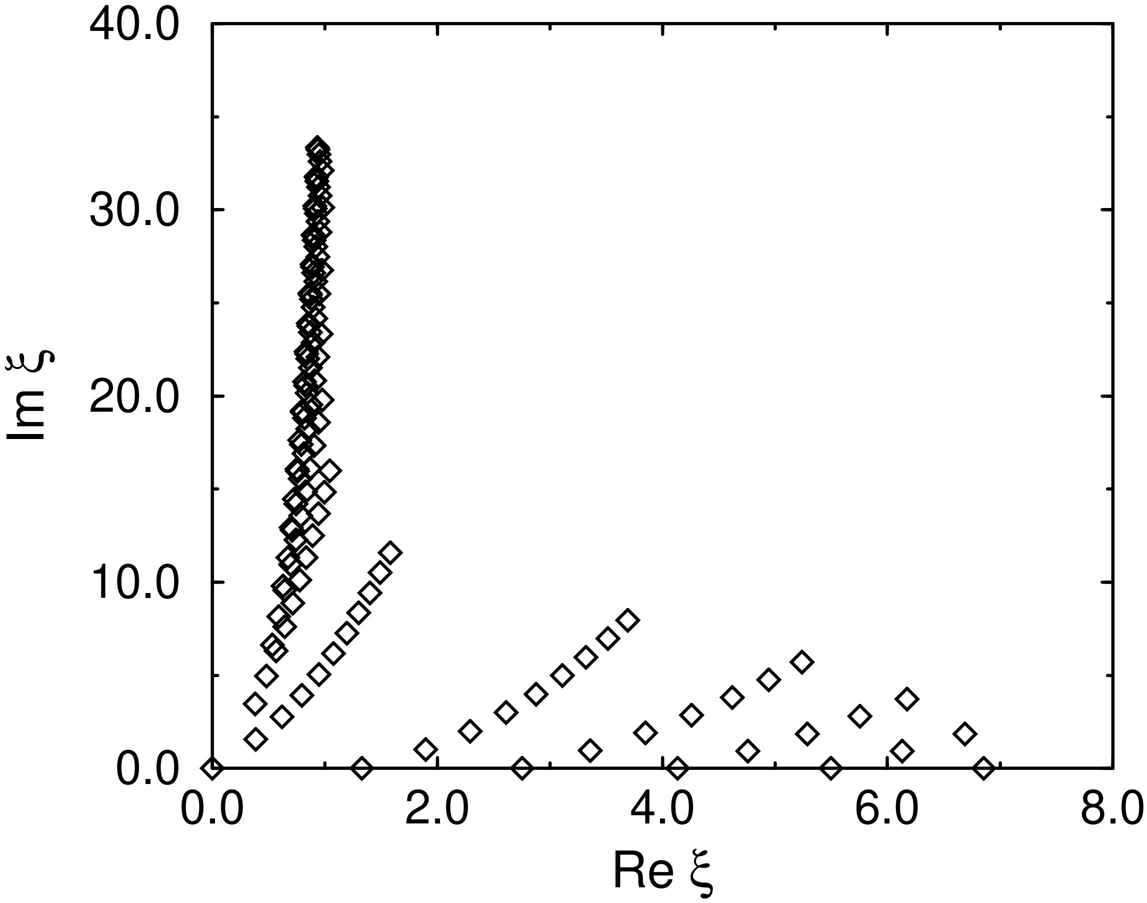

Figure 1: First 11×11 $(0 \leq k, l < 11)$ eigenvalues of the Liouville operator $\mathcal{L}$ in units of v_F/R, as given by Eq. (5.20). From Ref. [137].

In contrast to the diffusive case, this constant is negative: the level repulsion is enhanced with respect to the RMT.

