GEORGIA TECH ,
MATH 8823 BED,

SCHOOLS OF MATH & PHYSICS
RANDOM MATRIX THEORY

Random Matrix Theory

Series 1

Laguerre Polynomials :

The Laguerre polynomials are defined by the following two conditions :

1. $L_n(x) = x^n + O(x^{n-1})$
2. $\int_0^\infty dx e^{-x} L_n(x)\, x^l = 0$ whenever $0 \leq l \leq n-1$.

1). Using the orthogonality of such polynomials, prove the following properties :

1. **Recurrence relation :** $xL_n = L_{n+1} + (2n+1)L_n + n^2 L_{n-1}$.
2. **Derivative :** $xdL_n/dx = nL_n + n^2 L_{n-1} = xL_n - L_{n+1} - (n+1)L_n$.
3. **Normalization :** $\|L_n\| = n!$.
4. **Expansion :** $L_n(x) = x^n - n^2 x^{n-1} + O(x^{n-2})$.

2). Using the previous relations, compute the *generating functional* defined by :

$$F(x,y) \;=\; \sum_{n=0}^{\infty} \frac{y^n}{n!}\, L_n(x) \;.$$

3). Using the previous prelations prove that the Laguerre polynomials satisfy the following differential equation :

$$x\,\frac{d^2 L_n}{dx^2} \;+\; (1-x)\,\frac{dL_n}{dx} \;+\; n\,L_n \;=\; 0$$

4). Using the inner product

$$\langle f|g\rangle \;=\; \int_0^\infty dx e^{-x}\,\overline{f(x)}\,g(x) \;,$$

show that the differential operator

$$H \;=\; -x\frac{d^2}{dx^2} - (1-x)\,\frac{d}{dx} \;,$$

satisfies

$$\langle f|Hg\rangle \;=\; \int_0^\infty dx\; xe^{-x}\,\overline{\frac{df}{dx}}\,\frac{dg}{dx} \;.$$

Conclude by showing that the L_n's are the eigevectors of H and that H as no other part in its spectrum than simple eigenvalues. Give the list of the eigenvalues.