The high-frequency behavior of $R_2(\omega)$ is expressed in terms of the spectral determinant (3.56) by the formulas of Sec. 3.3.3. Calculating the spectral determinant for the present problem, we find [137, 138]

$$\begin{aligned} D(s) &= s^{-2} \prod_{kl\neq(00)} (1 - is\Delta/\gamma_{kl})^{-1}(1 + is\Delta/\gamma_{kl})^{-1} \\ &= \left(\frac{\pi}{2}\right)^6 \frac{1}{N} \prod_l \frac{1}{\tilde{J}_l(isN^{-1/2})\tilde{J}_l(-isN^{-1/2})}, \end{aligned} \tag{5.23}$$

where $N = (v_\mathrm{F}/R\Delta)^2 = (p_\mathrm{F}R/2)^2$ is the number of electrons below the Fermi level. For high frequencies $\omega \gg v_F/R$ this yields the following expression for the smooth (Altshuler-Shklovskii) and the oscillating part of the level correlation function:

$$R_{2,\mathrm{AS}}(\omega) = \left(\frac{\Delta R}{v_\mathrm{F}}\right)^2 \left(\frac{v_\mathrm{F}}{2\pi\omega R}\right)^{1/2} \cos\left(4\frac{\omega R}{v_\mathrm{F}} - \frac{\pi}{4}\right), \tag{5.24}$$

$$R_{2,\mathrm{osc}}(\omega) = \frac{\pi^4}{128}\left(\frac{\Delta R}{v_\mathrm{F}}\right)^2 \cos\left(\frac{2\pi\omega}{\Delta}\right). \tag{5.25}$$

It is remarkable that the amplitude of the oscillating part (5.25) does not depend on frequency. This is in contrast to the diffusive case, where in the high-frequency regime (ω above the Thouless energy) the oscillating part $R_\mathrm{osc}(\omega)$ is exponentially small, see Eq. (3.58).

5.3.2 The level number variance.

The smooth part of the level correlation function can be best illustrated by plotting the level number variance $\Sigma_2(s)$ [see Sec. 3.4.2]. A calculation [137] gives for $s \ll N^{1/2}$

$$\pi^2\Sigma_2(s) = 1 + \mathbf{C} + \ln(2\pi s) + as^2/(2N) \tag{5.26}$$

and for $s \gg N^{1/2}$

$$\pi^2\Sigma_2(s) = 1 + \mathbf{C} + \ln\frac{16N^{1/2}}{\pi^2} - \frac{\pi^2}{16}\left(\frac{2N^{1/2}}{\pi s}\right)^{1/2} \cos\left(\frac{4s}{N^{1/2}} - \frac{\pi}{4}\right). \tag{5.27}$$

Here $\mathbf{C} \approx 0.577$ is Euler's constant, and the numerical constant a is defined by Eq. (5.22). The first three terms at the r.h.s. of Eq. (5.26) represent the RMT contribution (curve 1 in Fig. 2). As seen from Fig. 2, the two asymptotics (5.26) and (5.27) perfectly match in the intermediate regime, $s \sim N^{1/2}$. Taken together, they provide a complete description of $\Sigma_2(s)$. According to Eq.(5.27), the level number variance saturates at the value $\Sigma_2^{(0)} = \pi^{-2}(1 + \mathbf{C} + \ln(16N^{1/2}/\pi^2))$, in contrast to the behavior found for diffusive systems [see Eq. (3.70)] or ballistic systems with weak bulk disorder [148, 149]. The saturation occurs at energies $s \sim N^{1/2}$, or in dimensionful units $E \sim v_\mathrm{F}/R$. This saturation of $\Sigma_2(s)$, as well as its oscillations on the scale set by short periodic orbits, is expected for a generic chaotic billiard [47]. It is also in good agreement with the results for $\Sigma_2(s)$ found numerically for a tight-binding model with moderately strong disorder on boundary sites [142].

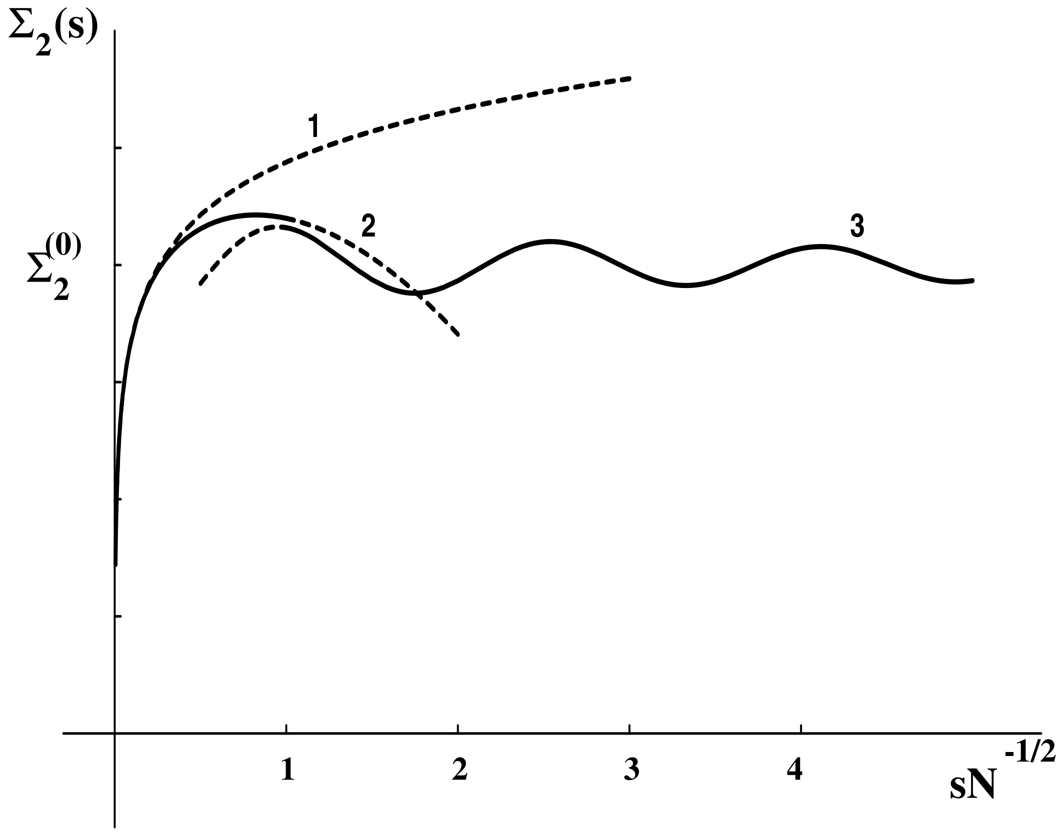

Figure 2: Level number variance $\Sigma_2(E)$ as a function of energy; $s = E/\Delta$. Curve 1 shows the RMT result, while curves 2 and 3 correspond to asymptotic regimes of low (5.26) and high (5.27) frequencies. The saturation value $\Sigma_2^{(0)}$ is given in the text. From Ref. [137].

5.3.3 Eigenfunction statistics.

Let us recall that the system-specific contributions to fluctuations and correlations of the eigenfunction amplitudes were described in the diffusive case in terms of the diffusion propagator $\Pi(\mathbf{r},\mathbf{r}')$, see Sec. 4.4, 4.5.3. Its ballistic counterpart $\Pi_{\rm B}$ is defined as follows

$$\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2) = \int d\mathbf{n}_1 d\mathbf{n}_2\, g(\mathbf{r}_1,\mathbf{n}_1;\mathbf{r}_2,\mathbf{n}_2),$$
$$\mathcal{L}g(\mathbf{r}_1,\mathbf{n_1};\mathbf{r}_2,\mathbf{n}_2) = (\pi\nu)^{-1}\left[\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{n}_1-\mathbf{n}_2) - V^{-1}\right]. \qquad (5.28)$$

Equivalently, the function $\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2)$ can be defined as

$$\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2) = \int_0^\infty dt \int d\mathbf{n}_1\, \tilde{g}(\mathbf{r}_1,\mathbf{n}_1,t;\mathbf{r}_2)\ , \qquad (5.29)$$

where $\tilde{g}$ is determined by the evolution equation

$$\left(\frac{\partial}{\partial t} + v_F\mathbf{n_1}\nabla_1\right)\tilde{g}(\mathbf{r}_1,\mathbf{n}_1,t;\mathbf{r}_2) = 0\ , \qquad t>0 \qquad (5.30)$$

with the boundary condition

$$\tilde{g}|_{t=0} = (\pi\nu)^{-1}[\delta(\mathbf{r}_1-\mathbf{r}_2) - V^{-1}]. \qquad (5.31)$$

Equation (5.28) is a natural "ballistic" counterpart of Eq.(3.42). Note, however, that $\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2)$ contains a contribution $\Pi_{\rm B}^{(0)}(\mathbf{r}_1,\mathbf{r}_2)$ of the straight line motion from $\mathbf{r}_2$ to $\mathbf{r}_1$ (equal to $1/(\pi p_{\rm F}|\mathbf{r_1}-\mathbf{r_2}|)$ in 2D and to $1/2(p_{\rm F}|\mathbf{r_1}-\mathbf{r_2}|)^{\mathbf{2}}$ in 3D), which is nothing but the smoothed version of the function $f_{\rm F}^2(|\mathbf{r}_1-\mathbf{r}_2|)$. For this reason, $\Pi(\mathbf{r}_1,\mathbf{r}_2)$ in the formulas of Sec. 4 should be replaced in the ballistic case by $\Pi(\mathbf{r}_1,\mathbf{r}_2) = \Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2) - \Pi_{\rm B}^{(0)}(\mathbf{r}_1,\mathbf{r}_2)$. At large distances $|\mathbf{r}_1-\mathbf{r}_2| \gg \lambda_F$ the (smoothed) correlation function takes in the leading approximation the form

$$V^2\langle|\psi(\mathbf{r}_1)|^2|\psi(\mathbf{r}_2)|^2 = 1 + \frac{2}{\beta}\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2). \qquad (5.32)$$

For related results obtained in the semiclassical approach see Refs. [150, 151].

Equation (5.32) shows that correlations in eigenfunction amplitudes in remote points are determined by the classical dynamics in the system. It is closely related to the phenomenon of scarring of eigenfunctions by the classical orbits [152, 153]. Indeed, if $\mathbf{r}_1$ and $\mathbf{r}_2$ belong to a short periodic orbit, the function $\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2)$ is positive, so that the amplitudes $|\psi_k(\mathbf{r}_1)|^2$ and $|\psi_k(\mathbf{r}_2)|^2$ are positively correlated. This is a reflection of the "scars" associated with this periodic orbit and a quantitative characterization of their strength in the coordinate space. Note that this effect gets smaller with increasing energy E of eigenfunctions. Indeed, for a strongly chaotic system and for $|\mathbf{r}_1-\mathbf{r}_2| \sim L$ (L being the system size), we have in the 2D case $\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2) \sim \lambda_F/L$, so that the magnitude of the correlations decreases as $E^{-1/2}$.

For the model of a circular billiard with diffuse surface scattering a direct calculation gives [137]

$$\Pi_{\rm B}(\mathbf{r}_1,\mathbf{r}_2) \;=\; \Pi_1(\mathbf{r}_1,\mathbf{r}_2)+\Pi_2(\mathbf{r}_1,\mathbf{r}_2), \tag{5.33}$$

$$\begin{aligned}
\Pi_1(\mathbf{r}_1,\mathbf{r}_2) \;&=\; \Pi_{\rm B}^{(0)}(\mathbf{r}_1-\mathbf{r}_2)-V^{-1}\int d^2\mathbf{r}'_1\Pi_{\rm B}^{(0)}(\mathbf{r}'_1-\mathbf{r}_2) \\
&-V^{-1}\int d^2\mathbf{r}'_2\Pi_{\rm B}^{(0)}(\mathbf{r}_1-\mathbf{r}'_2)+V^{-2}\int d^2\mathbf{r}'_1 d^2\mathbf{r}'_2\Pi_{\rm B}^{(0)}(\mathbf{r}'_1-\mathbf{r}'_2); \\
\Pi_2(\mathbf{r}_1,\mathbf{r}_2) \;&=\; \frac{1}{4\pi p_F R}\sum_{k=1}^{\infty}\frac{4k^2-1}{4k^2}\left(\frac{r_1 r_2}{R^2}\right)^k\cos k\,(\theta_1-\theta_2)
\end{aligned} \tag{5.34}$$

where $\Pi_{\rm B}^{(0)}(\mathbf{r})=1/(\pi p_F|\mathbf{r}|)$, and (r,θ) are the polar coordinates. This formula has a clear interpretation. The function $\Pi_{\rm B}$ can be represented as a sum over all paths leading from $\mathbf{r}_1$ to $\mathbf{r}_2$, with possible surface scattering in between. In particular, Π_1 corresponds to direct trajectories from $\mathbf{r}_1$ to $\mathbf{r}_2$ with no reflection from the surface, while the contribution Π_2 is due to the surface scattering. The first term in the numerator $4k^2-1$ comes from trajectories with only one surface reflection, while the second sums up contributions from multiple reflections.

5.4 σ-model for the kicked rotor.

Up to now we have considered autonomous systems, with the Hamiltonian $\hat{H}$ having no time dependence. The supersymmetric σ-model approach is, however, also applicable to periodically driven systems ($\hat{H}(t)$ periodic in time t), as we are going to discuss. The standard system of the latter type is the quantum kicked rotor (QKR) [154, 155, 30] defined by the Hamiltonian

$$\hat{H}=\frac{\hat{l}^2}{2I}+\tilde{k}\cos\theta\sum_{m=-\infty}^{\infty}\delta(t-mT)\ . \tag{5.35}$$

Here $\hat{l}=-i\hbar\partial/\partial\theta$ is the angular momentum operator conjugate to the angle variable θ; I is the moment of inertia, T the kick period, and $\tilde{k}$ the kick strength. The classical version of this problem is characterized by a single dimensionless parameter $K=\tilde{k}\tau/I$. For a sufficiently large K the motion becomes globally chaotic and the system exhibits unbounded diffusion in the phase space (in the direction of the angular momentum), the diffusion coefficient being

$$D\equiv\left.\frac{\langle[l(t)-l(0)]^2\rangle}{2t}\right|_{t\to\infty}\simeq\frac{\tilde{k}^2}{4T}\qquad\text{for } K\gg 1. \tag{5.36}$$

In contrast to the classical problem, the quantum system depends non-trivially on both parameters $\tilde{k}$ and T, since there are two dimensionless combinations which can be formed: $k=\tilde{k}/\hbar$ and $\tau=\hbar T/I$. The classical limit corresponds thus to $k\to\infty$,

$\tau \to 0$ at fixed $k\tau = K =$ const. After this short reminder of the classical-quantum correspondence for the kicked rotor problem, we set $\hbar = 1$ as in the other parts of this article.

Numerical simulations of the QKR have shown that at long times the classical diffusion is suppressed, the phenomenon known as "dynamical localization". In Ref. [156] an analogy between this problem and the Anderson localization in 1D systems was drawn. Later studies revealed a close connection between the QKR problem and the RBM ensemble (see Sec. 2.5.4). The evolution (Floquet) operator of the QKR has in the angular momentum representation the form

$$U_{ll'} = (-i)^{l-l'} \exp\left\{-i\frac{\tau}{2}l^2\right\} J_{l-l'}(k) , \tag{5.37}$$

where $J_m(k)$ is the Bessel function. Since $J_m(k) \simeq (2\pi m)^{-1/2}(ek/2m)^m$ for $m \gg 1, k$, we see that the matrix elements of U are exponentially small for $|l-l'| \gg k$. Therefore the matrix U is indeed banded, with the bandwidth $b \sim k \gg 1$. Furthermore, the second factor on the r.h.s. of (5.37) mimics a generator of pseudorandom numbers (of absolute value unity). Indeed, numerical simulations [30, 31, 157] have demonstrated that the statistical properties of the two models (QKR and RBM) are very close to each other.

On the other hand, the RBM ensemble was shown to belong to the class of quasi-1D disordered systems described by the 1D σ-model [32], see also Sec. 4.3.1. The solution of the 1D σ-model made possible a detailed analytical study of eigenfunctions of such systems [70], see Sec. 4.3. While the pseudo-RBM properties of the QKR evolution operator $U_{nn'}$, as well as the numerical simulations, suggested strongly that these results are also applicable to QKR, a formal derivation of this fact has been missing until recently. This gap was filled by Altland and Zirnbauer (AZ) [158] who achieved a mapping of the QKR onto the 1D σ-model. The idea of their calculation is essentially the same as the one described in Sec. 5.2.3. Performing the averaging over the quasienergy spectrum with making use of the transformation (5.11) and carrying out the semiclassical expansion, AZ obtained the 1D version of the σ-model (3.16) (or its orthogonal-class counterpart for unbroken time-reversal symmetry), with D equal to the classical diffusion constant of the rotor (5.36) and $\nu = T/2\pi$. This is precisely what could have been expected from the analogy with disordered wires or the RBM ensemble; the above value of ν being the density of quasienergies ω_k corresponding to the eigenvalues $e^{i\omega_k T}$ of the Floquet operator. This mapping allowed AZ to conclude that the statistical properties of the QKR are identical to those of the quasi-1D disordered systems. In particular, the localization length $L_{\rm loc}$ of the QKR governing the decay of a typical eigenfunction, $|\psi^2(l)|_{\rm typ} \propto \exp(-|l-l_0|/L_{\rm loc})$, is found to be $L_{\rm loc} = \beta\pi\nu D = (\beta/8)k^2$, in agreement with numerical simulations of Shepelyansky [159].[38]

[38]Shepelyansky considered also another localization length – that of a steady state – and concluded that it is larger by a factor of 2. This is in disagreement with the field-theoretical results, which give the same value $L_{\rm loc} = (\beta/8)k^2$ for the both lengths. Presumably, the discrepancy is due to insufficient numerical accuracy of evaluation of the steady state asymptotic decay rate for $k^2 \gg 1$ in [159].

It should be noted, however, that the problem of sufficiency of the energy averaging discussed in Sec. 5.2.3 is equally applicable here. In [160] AZ acknowledged that an additional averaging over an ensemble of rotors having the same classical limit (*i.e.* of the type described in Sec. 5.2.3) was needed to justify the semiclassical expansion which is in the heart of the derivation in [158]. In fact, the conclusion of the existence of such an implicit ensemble averaging in [158] can also be supported by the following argument. Following [158], one can calculate, *e.g.*, the distribution function of the quantity of the type (4.31), $v \propto |\psi_\alpha^2(l_1)\psi_\alpha^2(l_2)|$, which will have the log-normal distribution (4.32). In particular, the far tail of this distribution is crucially important for identifying the localization length of a typical eigenstate on the basis of the average value $\langle v \rangle$ (which has a 4 times smaller decay rate). However, to be able to find the whole distribution, one should average over an exponentially large $[\gg \exp(|l_1 - l_2|/4L_{\rm loc})]$ number of eigenfunctions, while the energy averaging alone reduces effectively to an averaging over $\sim |l_1 - l_2|$ eigenfunctions only and is thus insufficient.

5.5 Concluding remarks.

We finish these notes by comparing briefly the supersymmetric and the semiclassic (periodic orbit) approach to the spectral statistics of chaotic systems. The main advantage of the supersymmetry method is that it allows one to get the RMT results in the leading approximation. In contrast, the semiclassical approach is only justified for times much shorter than the Heisenberg one, $\tau/2\pi \ll 1$. Obtaining the GUE results within this approach requires using an *ad hoc* regularization prescription [136, 161]. Even the problem of calculating the perturbative (in τ) corrections to the leading behavior $K(\tau) = \tau/\pi$ in GOE (for GUE the perturbative corrections are identically zero) has not been solved semiclassically. Therefore, the σ-model approach is the only known method allowing to obtain the RMT results in a controlled way.

On the other hand, the ballistic σ-model approach is also not free from problems. In particular, construction of a regular expansion in $1/g$ has not been achieved in this case (in contrast to the diffusive σ-model). While the leading non-universal contributions (discussed above) to the energy level and eigenfunction statistics are only determined by the density relaxation modes ϕ_μ and their eigenvalues γ_μ, the higher-order (in $1/g$) terms are induced by interaction of these modes. In the case of the diffusive σ-model, the corresponding contributions are known as weak localization corrections and can be calculated systematically. At the same time, recent attempts to perform such a calculation in the ballistic case [162, 163] did not produce unambiguous results, because of the singular nature of the expressions obtained. This problem (as well as that of repetitions, see Sec. 5.2.5) remains a challenge for future research.

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